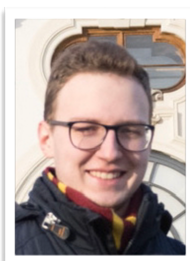
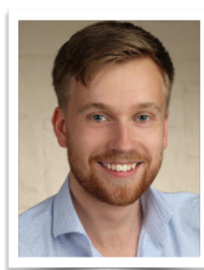
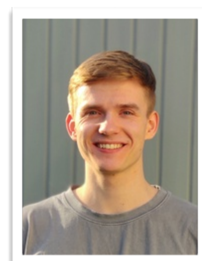
Real-time methods for spectral functions

Johannes Roth, Dominik Schweitzer, Leon Sieke, & LvS

Phys. Rev. D 105 (2022) 116017 [arXiv:2112.12568 [hep-ph]]







Bundesministerium für Bildung und Forschung

Riva del Sole, 12 September 2022

HFHF Helmholtz Forschungsakademie Hessen für FAIR

CRC-TR 211 Strong-interaction matter under extreme conditions

Outline

- spectral functions
- real-time methods
- field theory applications

Spectral Functions

commutator of interacting fields: **free fields**

$$\langle [\phi(x), \phi(0)] \rangle = \int_0^\infty dm^2 \; \rho(m^2) \; i\Delta(x; m^2)$$

Fourier transform:

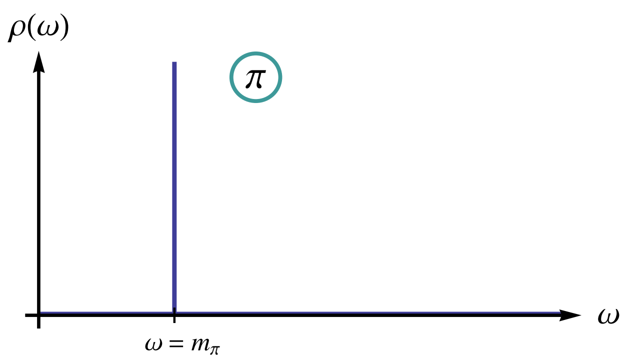
$$\rho(\omega, \vec{p}) = \int d^4x \; e^{ipx} \, i \langle [\phi(x), \phi(0)] \rangle$$

spectral function:

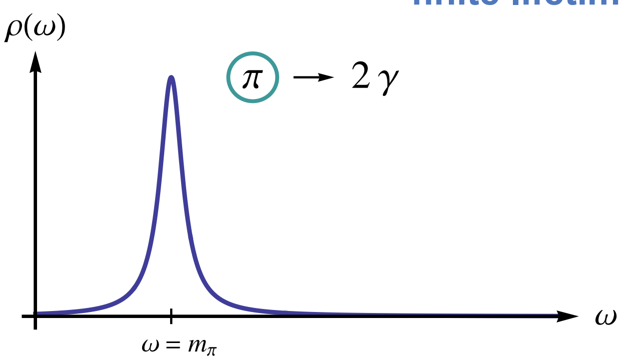
$$\Rightarrow \qquad \rho(\omega, \vec{p}) = 2\pi i \, \epsilon(\omega) \theta(p^2) \, \rho(p^2)$$

$$\rho(p^2) = (2\pi)^3 \sum_\psi \delta^4(p - q_\psi) \, \left| \langle \Omega | \phi(0) | \psi \rangle \right|^2 \, , \quad p_0 > 0$$

free fields (stable pion):

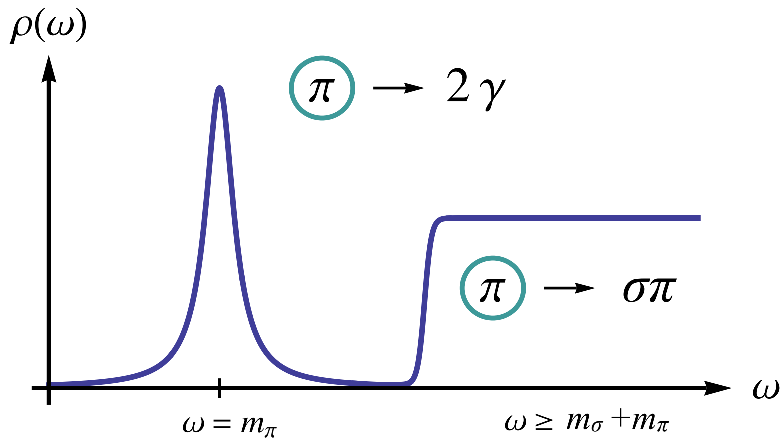


finite lifetime/width



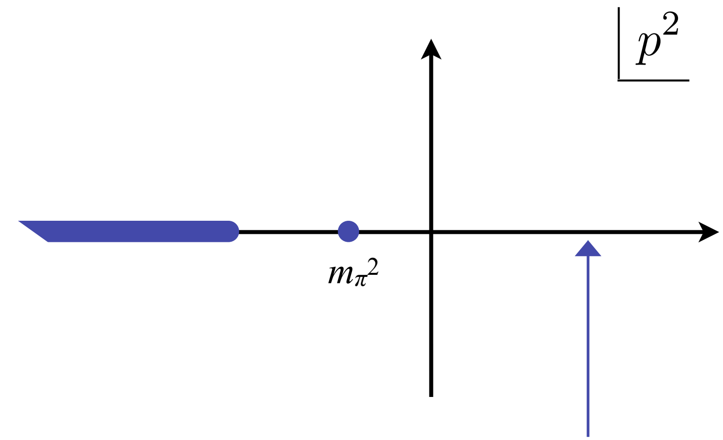
Spectral Functions

two-particle thresholds:



discontinuity at cut of propagator:

$$D(p) = \int_0^\infty dm^2 \, \rho(m^2) \, \frac{1}{p^2 + m^2}$$



retarded, imaginary part: $\rho(p^2) = -\frac{1}{\pi} \, \mathrm{Im} \, D_R(p)$

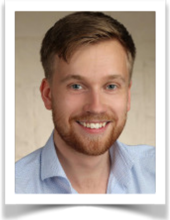
Euclidean space:

$p^2 > 0$

Euclidean data: $D(t, \vec{p} = 0) = \int_0^\infty dm \; \rho(m^2) \, \exp\{-mt\}$

(inverse Laplace, try e.g. MEM, but ill-posed numerical problem)

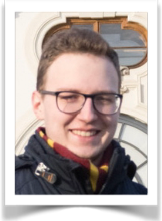
Real-Time Methods



(i) Classical-statistical simulations

(ii) Gaussian state approximation

(iii) Real-time FRG



QM Example

anharmonic oscillator:

$$\hat{H}_S = \frac{\hat{p}^2}{2} + \frac{\omega_0^2}{2}\hat{x}^2 + \frac{\lambda}{4!}\hat{x}^4$$

- **spectral function:** $\rho(t-t') = \mathrm{i}\langle[\hat{x}(t),\hat{x}(t')]\rangle_\beta$
- **Fourier transform:**

$$\rho(\omega) = \frac{1}{Z}\sum_{m,n} \mathrm{e}^{-\beta E_n}\Big(\delta(\omega - E_m + E_n) - \delta(\omega + E_m - E_n)\Big)\,|\langle n|\hat{x}|m\rangle|^2$$

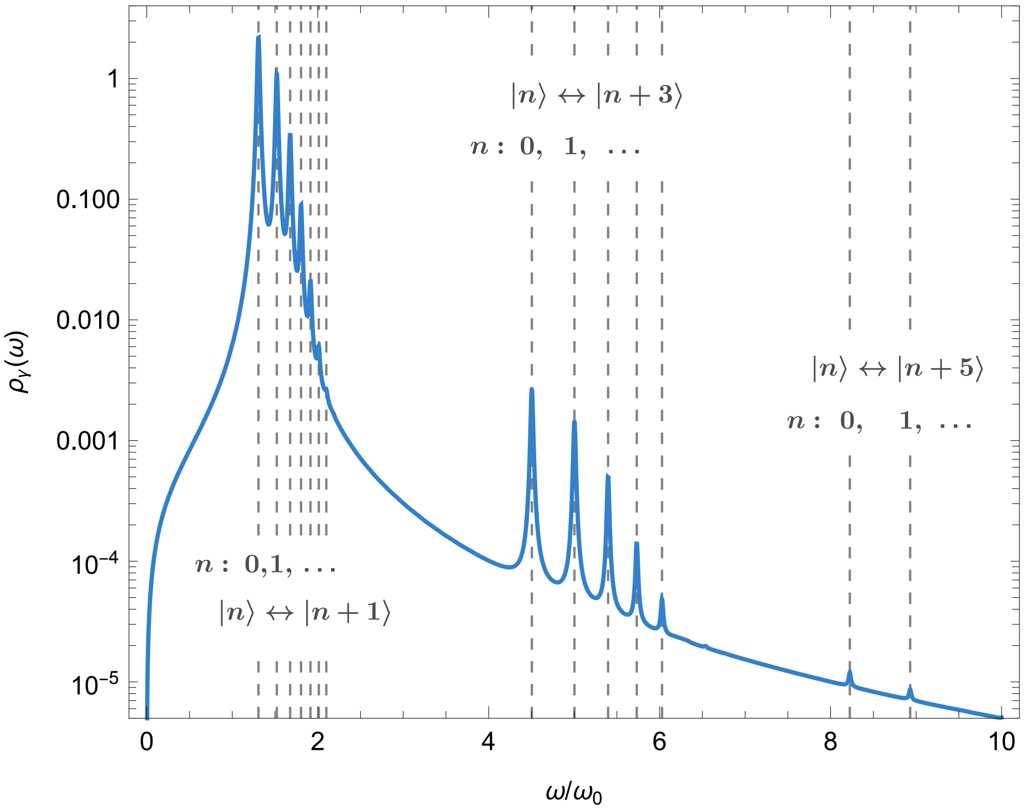
- **Ohmic damping (Caldeira-Leggett):**

$$\rho_\gamma(\omega) = \frac{1}{Z}\sum_{m,n} \mathrm{e}^{-\beta E_n}\,|\langle n|\hat{x}|m\rangle|^2\, 2\Delta E_{mn} \times \frac{1}{\pi}\,\frac{\gamma\omega}{(\omega^2 - \Delta E_{mn}^2)^2 + \gamma^2\omega^2}$$

QM Example

anharmonic oscillator:

$$\rho_\gamma(\omega) = \frac{1}{Z} \sum_{m,n} \mathrm{e}^{-\beta E_n} \, |\langle n|\hat{x}|m\rangle|^2 \, 2\Delta E_{mn} \times \frac{1}{\pi} \frac{\gamma\omega}{(\omega^2 - \Delta E_{mn}^2)^2 + \gamma^2\omega^2}$$



(i) Classical-Statistical Limit

- **classical Langevin:**

$$\frac{\mathrm{d}}{\mathrm{d}t} X = P, \quad \frac{\mathrm{d}}{\mathrm{d}t} P = -\omega_0^2 X - \frac{\lambda}{6} X^3 - \gamma P + \xi(t)$$

$$\langle \xi(t)\xi(t')\rangle_\beta = 2\gamma T \, \delta(t-t')$$

- **classical SF from FDR:**

$$\rho_{\mathrm{c}}(t-t') = -\frac{1}{2T} \big\langle P(t)X(t') - X(t)P(t') \big\rangle_\beta$$

$$\rho_{\mathrm{c}}(t-t') = -\frac{1}{T}\, \partial_t F_{\mathrm{c}}(t-t')$$

- **classical-statistical field theory simulations**
  - high-temperature (Rayleigh-Jeans) limit: G. Aarts, PLB 518 (2001) 315
  - critical SFs, dynamic scaling functions:

    J. Berges, S. Schlichting, D. Sexty, NPB 832 (2010) 228

    S. Schlichting, D. Smith, L.v.S., NPB 950 (2020) 114868

    D. Schweitzer, S. Schlichting, L.v.S., NPB 960 (2020) 115165; arXiv:2110.01696

- **Heisenberg-Langevin:**

G.-L. Ingold, Lect. Notes Phys., 611 (2002) 1 - 53

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\hat{x}(t) = \hat{p}(t), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\,\hat{p}(t) = -\int_0^t \mathrm{d}t'\,\gamma(t-t')\hat{p}(t') - V'(\hat{x}(t)) + \hat{\xi}(t)$$

- **Ohmic bath:**

$$J_\Lambda(\omega) = 2\gamma\omega\,\Theta(\Lambda - |\omega|)$$

$$\gamma(t) = 2\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\,\frac{J(\omega)}{\omega}\,\cos(\omega t) \;=\; \frac{2\gamma\Lambda}{\pi}\,\frac{\sin(\Lambda t)}{\Lambda t} \xrightarrow{\Lambda\to\infty} 2\gamma\delta(t)$$

$$\langle \xi(t)\xi(t')\rangle_\beta = \int_0^\infty \frac{\mathrm{d}\omega}{\pi} J(\omega)\, n_\mathrm{B}(\omega)\,\cos(\omega(t-t'))$$

$$\xi(t) \equiv \langle \hat{\xi}(t)\rangle$$

$$\to \gamma T\left(-\frac{\pi T}{\sinh^2(\pi T(t-t'))} + \frac{1}{\pi T(t-t')^2}\right)$$

- **Frequency:**

$$\langle \xi(-\omega)\xi(\omega)\rangle_\beta = 2\gamma\omega\, n_\mathrm{B}(\omega)\,, \qquad \omega > 0$$

colored noise

$$\to 2\gamma T\,, \qquad T \gg \omega$$

classical limit

Gaussian-State Approximation

P. Buividovich, M. Hanada, A. Schäfer, PRD 99 (2019) 046011

- **Wigner function:**

$$w(x,p) = \int \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}py} \langle x + y/2 | \hat{\rho} | x - y/2 \rangle$$

- **GSA:**

$$= \mathcal{N} \exp \left\{ -\frac{1}{2} \begin{pmatrix} x - X \\ p - P \end{pmatrix}^T \begin{pmatrix} \sigma_{xx} & \sigma_{xp} \\ \sigma_{xp} & \sigma_{pp} \end{pmatrix}^{-1} \begin{pmatrix} x - X \\ p - P \end{pmatrix} \right\}$$

- **Harmonic oscillator:**

mixed thermal state

$$\hat{\rho}_{\mathrm{HO}} = e^{-\beta \hat{H}}/Z = Z^{-1} \sum_n \mathrm{e}^{-\beta\omega_0(n+1/2)} |n\rangle \langle n|$$

$$w_{\mathrm{HO}}(x,p) = \frac{2}{F(\omega_0)}\, \mathrm{e}^{-\frac{p^2+\omega_0^2 x^2}{\omega_0 F(\omega_0)}}, \qquad F(\omega) = \coth \frac{\beta\omega}{2}$$

thermal distribution

Gaussian-State Approximation

- **General Gaussian state:**

coherent states

$$\hat{\rho}_G = \tilde{N} \int \mathrm{d}X\,\mathrm{d}P\,\exp\left\{-\frac{X^2}{2\sigma^c_{xx}} - \frac{P^2}{2\sigma^c_{pp}}\right\} |X,P\rangle\langle X,P|$$

$$\sigma^c_{xx} = n_\mathrm{B}(\omega_0)/\omega_0, \qquad \sigma^c_{pp} = \omega_0\, n_\mathrm{B}(\omega_0)$$

$$\neq F(\omega_0)/2\omega_0$$

$F(\omega_0) = 2n_\mathrm{B}(\omega_0) + 1$

mixed thermal state

- **Gaussian WF with:**

$$\Sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xp} \\ \sigma_{xp} & \sigma_{pp} \end{pmatrix}, \qquad f = \sqrt{\sigma_{xx}\sigma_{pp} - \sigma^2_{xp}}$$

covariance matrix

symplectic EV

$$= 1/2$$

if and only if pure

- **von Neumann entropy:**

$$S = -\mathrm{Tr}(\hat{\rho}\ln\hat{\rho}) = \left(f+\tfrac{1}{2}\right)\ln\left(f+\tfrac{1}{2}\right) - \left(f-\tfrac{1}{2}\right)\ln\left(f-\tfrac{1}{2}\right)$$

(ii) Gaussian-State Approximation

- **HLE in GSA:**

$$\frac{\mathrm{d}}{\mathrm{d}t} X = P, \quad \frac{\mathrm{d}}{\mathrm{d}t} P = -\left(\omega_0^2 + \frac{\lambda}{2}\sigma_{xx}\right) X - \frac{\lambda}{6} X^3 - \gamma P + \xi(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{xx} = 2\sigma_{xp},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{xp} = \sigma_{pp} - \sigma_{xx}\mathcal{C}(X,\sigma_{xx}) - \gamma\sigma_{xp} + \langle\!\langle \hat{x}(t)\hat{\xi}(t)\rangle\!\rangle,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{pp} = -2\sigma_{xp}\mathcal{C}(X,\sigma_{xx}) - 2\gamma\sigma_{pp} + 2\,\langle\!\langle \hat{p}(t)\hat{\xi}(t)\rangle\!\rangle$$

- **adiabatic approximation:**

$$\mathcal{C}(t) \equiv \mathcal{C}(X,\sigma_{xx}) = \omega_0^2 + \frac{\lambda}{2}(X^2(t) + \sigma_{xx}(t))$$

$$\mathcal{C}(t) = \mathcal{C}_0(T) + \delta\mathcal{C}(t)$$

$$\mathcal{C}_0(T) \equiv \langle \mathcal{C}(X,\sigma_{xx})\rangle_\beta = \omega_0^2 + \frac{\lambda}{2}\langle \hat{x}^2\rangle_\beta$$

time independent

(iii) Real-Time FRG

- **Keldysh action:**

$$S[\phi] = \frac{1}{2}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\phi^T(-\omega)\begin{pmatrix} 0 & \omega^2 - \mathrm{i}\gamma\omega - \omega_0^2 \\ \omega^2 + \mathrm{i}\gamma\omega - \omega_0^2 & 2\mathrm{i}\gamma\omega\coth\left(\frac{\omega}{2T}\right)\end{pmatrix}\phi(\omega)$$

$$- \frac{2\lambda}{4!}\int_{-\infty}^{\infty} dt\, \left(\phi^c(t)\phi^c(t)\phi^c(t)\phi^q(t) + \phi^c(t)\phi^q(t)\phi^q(t)\phi^q(t)\right)$$

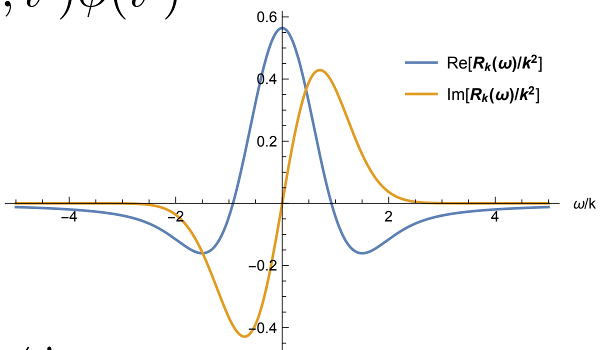
- **causal regulator for FRG:**

$$\Delta S_k[\phi] = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}t'\, \phi^T(t)R_k(t,t')\phi(t')$$

via scale-dependent "heat-bath"

$$R_k^{R/A}(\omega) = -\int_0^{\infty}\frac{d\omega'}{2\pi}\frac{2\omega' J_k(\omega')}{(\omega \pm i\varepsilon)^2 - \omega'^2}$$

$$R_k^K(\omega) = \left(R_k^R(\omega) - R_k^A(\omega)\right)\coth\frac{\omega}{2T} = iJ_k(\omega)\coth\frac{\omega}{2T}$$



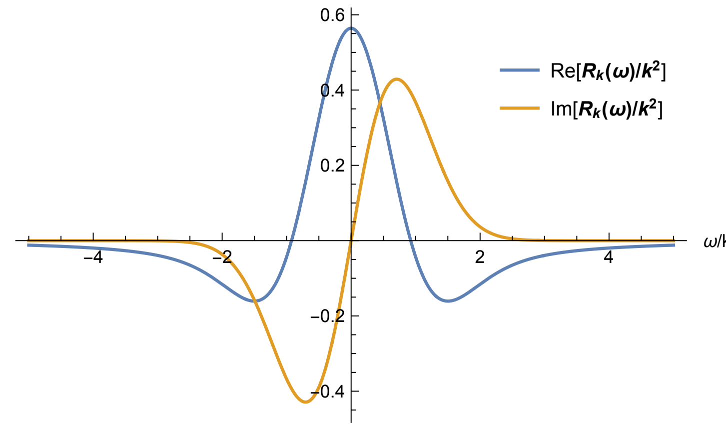
$$J_k(\omega) = 2k\omega\,\exp\{-\omega^2/k^2\}$$

(iii) Real-Time FRG

- **causal regulator for FRG:**

$$R_k^{R/A}(\omega) = -\int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega' J_k(\omega')}{(\omega \pm i\varepsilon)^2 - \omega'^2} \quad - \alpha k^2$$

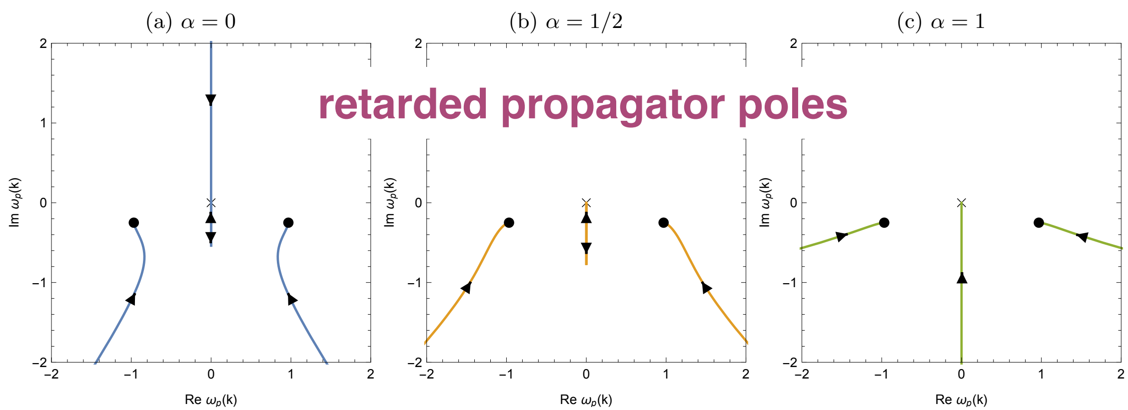
avoid unphysical regulator poles



$$J_k(\omega) = 2k\omega\, \exp\left\{-\omega^2/k^2\right\}$$

- **simple example:**

retarded propagator poles



Drude model with frequency $\omega_{\mathrm{D}} = k$

$$R_k^{R/A}(\omega) = \frac{1}{2}\frac{k^2}{1 \mp \mathrm{i}\omega/k} - \alpha k^2$$

$$\omega_{p,\pm}(k=0) = -\mathrm{i}\frac{\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

$$G_k^{R/A}(\omega) = -\frac{1}{\omega^2 \pm \mathrm{i}\gamma\omega - \omega_0^2 + R_k^{R/A}(\omega)}$$

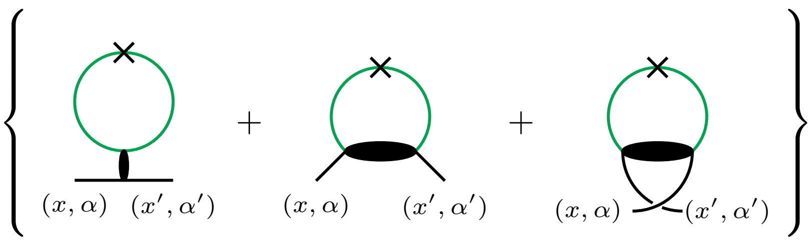
(iii) Real-Time FRG

- **Effective average action:**
  truncation

$$\Gamma_k[\phi] = \frac{1}{2}\int_{xx'} \phi^T(x) \begin{pmatrix} 0 & \Gamma_k^{(2),A}(x,x') \\ \Gamma_k^{(2),R}(x,x') & \Gamma_k^{(2),K}(x,x') \end{pmatrix} \phi(x') + \frac{3}{4!}\int_{xx'} \phi^\alpha(x)\phi^\beta(x)\Gamma_k^{\alpha\beta;\beta'\alpha'}(x,x')\phi^{\beta'}(x')\phi^{\alpha'}(x')$$
$$- \frac{1}{6!}\int_x \left(\frac{3}{2}\mu_k(\phi^c(x))^5\phi^q(x) + 5\mu_k(\phi^c(x))^3(\phi^q(x))^3 + \frac{3}{2}\mu_k\phi^c(x)(\phi^q(x))^5\right) + O(\phi^8),$$

- two-loop exact:

$$\partial_k \Gamma_k^{\alpha\alpha'}(x,x') = -\frac{\mathrm{i}}{2}$$



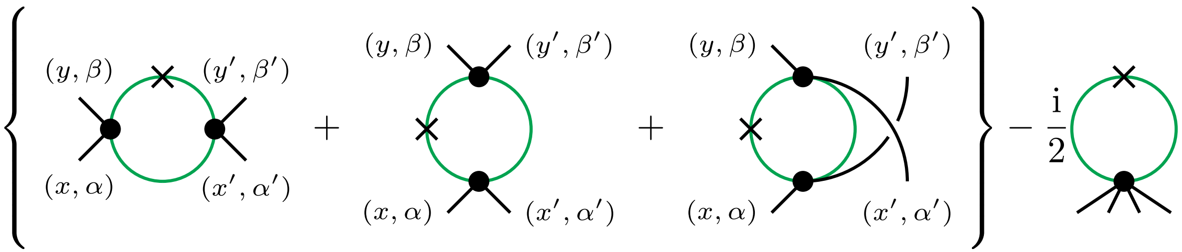
use for 2-point function

S. Huelsmann, S. Schlichting, Ph. Scior, PRD 102 (2020) 096004

(iii) Real-Time FRG

- **one-loop structure:**

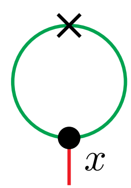
$$\partial_k \Gamma_k^{\alpha\beta\beta'\alpha'}(x,y,y',x') = -\mathrm{i}$$



for 4-point function

- **flow of local vertices:**

$$\partial_k \frac{\delta \Gamma_k[\phi]}{\delta \phi^q(x)} = -\frac{i}{2}$$



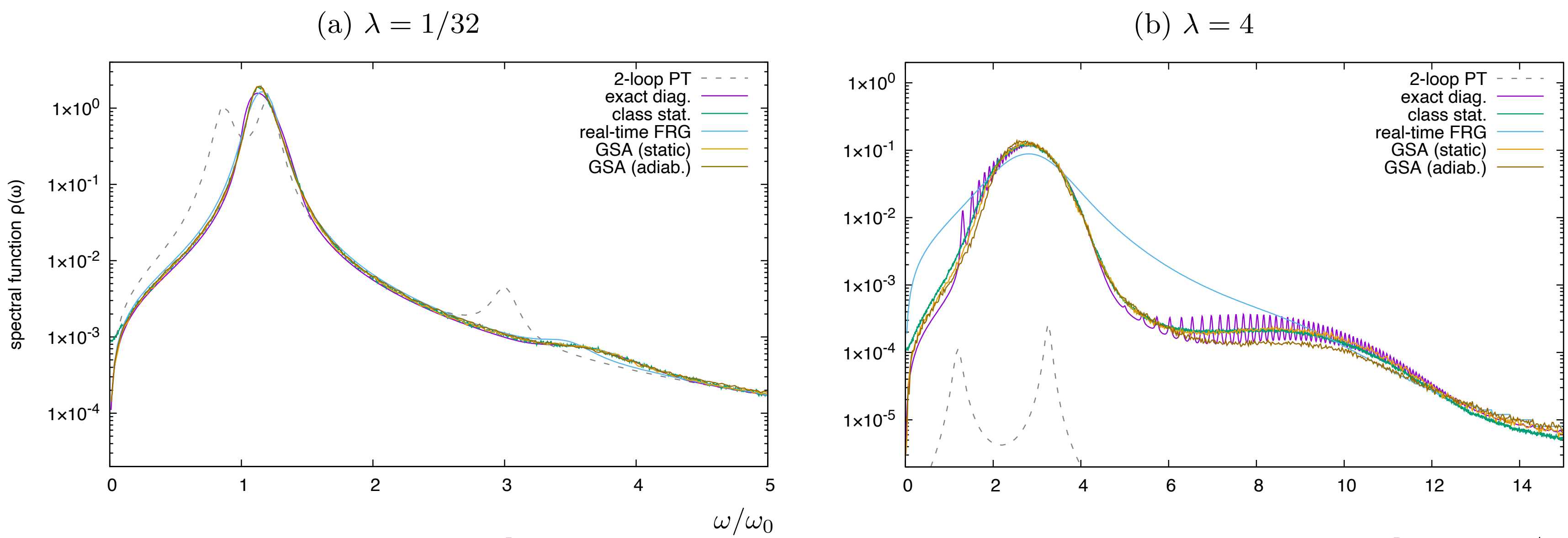
use for 6-point function

- **combined vertex and loop expansion**

here order $Q = 2l + n = 6$

Results

- **verify high-temperature (classical) limit:** $T = 32\,\omega_0\,,\ \gamma = 0.06\,\omega_0$



weak coupling **not-so-weak coupling**

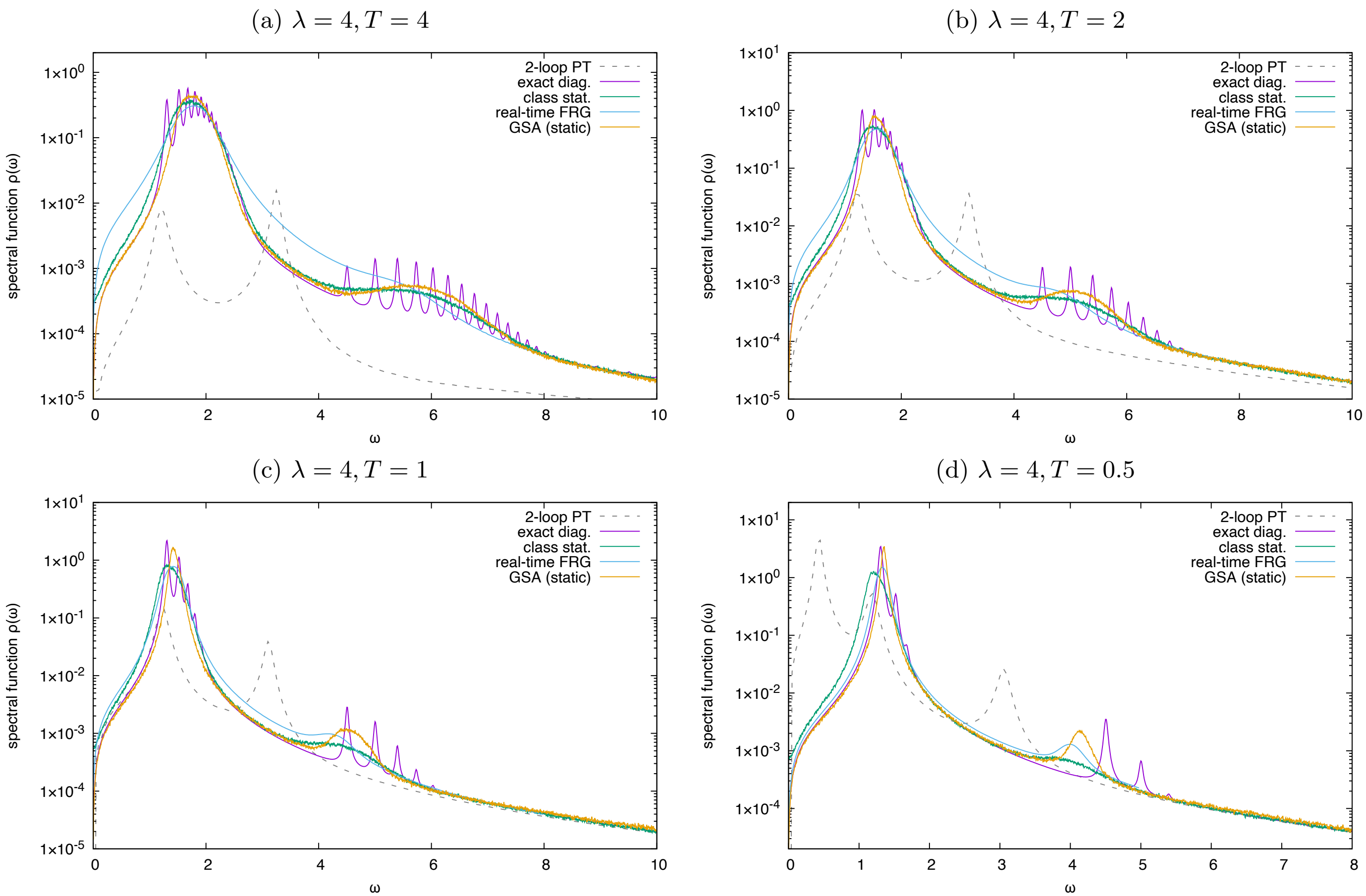
- relevant *thermal coupling*:

$$\lambda T = 1 \qquad\qquad \lambda T = 128$$

- also for FRG, with loop expansion need: $\lambda T \lesssim 4$

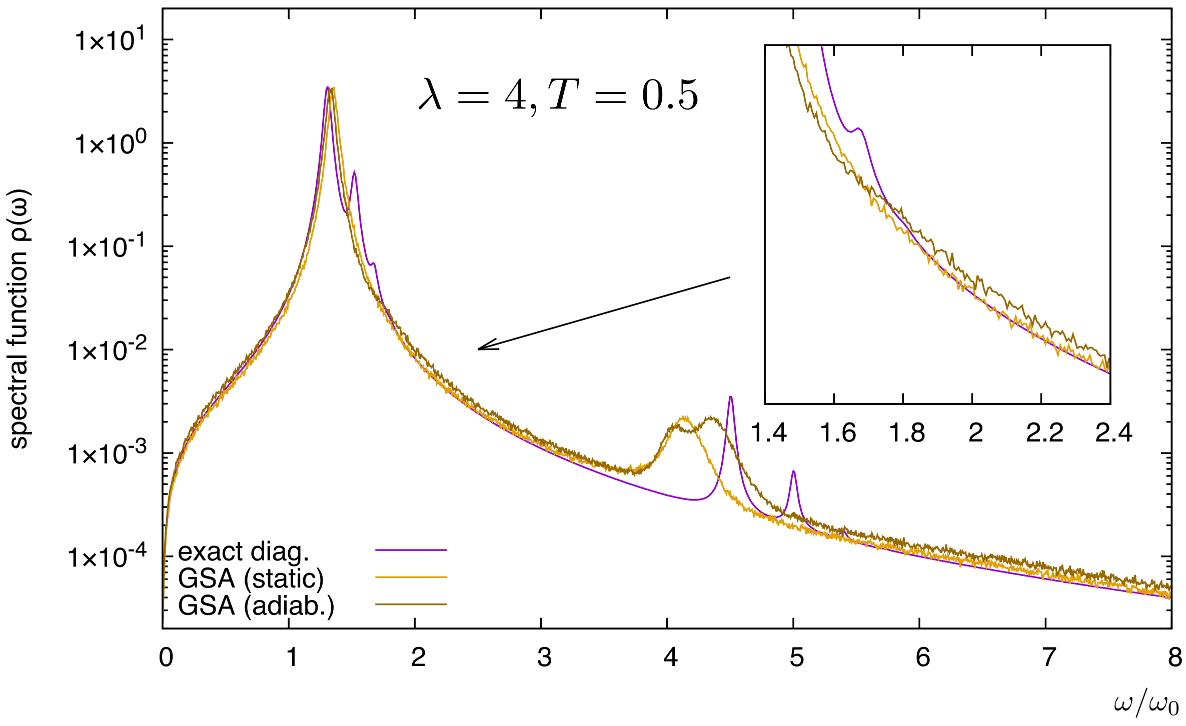
Results

- **lower temperatures:**



Results

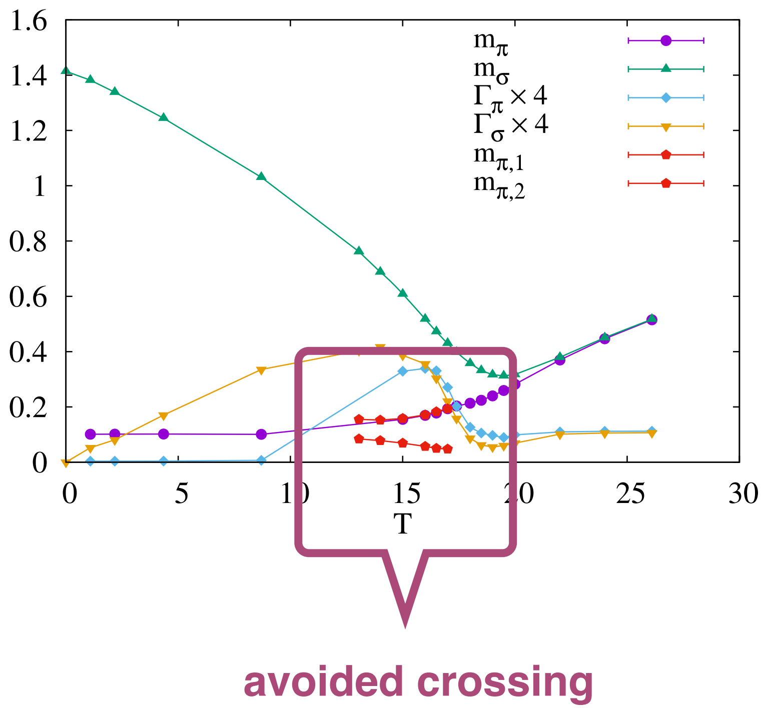
- compare static vs adiabatic GSA:

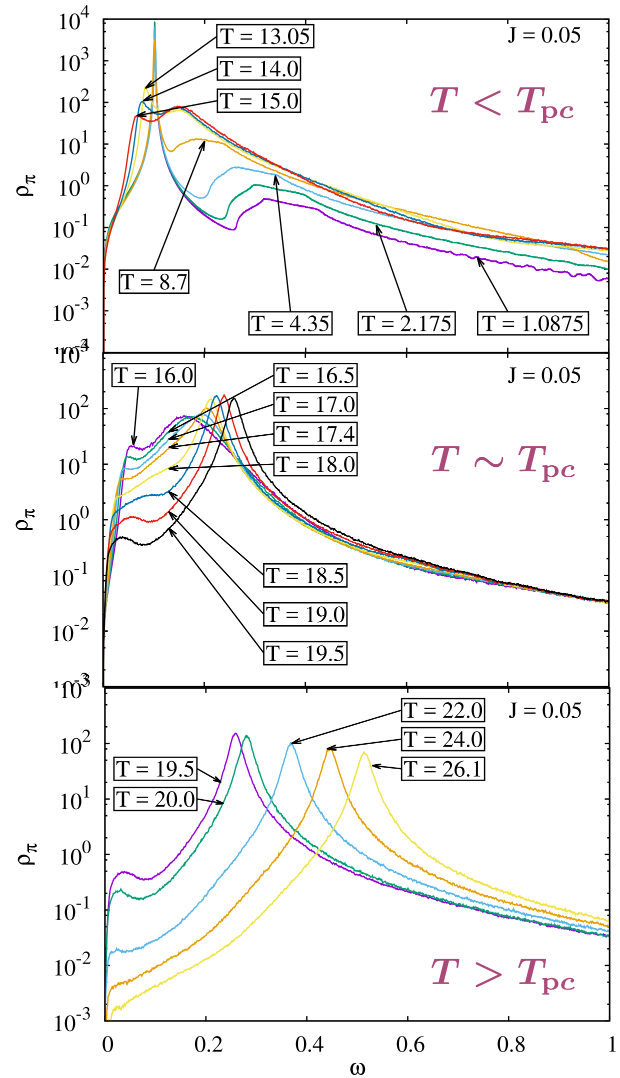


Field Theory Applications

- **classical-statistical SFs:**

e.g. pion in O(4) model





Schlichting, Smith, von Smekal, NPB 950 (2020) 114868

Classical-Statistical SFs

Z_2 scalar fields with Langevin (Model A) dynamics

$$\ddot{\phi}(\boldsymbol{x},t) = -\frac{\delta\mathfrak{H}[\phi]}{\delta\phi(\boldsymbol{x},t)} - \gamma\dot{\phi}(\boldsymbol{x},t) + \sqrt{2\gamma T}\,\eta(\boldsymbol{x},t)$$

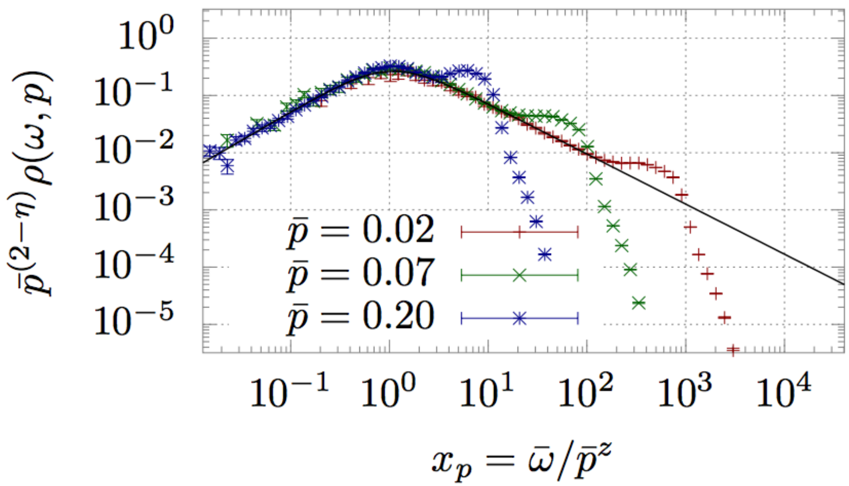
$$\big\langle \eta(\boldsymbol{x}',t')\eta(\boldsymbol{x},t)\big\rangle = \delta(\boldsymbol{x}'-\boldsymbol{x})\delta(t'-t)$$

- **dynamic scaling functions:**
  - **use scaling relation of critical SF:**

$$\rho\left(\omega,p,\tau\right) = s^{2-\eta}\rho\left(s^z\omega, sp, s^{\frac{1}{\nu}}\tau\right)$$

to determine new universal scaling funct.'s f_ω, f_p, $f_\tau^\pm$, e.g.

$$\rho\left(\omega,p,\tau\right) = \bar{p}^{-(2-\eta)}\,f_p\Big(\bar{\omega}/\bar{p}^z, \tau/\bar{p}^{1/\nu}\Big)$$



dynamic (two-variable) scaling function

Schweitzer, Schlichting, von Smekal, NPB 960 (2020) 115165

Classical-Statistical SFs

Z_2 scalar fields with diffusive (Model B) dynamics

$$\ddot{\phi}(\boldsymbol{x},t) = \mu\nabla^2 \frac{\delta\mathfrak{H}[\phi]}{\delta\phi(\boldsymbol{x},t)} - \gamma\dot{\phi}(\boldsymbol{x},t) + \sqrt{2\gamma T}\,\eta(\boldsymbol{x},t)$$

$$\big\langle \eta(\boldsymbol{x}',t')\eta(\boldsymbol{x},t)\big\rangle = -\mu\nabla^2\delta(\boldsymbol{x}'-\boldsymbol{x})\delta(t'-t)$$

conserved order parameter

$$Q = \int \mathrm{d}^d x\,\phi(\boldsymbol{x},t) \quad \text{with} \quad \dot{Q} = 0$$

- **spectral functions:**

$$\rho_{\mathrm{BW}}(\omega,\boldsymbol{p}) = \frac{\mu\boldsymbol{p}^2\Gamma_p\,\omega}{\left(\omega^2-\omega_p^2\right)^2 + \Gamma_p^2\omega^2}$$

non-critical:

$$\omega_p^2 = \mu\boldsymbol{p}^2\big(m^2(T)+\boldsymbol{p}^2\big)$$

$$\Gamma_p(\gamma) = \Gamma_p(0) + \gamma$$

$$\Gamma_p(0) = \bar{\Gamma}(T)\cdot\begin{cases} |\boldsymbol{p}|, & T \ll T_c \\ \boldsymbol{p}^2, & T \gg T_c \end{cases}$$

critical:

$$\omega_p^2 = \omega_0^2\,\bar{p}^{z_\omega}$$

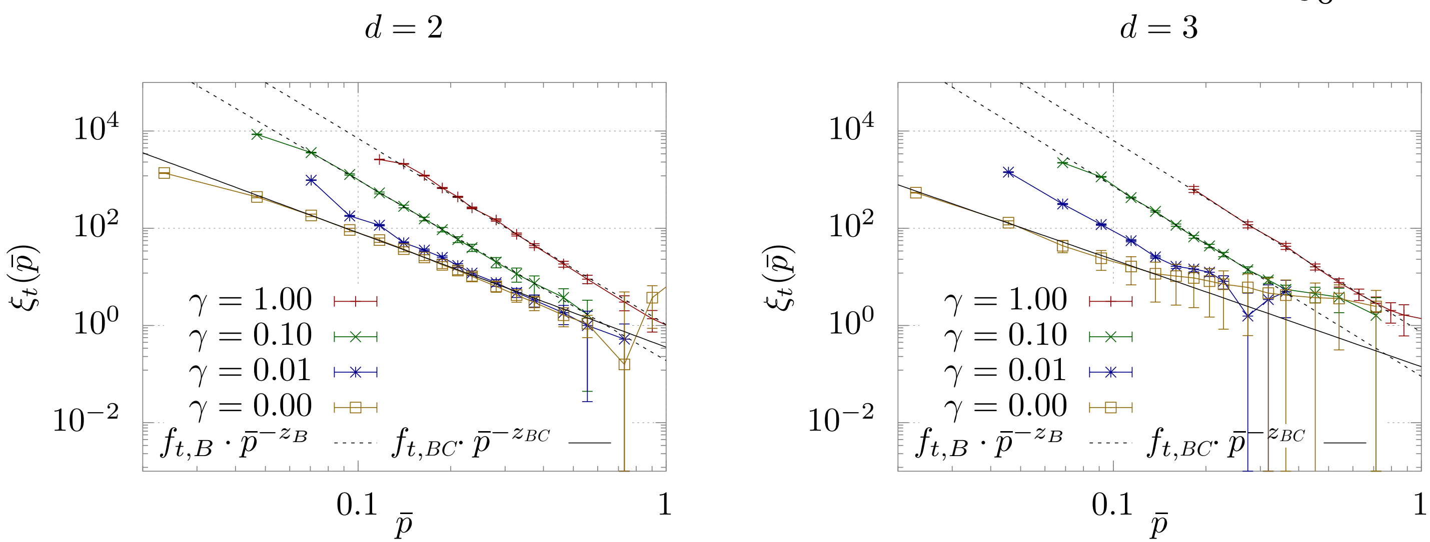
$$\Gamma_p = \Gamma_0\,\bar{p}^{z_\Gamma}$$

Schweitzer, Schlichting, von Smekal, arXiv:2110.01696

Classical-Statistical SFs

- **critical dynamics of relativistic diffusion:**

$$\xi_t(p) = \frac{\int_0^\infty t\,\rho(t,p,T_c)\,\mathrm{d}t}{\int_0^\infty \rho(t,p,T_c)\,\mathrm{d}t}$$



d	$z(\gamma = 1.0)$	$z(\gamma = 0.1)$	$z(\gamma = 0.0)$
2	3.83(10)	3.716(17)	2.354(23)
3	3.95(8)	3.91(6)	2.20(13)

Model B: $z = 4 - \eta$

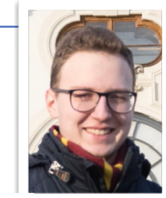
$\stackrel{!}{=} \mathbf{3.75}$

$\stackrel{!}{\approx} \mathbf{3.96}$

$\xi_t \sim \xi^z$

Schweitzer, Schlichting, von Smekal, arXiv:2110.01696

Real-Time FRG - Critical Dynamics



J. Roth, MSc Thesis, JLU, March 2022

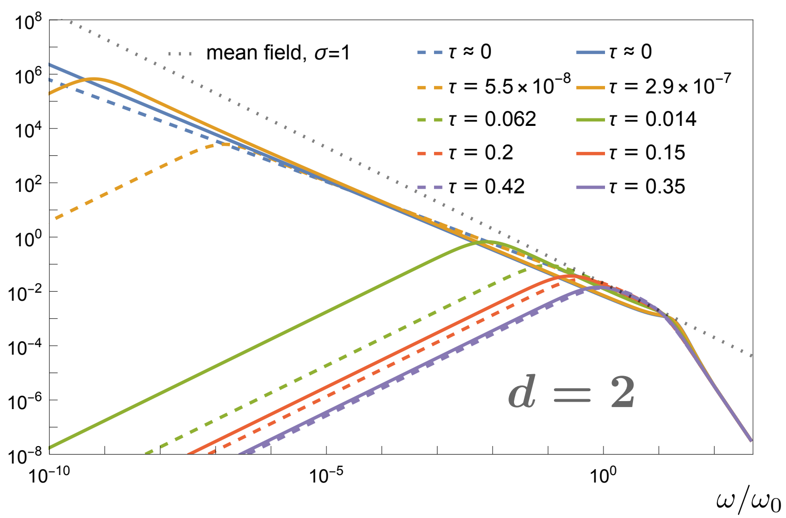
- **effective average action:**

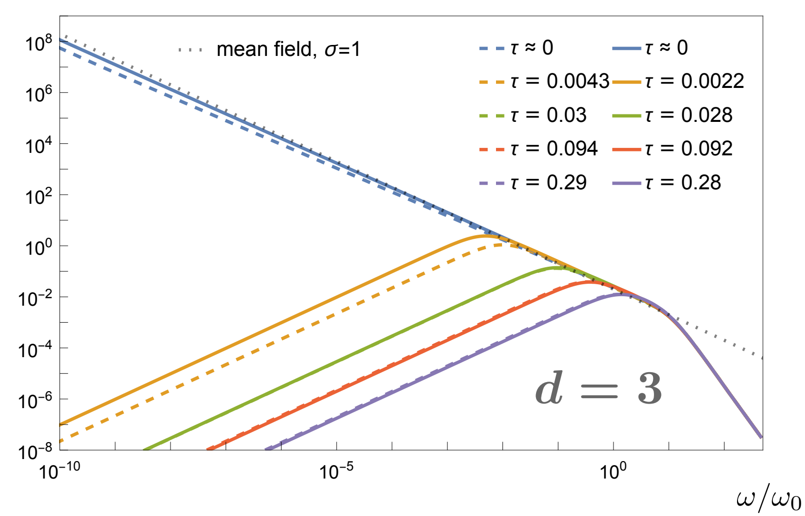
Martin-Siggia-Rose **Model A**

$$\Gamma_k = \frac{1}{2}\int_p \Delta\phi^T(-p) \begin{pmatrix} 0 & Z_k^{\parallel}(\omega)\,\omega^2 - Z_k^{\perp}\boldsymbol{p}^2 - m_k^2 - i\gamma_k(\omega)\omega \\ \text{c.c. of adv.} & 4i\gamma_k(\omega)T \end{pmatrix} \Delta\phi(p)$$

$$-\frac{\kappa_k}{\sqrt{8}}\int_x \left(\phi^c - \phi^c_{0,k}\right)^2 \phi^q - \frac{\lambda_k}{12}\int_x \left(\phi^c - \phi^c_{0,k}\right)^3 \phi^q\,, \qquad \eta_k^{\perp} = -k\partial_k \log Z_k^{\perp}$$

- **critical SFs:** $\rho(\omega,0) \sim \omega^{-\sigma}$ $\qquad \sigma = \dfrac{2-\eta_\perp}{z}$





Summary

- real-time methods for spectral functions:
  - (i) classical-statistical simulations
  - (ii) Gaussian state approximation
  - (iii) real-time FRG

  tested in (open) QM system

- field theory applications:

  study critical dynamics

  Models A, B, C

J. Roth, MSc Thesis, JLU, March 2022
J. Roth & LvS, in preparation

Thank you for your attention!