Inhomogeneous phases in the 3+1-dimensional mean-field Nambu-Jona-Lasinio model on the lattice

Laurin Pannullo, Marc Wagner, Marc Winstel

Goethe University Frankfurt

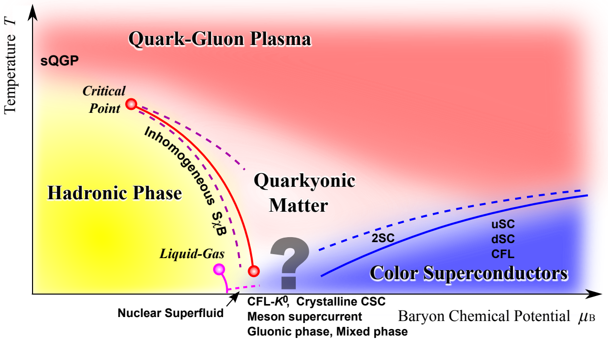
HFHF Theory Retreat, 16.09.2022

Don’t investigate
Inhomogeneous phases in the 3+1-dimensional mean-field Nambu-Jona-Lasinio model on the lattice

Laurin Pannullo, Marc Wagner, Marc Winstel
Goethe University Frankfurt

HFHF Theory Retreat, 16.09.2022

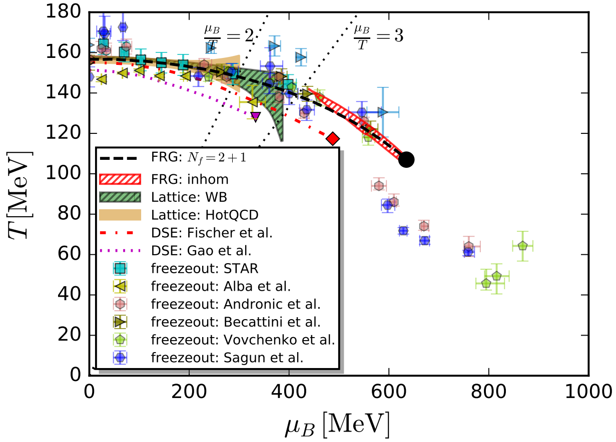
Motivation I



[K. Fukushima, T. Hatsuda, *Reports on Prog. Phys.* **74** (2011)]

- QCD phase diagram – a plot full of conjectures
- What goes on at finite or large μ_B?

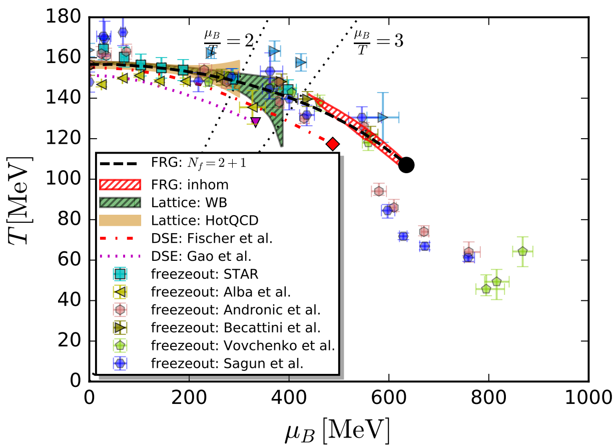
Motivation I



[W.-j. Fu *et al.*, *Phys. Rev. D.* **101** (2020)]

- QCD phase diagram – a plot full of conjectures
- What goes on at finite or large μ_B?
  - Do first principal calculations $\Rightarrow$ very hard / impossible

Motivation I



[W.-j. Fu *et al.*, *Phys. Rev. D.* **101** (2020)]

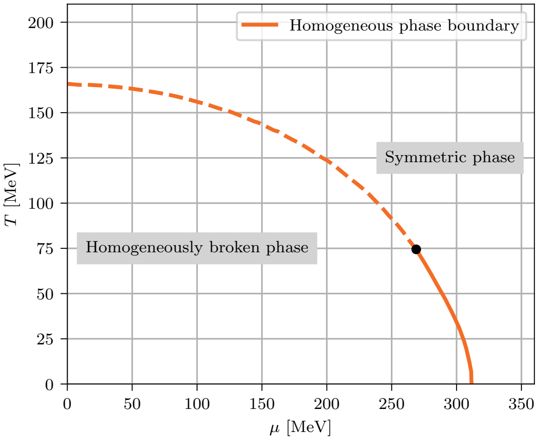
- QCD phase diagram – a plot full of conjectures
- What goes on at finite or large μ_B?
  - Do first principal calculations $\Rightarrow$ very hard / impossible
  - Use models of QCD $\Rightarrow$ a lot easier; questionable physical relevance of predictions

Motivation II

- Nambu-Jona-Lasinio model

$$\mathcal{L} = \bar{\psi}\left(\not{\partial} + \gamma_0 \mu + \sigma + \mathrm{i}\gamma_5\, \boldsymbol{\tau}\cdot\boldsymbol{\pi}\right)\psi + \frac{\sigma^2 + \boldsymbol{\pi}^2}{4G}$$

- phase diagram that resembles our QCD expectations



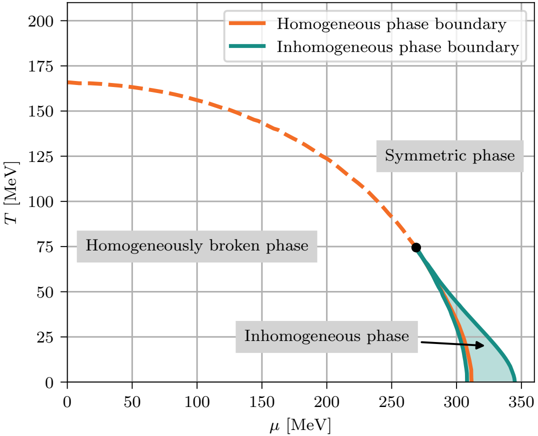
[D. Nickel, *Phys. Rev. D.* **80** (2009)]

Motivation II

- Nambu-Jona-Lasinio model

$$\mathcal{L} = \bar{\psi}\left(\partial\!\!\!/ + \gamma_0 \mu + \sigma + \mathrm{i}\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{\pi}\right)\psi + \frac{\sigma^2 + \boldsymbol{\pi}^2}{4G}$$

- phase diagram that resembles our QCD expectations
- features an inhomogeneous phase (IP) – a phase with a space-dependent chiral condensate



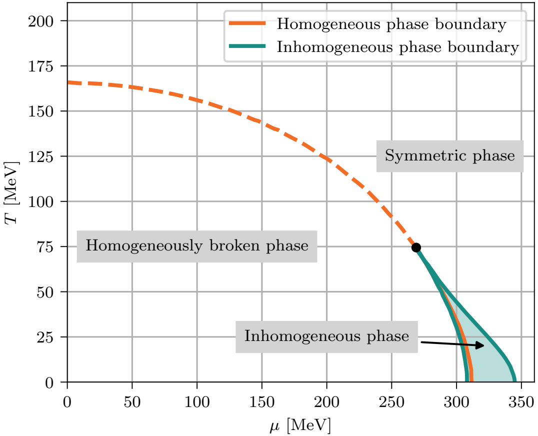
[D. Nickel, *Phys. Rev. D.* **80** (2009)]

Motivation II

- Nambu-Jona-Lasinio model

$$\mathcal{L} = \bar{\psi}\,(\partial\!\!\!/ + \gamma_0\mu + \sigma + \mathrm{i}\gamma_5\,\boldsymbol{\tau}\cdot\boldsymbol{\pi})\,\psi + \frac{\sigma^2 + \boldsymbol{\pi}^2}{4G}$$

- phase diagram that resembles our QCD expectations
- features an inhomogeneous phase (IP) – a phase with a space-dependent chiral condensate
- several problems with this result
  - Mean-field
    $\Rightarrow$ no bosonic quantum fluctuations
  - non-renormalizable model
    $\Rightarrow$ results may depend on the regularization



[D. Nickel, *Phys. Rev. D.* **80** (2009)]

Motivation III

- Main goal: Investigate IPs in models for QCD beyond the mean-field approximation
  - via FRG $\Rightarrow$ see Lennart Kurth's talk yesterday
  - via lattice Monte-Carlo (MC) simulations $\Rightarrow$ our approach

Motivation III

- Main goal: Investigate IPs in models for QCD beyond the mean-field approximation
  - via FRG $\Rightarrow$ see Lennart Kurth's talk yesterday
  - via lattice Monte-Carlo (MC) simulations $\Rightarrow$ our approach
- But first:
  - feasability of lattice MC simulations of these models at finite μ
    $\Rightarrow$ there have been MC simulations in the past [S. Hands, D. N. Walters, *Phys. Rev. D.* **69** (2004)]

Motivation III

- Main goal: Investigate IPs in models for QCD beyond the mean-field approximation
  - via FRG $\Rightarrow$ see Lennart Kurth's talk yesterday
  - via lattice Monte-Carlo (MC) simulations $\Rightarrow$ our approach
- But first:
  - feasability of lattice MC simulations of these models at finite μ
    $\Rightarrow$ there have been MC simulations in the past [S. Hands, D. N. Walters, *Phys. Rev. D.* **69** (2004)]
  - explore regulator dependence of IPs
  - suitability of lattice regularizations for IPs

Motivation III

- Main goal: Investigate IPs in models for QCD beyond the mean-field approximation
  - via FRG $\Rightarrow$ see Lennart Kurth's talk yesterday
  - via lattice Monte-Carlo (MC) simulations $\Rightarrow$ our approach
- But first:
  - feasability of lattice MC simulations of these models at finite μ
    $\Rightarrow$ there have been MC simulations in the past [S. Hands, D. N. Walters, *Phys. Rev. D.* **69** (2004)]
  - explore regulator dependence of IPs
  - suitability of lattice regularizations for IPs

$\Rightarrow$ Simple setup: Stability analysis of the $3+1$-dimensional NJL model in mean-field

Parameter fixing of the NJL model

coupling G

regulator Λ, e.g., Pauli-Villars mass, lattice spacing

$$\bar{\psi}\left(\not{\partial} + \gamma_0 \mu + \sigma + \mathrm{i}\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\pi}\right)\psi + \frac{\sigma^2 + \boldsymbol{\pi}^2}{4G}$$

Pion-decay constant $f_\pi = 88\,\mathrm{MeV}$

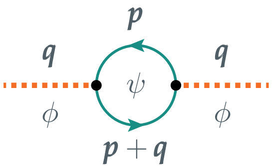
Constituent quark mass $M_0 \sim \langle \bar{\psi}\psi \rangle_{\mu=T=0}$
in range of 150-500 MeV

[S. P. Klevansky, *Rev. Mod. Phys.* **64** (1992)] [S. Hands, D. N. Walters, *Phys. Rev. D.* **69** (2004)]

Stability analysis

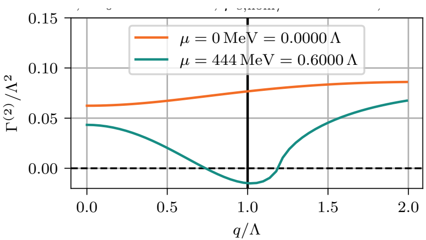
- Apply inhomogeneous perturbations $\delta\tilde{\phi}(\boldsymbol{q})$ to homogeneous fields
- Curvature of the action in direction $\delta\tilde{\phi}(\boldsymbol{q})$ is given by the bosonic two-point function $\Gamma^{(2)}_{\phi}(\boldsymbol{q})$
- Simple quantity in the mean-field approximation

$$\Gamma^{(2)}_{\phi}(\boldsymbol{q}) = \frac{1}{2G} +$$



$$= \frac{1}{2G} + \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \mathrm{Tr}\left[X_\phi\, S(p)\, X_\phi\, S(p+q)\right]$$

- negative values indicate instability for mode $\boldsymbol{q}$

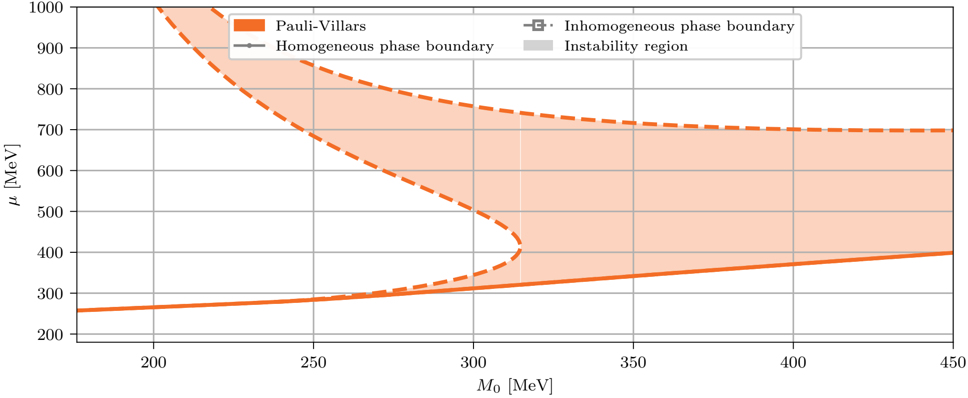


[M. Buballa *et al.*, *The Eur. Phys. J. Special Top.* **229** (2020)]

[A. Koenigstein *et al.* (2021)]

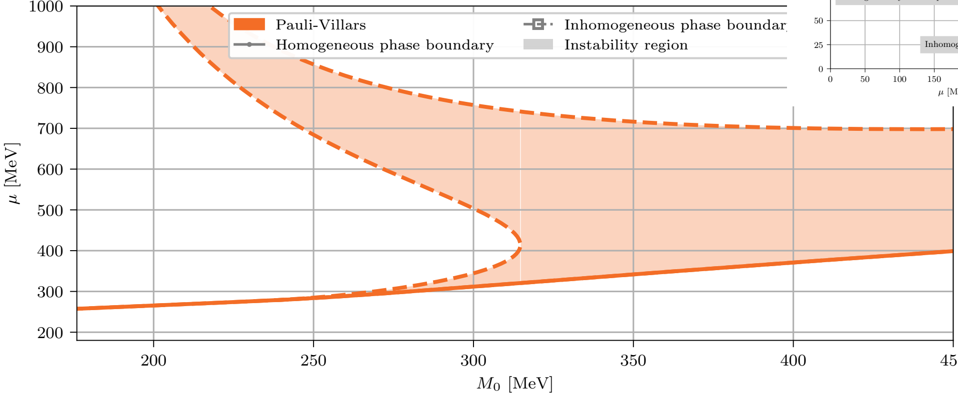
[M. Buballa *et al.*, *Phys. Rev. D.* **103** (2021)]

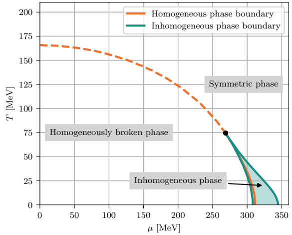
Quark mass scan



- Most common regularization scheme: Pauli-Villars (polynomial suppression of large momenta) [D. Nickel, *Phys. Rev. D.* **80** (2009)]

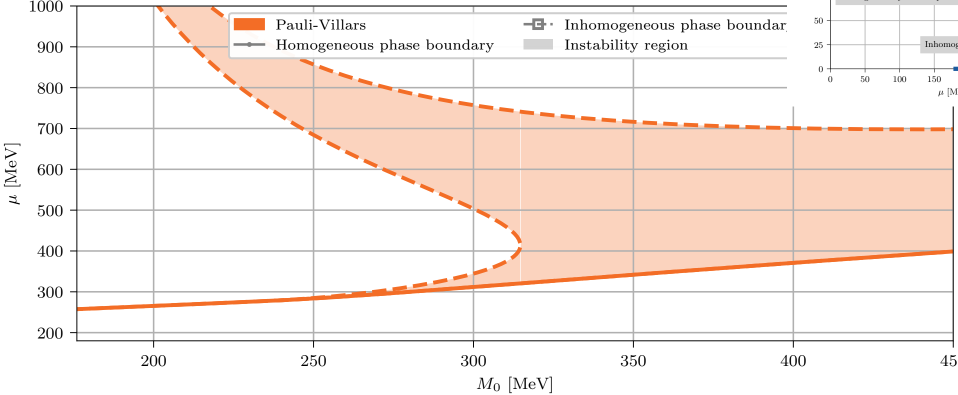
Quark mass scan

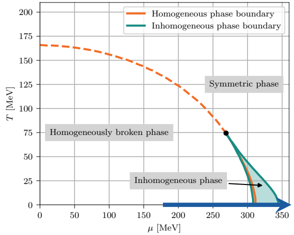




- Most common regularization scheme: Pauli-Villars (polynomial suppression of large momenta) [D. Nickel, *Phys. Rev. D.* **80** (2009)]

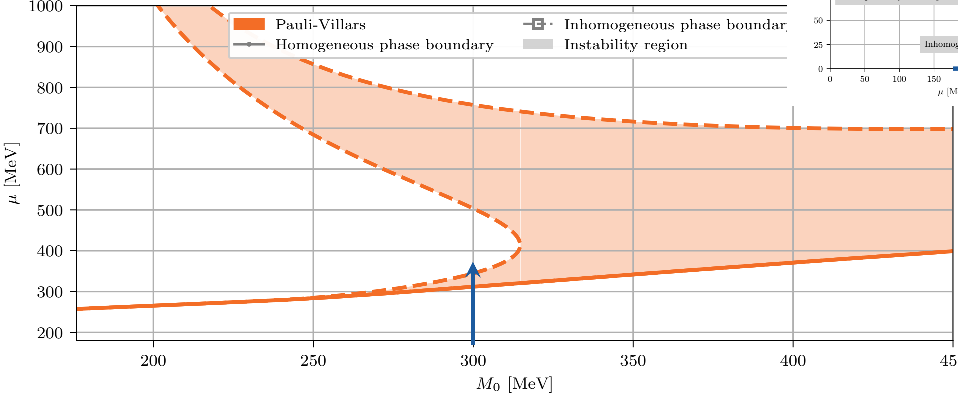
Quark mass scan

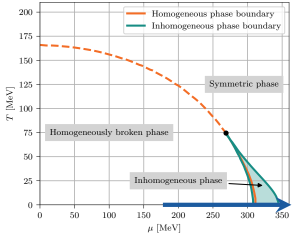




- Most common regularization scheme: Pauli-Villars (polynomial suppression of large momenta) [D. Nickel, *Phys. Rev. D.* **80** (2009)]

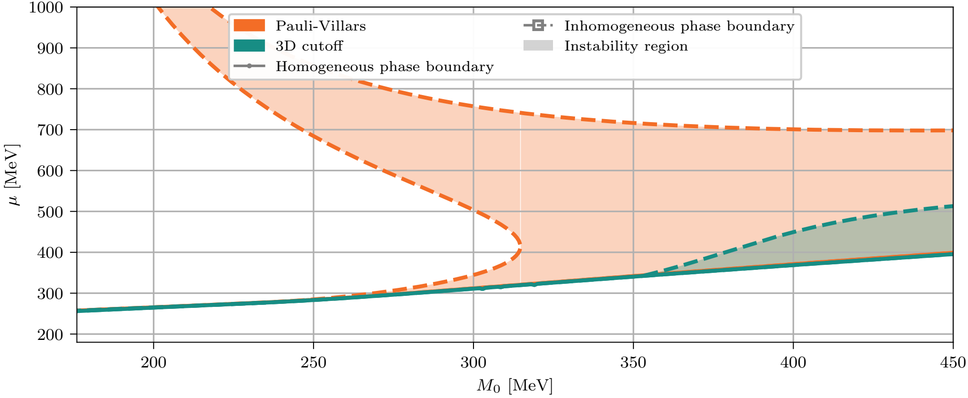
Quark mass scan





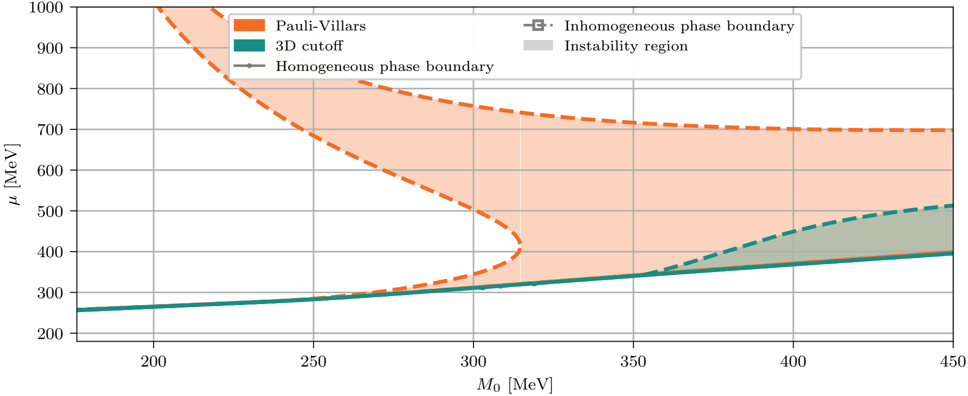
- Most common regularization scheme: Pauli-Villars (polynomial suppression of large momenta) [D. Nickel, *Phys. Rev. D.* **80** (2009)]

Quark mass scan



- 3D Cutoff (restriction of spatial loop momenta $|\boldsymbol{p}| < \Lambda$)
- Similar homogeneous phase boundary, but vastly different instability region

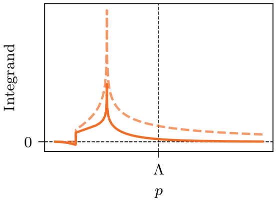
Quark mass scan



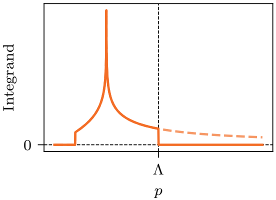
- General problem with cutoff schemes? Explicit breaking of translational invariance?
- Or general problem with results in this non-renormalizable model?

Lattice regularizations I

Pauli-Villars
polynomial supression



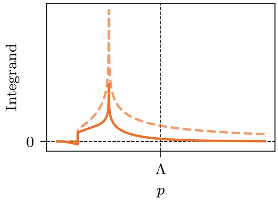
Momentum Cutoff
hard supression



Lattice regularizations I

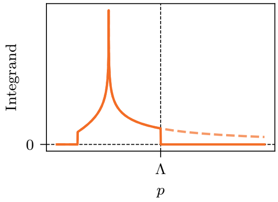
Pauli-Villars

polynomial supression



Momentum Cutoff

hard supression

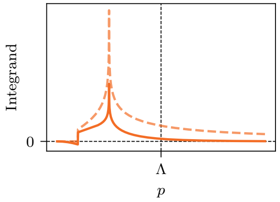


Lattice?

Lattice regularizations I

Pauli-Villars

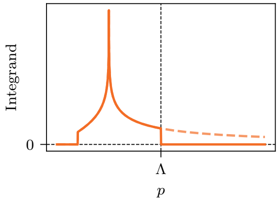
polynomial supression



Lattice?

Momentum Cutoff

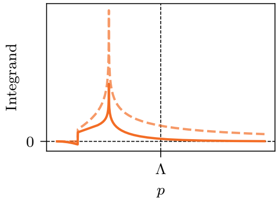
hard supression



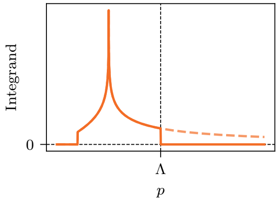
- something like a cutoff
  $\Rightarrow$ maximum momentum π/a
- but also periodic dispersion relation
  $\Rightarrow$ can give rise to doublers

Lattice regularizations I

Pauli-Villars
polynomial supression

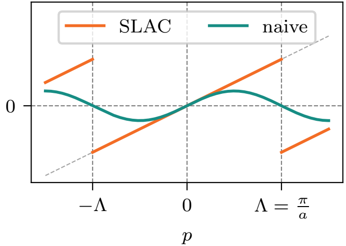


Momentum Cutoff
hard supression



Lattice?

- something like a cutoff
  $\Rightarrow$ maximum momentum π/a
- but also periodic dispersion relation
  $\Rightarrow$ can give rise to doublers



Lattice regularizations II

- Wilson fermions – explicit breaking of chiral symmetry at finite a **X**
- Domain wall fermions – too expensive for now **X**

Lattice regularizations II

- Wilson fermions – explicit breaking of chiral symmetry at finite a **X**
- Domain wall fermions – too expensive for now **X**
- SLAC – exact chiral symmetry, but problematic jump at the edge of the Brioullin zone **?**

Lattice regularizations II

- Wilson fermions – explicit breaking of chiral symmetry at finite a **X**
- Domain wall fermions – too expensive for now **X**
- SLAC – exact chiral symmetry, but problematic jump at the edge of the Brioullin zone **?**
- naive fermions – doubler mixing, but can be cured in the continuum limit **?**
  - $\Rightarrow$ problems with temporal doublers **X**
  - $\Rightarrow$ use Hybrid discretization with SLAC in the temporal direction and naive fermions in the spatial directions **?**

Lattice regularizations II

- Wilson fermions – explicit breaking of chiral symmetry at finite a **X**
- Domain wall fermions – too expensive for now **X**
- SLAC – exact chiral symmetry, but problematic jump at the edge of the Brioullin zone **?**
- naive fermions – doubler mixing, but can be cured in the continuum limit **?**
  - $\Rightarrow$ problems with temporal doublers **X**
  - $\Rightarrow$ use Hybrid discretization with SLAC in the temporal direction and naive fermions in the spatial directions **?**
- staggered fermions – *less* doubler mixing, but can be cured in the continuum limit; not full chiral symmetry **?**

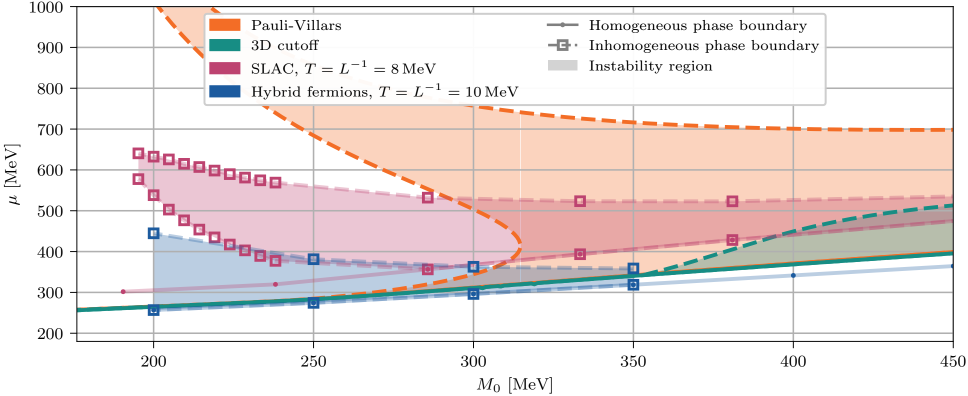
Lattice regularizations II

- Wilson fermions – explicit breaking of chiral symmetry at finite a **X**
- Domain wall fermions – too expensive for now **X**
- SLAC – exact chiral symmetry, but problematic jump at the edge of the Brioullin zone **?**
- naive fermions – doubler mixing, but can be cured in the continuum limit **?**
  - $\Rightarrow$ problems with temporal doublers **X**
  - $\Rightarrow$ use Hybrid discretization with SLAC in the temporal direction and naive fermions in the spatial directions **?**
- staggered fermions – *less* doubler mixing, but can be cured in the continuum limit; not full chiral symmetry **?**

Finished results with these

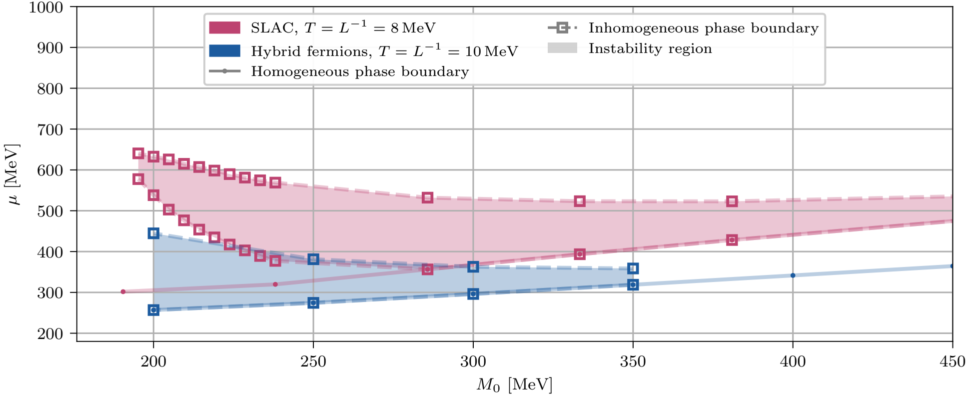
In progress

Quark mass scan



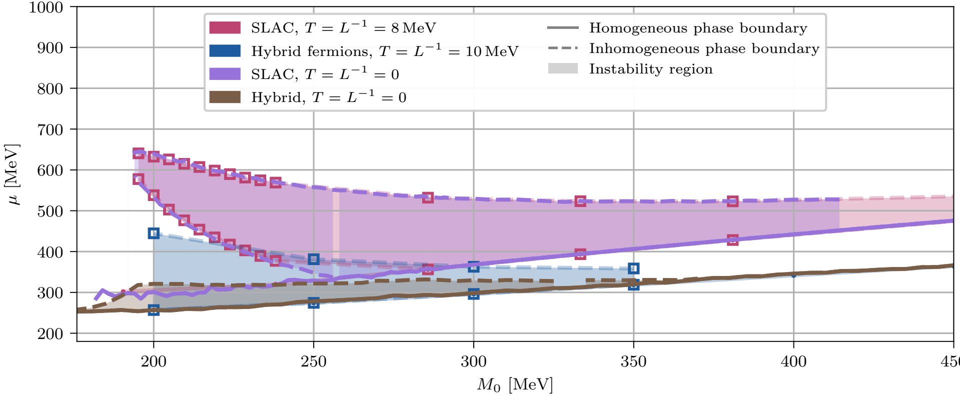
- Even more conflicting results between regularizations

Quark mass scan



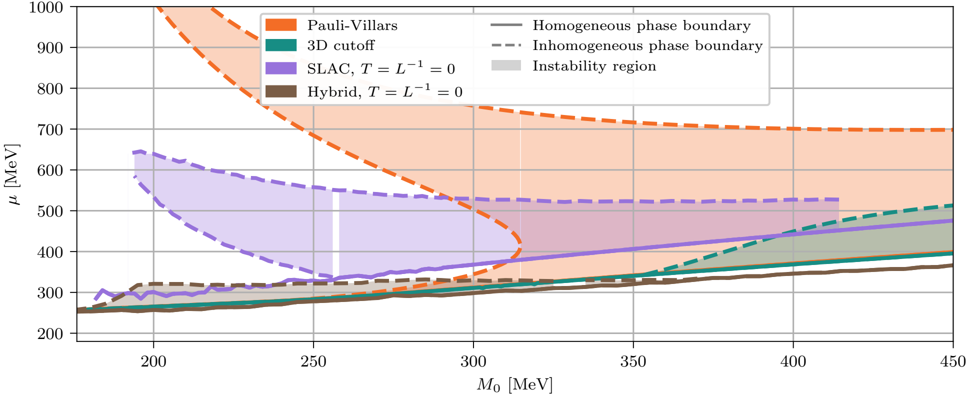
- Lattice results at finite space time volume
  $\Rightarrow$ mean-field stability analysis so simple that we can go on an infinite lattice
- SLAC unaffected, but Hybrid much smaller instability

Quark mass scan



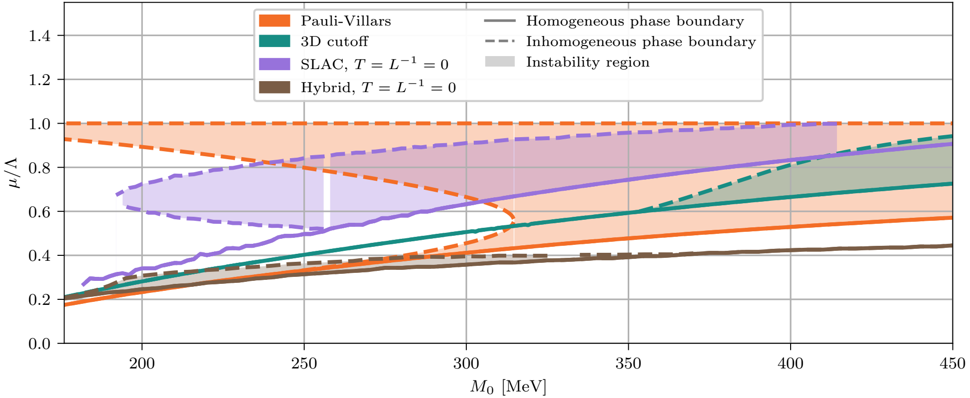
- Lattice results at finite space time volume
  $\Rightarrow$ mean-field stability analysis so simple that we can go on an infinite lattice
- SLAC unaffected, but Hybrid much smaller instability

Quark mass scan



- Lattice results at finite space time volume
  $\Rightarrow$ mean-field stability analysis so simple that we can go on an infinite lattice
- SLAC unaffected, but Hybrid much smaller instability

Quark mass scan

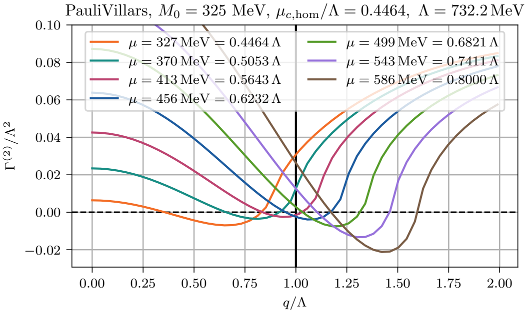


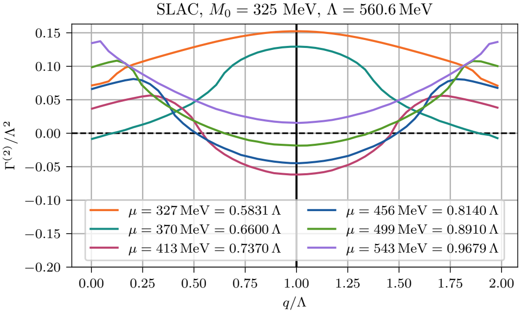
- Plot in cutoff units reveals that chemical potentials are in the order of the cutoff !
- IPs vanish when moving to larger cutoffs

A closer look at SLAC I

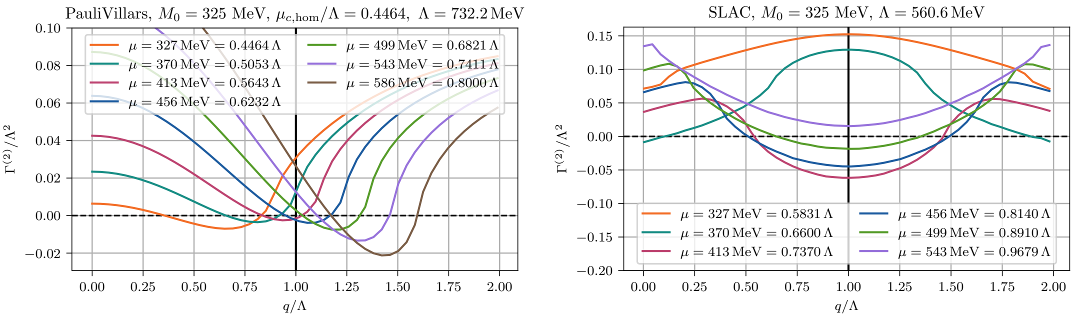
- Shouldn't SLAC be quite similar to the 3D cutoff results? After all SLAC fermions have the continuum dispersion relation...
- Yes, but they also have something like a doubler due to the discontinuity at the edge of the Brioullin zone
- This discontinuity could be probed by the bosonic field **!**
  $\Rightarrow$ But this is actually not the main problem here.

A closer look at SLAC II - Two-point functions



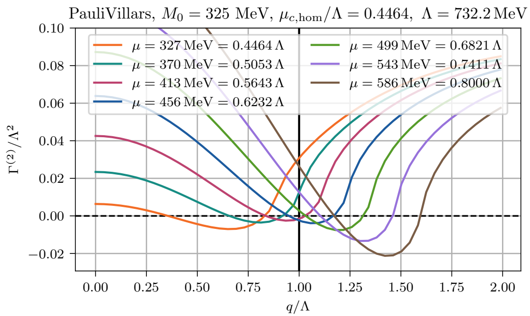


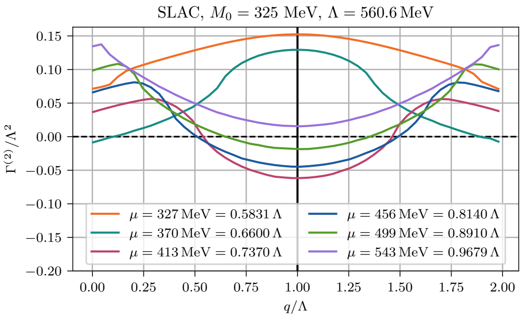
A closer look at SLAC II - Two-point functions



- SLAC has non-zero slope at $q = 0$
- caused by the jump at the edge of the Brioullin zone
- minimum runs into maximum q

A closer look at SLAC II - Two-point functions



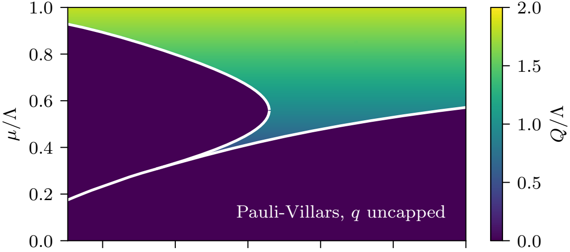


- SLAC has non-zero slope at $q = 0$
- caused by the jump at the edge of the Brioullin zone
- minimum runs into maximum q

$\Rightarrow$ Take a look at Q in the $M_0 - \mu$-plane

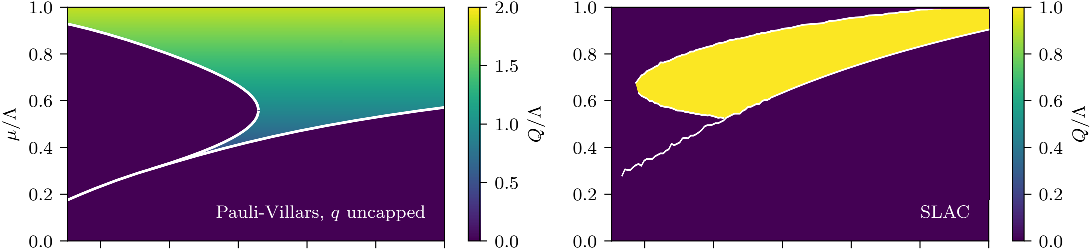
$$Q = \begin{cases} \mathrm{argmin}_q\, \Gamma^{(2)} & \text{if } \min_q \Gamma^{(2)} < 0 \\ 0 & \text{else} \end{cases}$$

Q scan

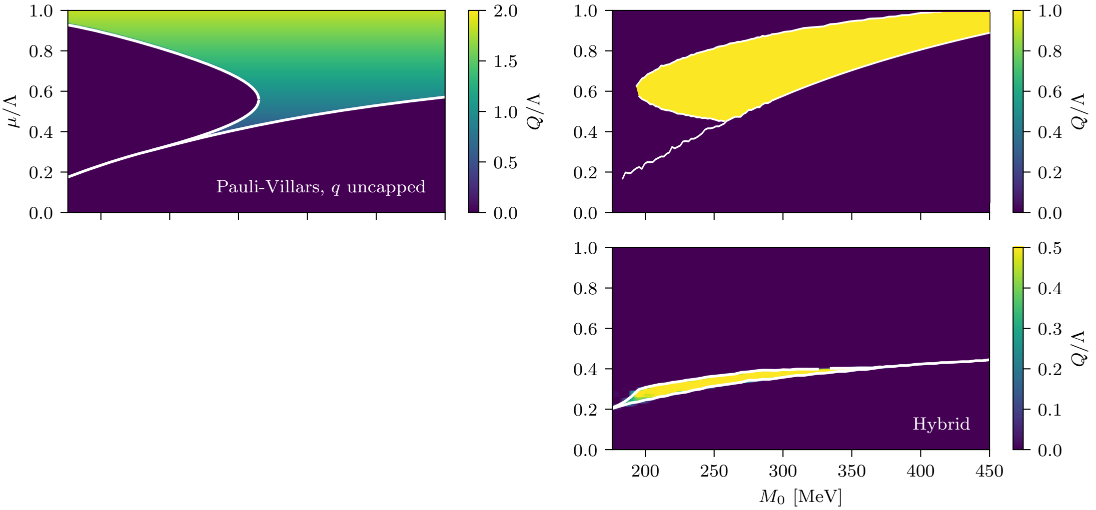


1.0
0.8
0.6
0.4
0.2
0.0
μ/Λ
Pauli-Villars, q uncapped
2.0
1.5
1.0
0.5
0.0
Q/Λ

Q scan

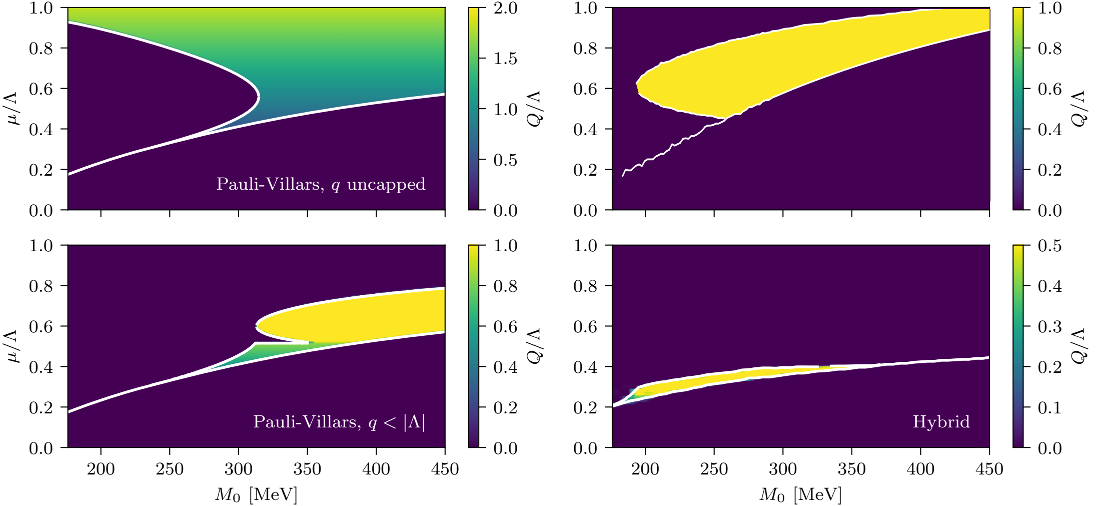

1.0
0.8
0.6
0.4
0.2
0.0
μ/Λ
Pauli-Villars, q uncapped
2.0
1.5
1.0
0.5
0.0
Q/Λ
1.0
0.8
0.6
0.4
0.2
0.0
SLAC
1.0
0.8
0.6
0.4
0.2
0.0
Q/Λ

Q scan



1.0
0.8
0.6
0.4
0.2
0.0
μ/Λ
Pauli-Villars, q uncapped
2.0
1.5
1.0
0.5
0.0
Q/Λ
1.0
0.8
0.6
0.4
0.2
0.0
Q/Λ
0.5
0.4
0.3
0.2
0.1
0.0
Q/Λ
Hybrid
200
250
300
350
400
450
M_0 [MeV]

Q scan



Pauli-Villars, q uncapped
Pauli-Villars, q < |Λ|
Hybrid
μ/Λ
Q/Λ
M0 [MeV]
1.0
0.8
0.6
0.4
0.2
0.0
2.0
1.5
0.5
0.3
0.1
200
250
300
350
400
450

Summary and Outlook

Summary:

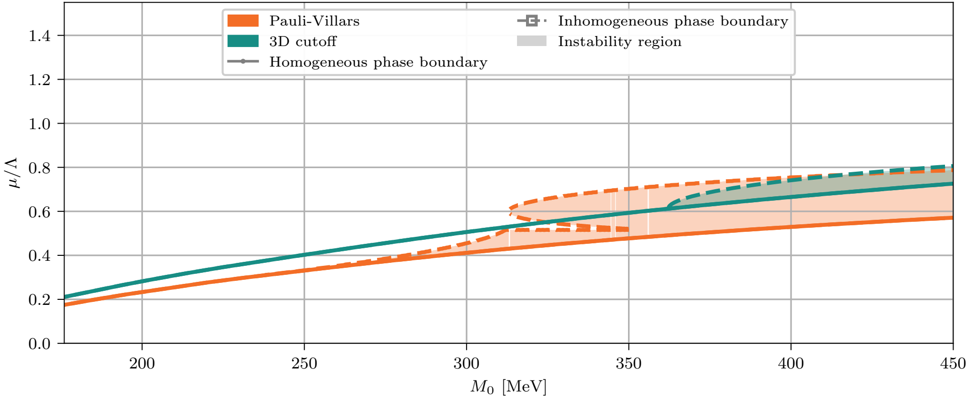
- The parameter regions are very unfavorable - especially for the lattice.
- A *straightforward* lattice investigation of inhomogeneous phases in the $3+1$-dimensional NJL model is most likely pointless.

Outlook:

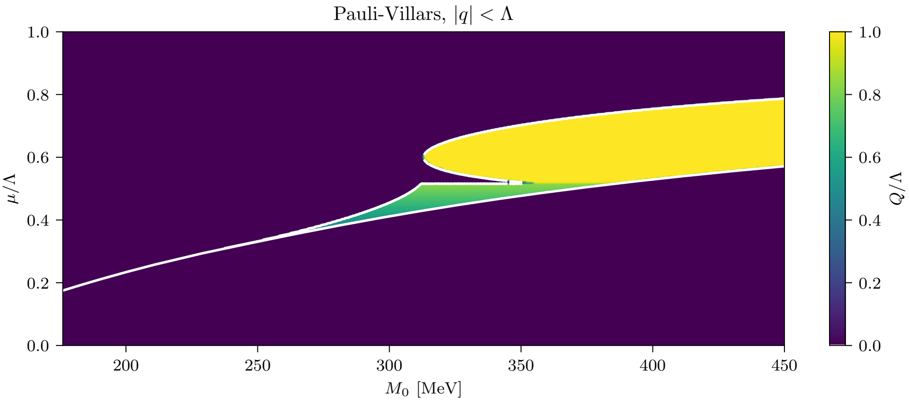
- Making the 3D Cutoff RG consistent yielded promising results
  $\Rightarrow$ exploring a similar treatment of the lattice discretizations
  $\Rightarrow$ Most likely not applicable in real world
- Finish results with staggered fermions
- Investigate the Quark-Meson model as it might have more favorable parameter regions and is 'renormalizable'

Appendix

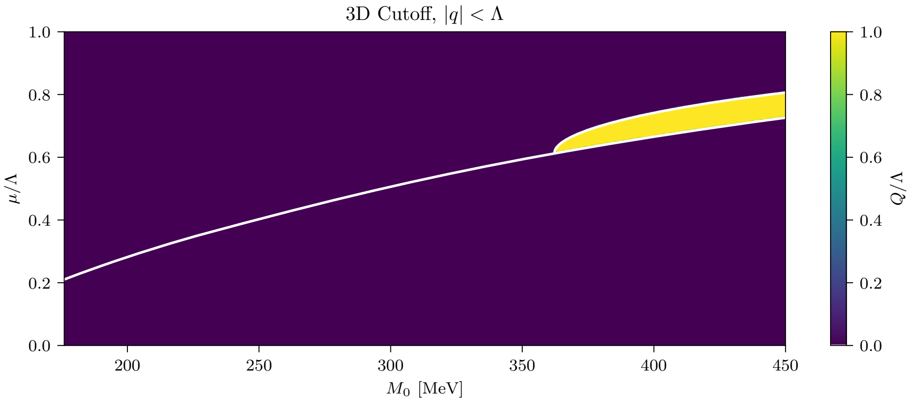
Continuum capped q



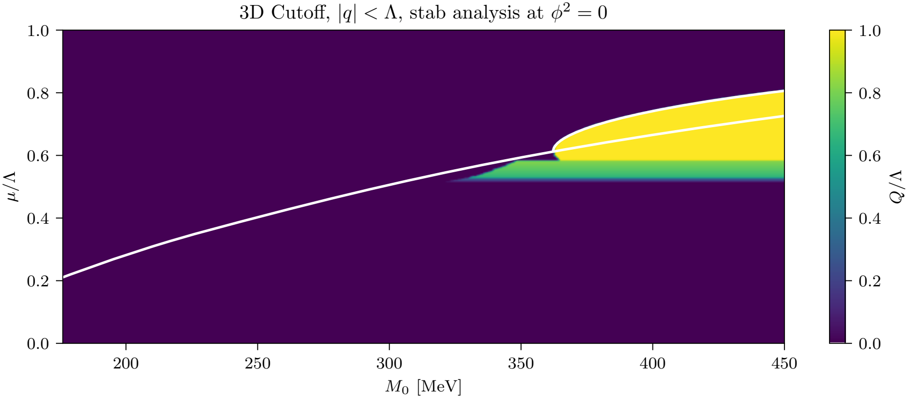
Continuum capped q



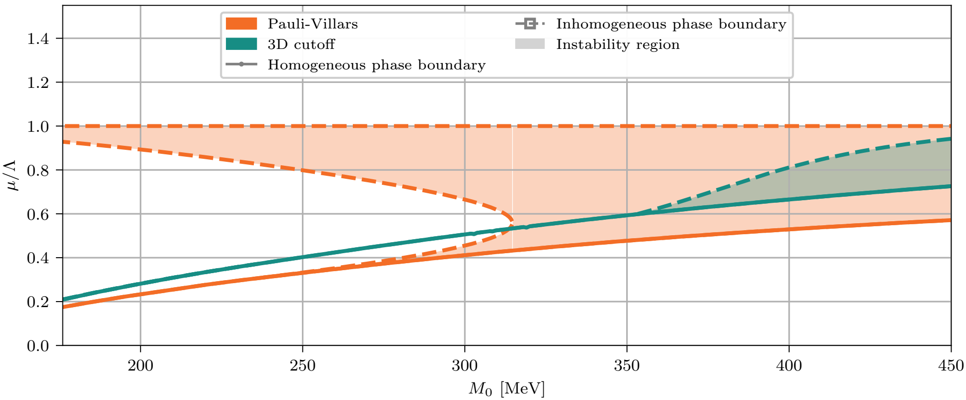
Continuum capped q



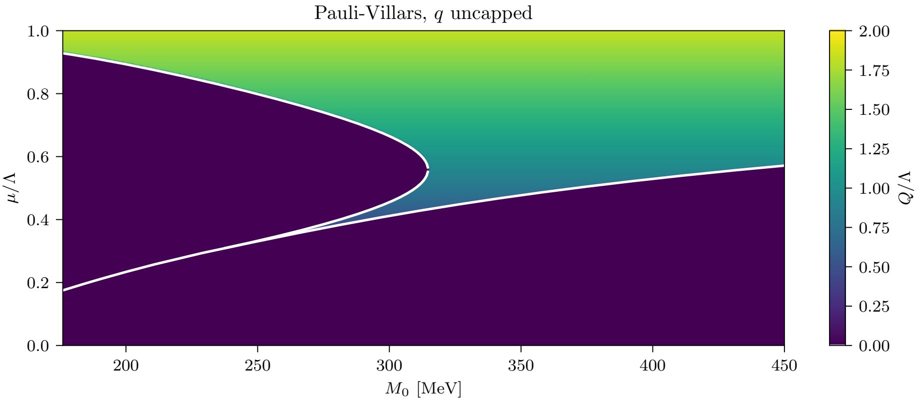
Continuum capped q



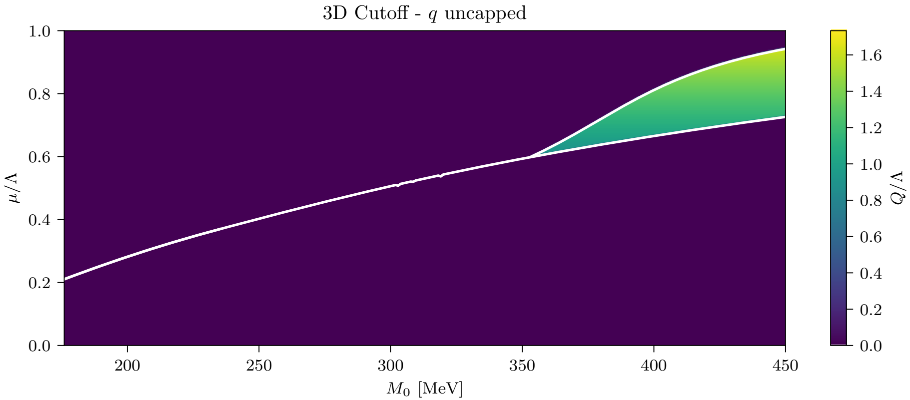
Continuum Uncapped q



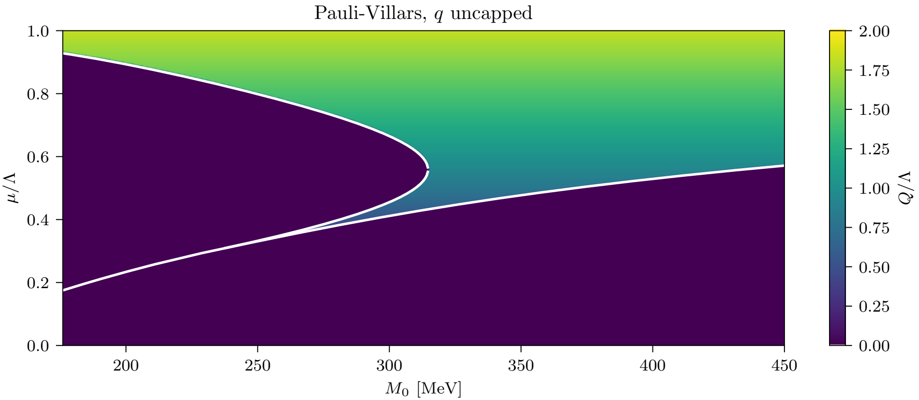
Continuum Uncapped q



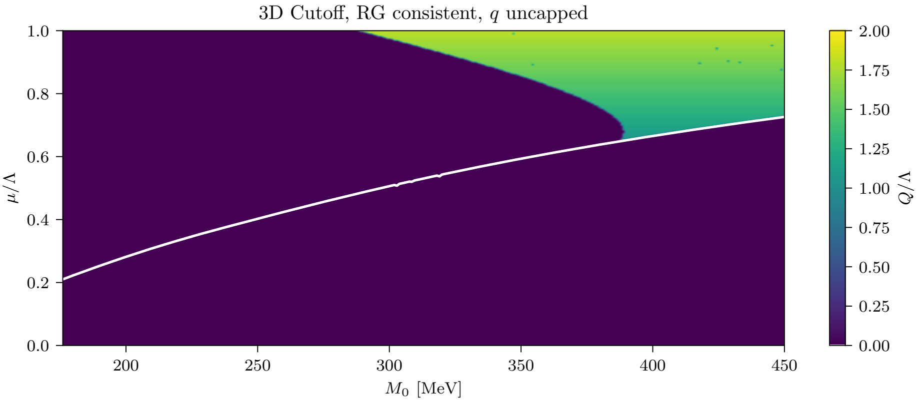
Continuum Uncapped q



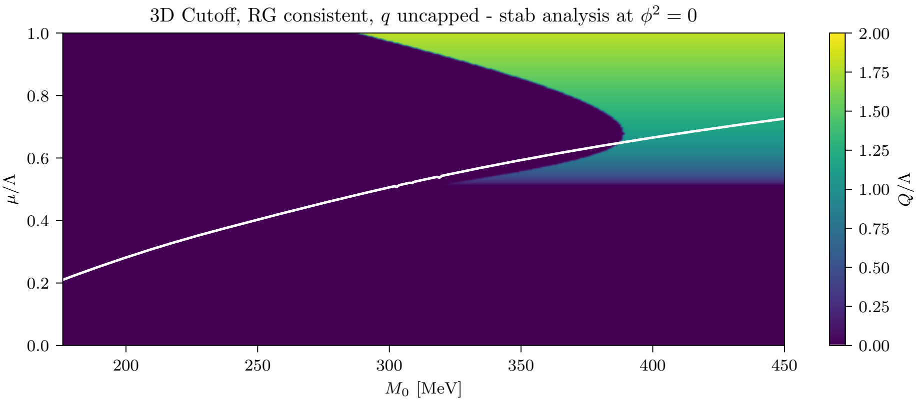
RG consistent cutoff



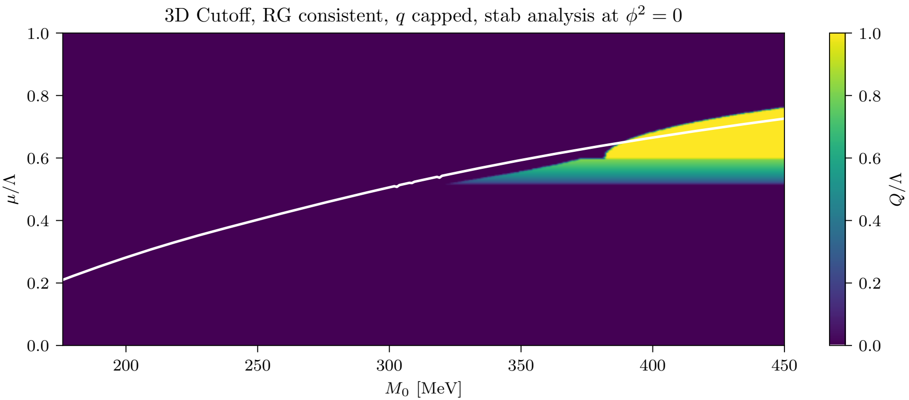
RG consistent cutoff



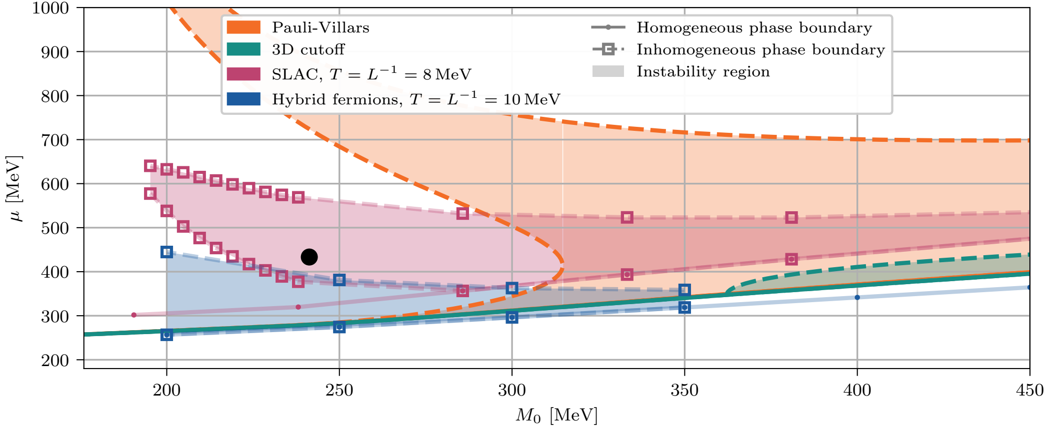
RG consistent cutoff



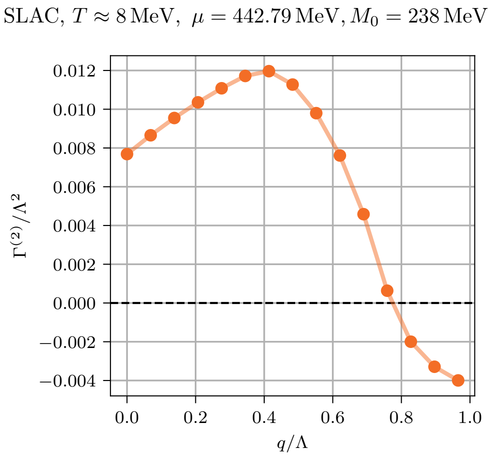
RG consistent cutoff

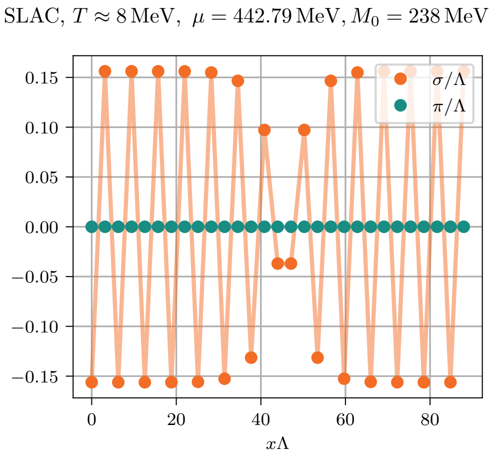


A closer look at SLAC IV - Minimum configuration



A closer look at SLAC IV - Minimum configuration



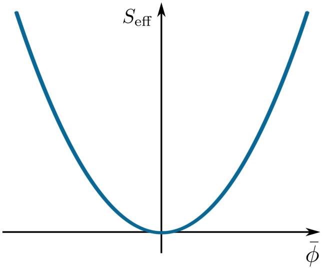


Stability analysis

- Homogeneous fields

$$\phi(x) = \bar{\phi}$$

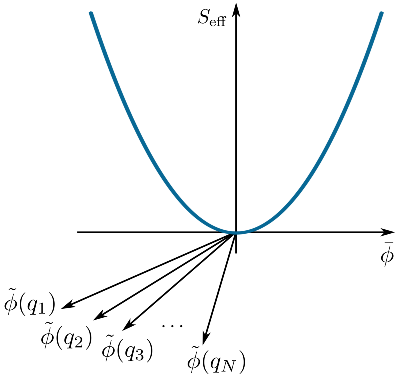
- Minimum is easy to obtain.



Stability analysis

- In general fields have full space dependence

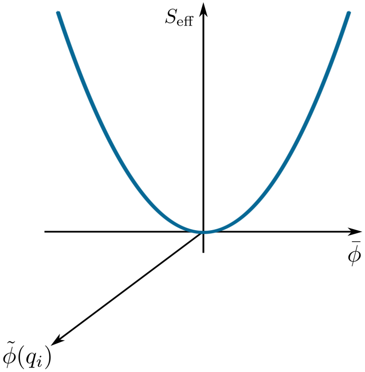
$$\phi(x) = \bar{\phi} + \phi_s(x)$$
$$= \bar{\phi} + \sum\!\!\!\!\!\!\!\int_j \tilde{\phi}_s(q_j)\, \mathrm{e}^{\mathrm{i}xq_j}$$



Stability analysis

- In general fields have full space dependence

$$\begin{aligned}\phi(x) &= \bar{\phi} + \phi_s(x) \\ &= \bar{\phi} + \sum\!\!\!\!\!\!\int_j \tilde{\phi}_s(q_j)\, \mathrm{e}^{\mathrm{i}xq_j}\end{aligned}$$

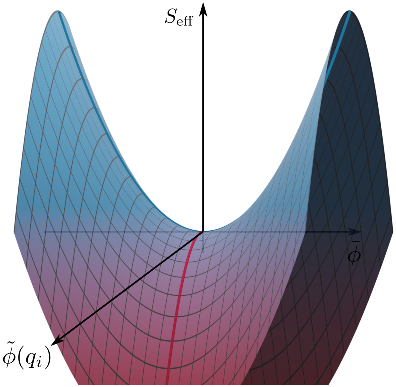


Stability analysis

- In general fields have full space dependence

$$\phi(x) = \bar{\phi} + \phi_s(x)$$
$$= \bar{\phi} + \sum\!\!\!\!\!\!\int_j \tilde{\phi}_s(q_j)\, \mathrm{e}^{\mathrm{i}xq_j}$$

- Former homogeneous minimum might only be saddle point
- Full dependence of S_{eff} on $\phi(x)$ extremely difficult or impossible



Stability analysis

- Consider only inhomogeneous perturbations

$$\phi(x) = \bar{\phi} + \delta\phi_s(x)$$
$$= \bar{\phi} + \sum\!\!\!\!\!\!\!\int_j \delta\tilde{\phi}_s(q_j)\, \mathrm{e}^{\mathrm{i}xq_j}$$

- investigate curvature at homogeneous minimum