Inhomogeneous phases beyond mean field

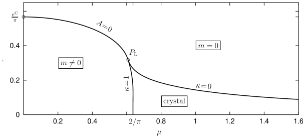
Lennart Kurth

TU Darmstadt - CRC-TR 211

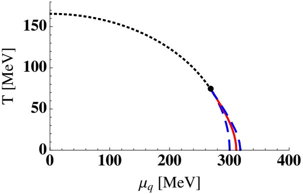
Motivation

- In many effective models one finds inhomogeneous phases in the mean field approximation.

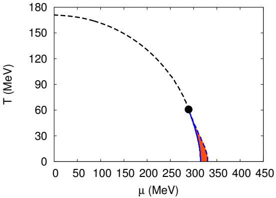
GN



NJL



QM



Goal

Find out if there are inhomogeneous phases in the quark meson model for some reasonable approximation that goes beyond mean field

O. Schnetz, M. Thies, and K. Urlichs, Annals Phys. 314 (2004) 425-447
D. Nickel, Phys. Rev. **D80**, 074025 (2009)
S. Carignano, M. Buballa, and B.-J. Schaefer, Phys.Rev. **D90**, 014033 (2014)

Basic concepts

The quark-meson model

$$S = \int d^4x \bigg(\bar{\psi}(Z^{\mathrm{F}} \partial\!\!\!/ + g\sigma + g\mathrm{i}\gamma_5 \tau \cdot \pi)\psi + \frac{Z^{\mathrm{B}}}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) + \frac{Z^{\mathrm{B}}}{2}(\partial_\mu \pi)\cdot(\partial^\mu \pi) + \frac{\kappa}{2}(\sigma^2+\pi^2) + \frac{\lambda}{4}(\sigma^2+\pi^2)^2 \bigg)$$

- 4 Bosons: σ, π and $2N_{\mathrm{c}}$ Fermions: $\bar{\psi}, \psi$
- 3 couplings: g, κ, λ and 2 wave function renormalizations: Z^{F}, Z^{B}
- Chirally symmetric
- Renormalizable

Inhomogeneous phases

- If

$$\Gamma[\text{some spatially varrying } \phi] < \Gamma[\text{any spatially contant } \phi]$$

  then the system is in an inhomogeneous phase
- Typically at large μ and small T
- ϕ is macroscopic order parameter
  $\rightarrow$ do not confuse with microscopic degrees of freedom
- Can be found by
  - ansatz that allows for inhomogeneity
  - stability analysis

Stability analysis

- "Taylor expand" Γ around homogeneous $\bar{\phi}$

$$\Gamma[\phi] = \Gamma\left[\bar{\phi}\right] + \int \mathrm{d}p \, \frac{\delta \Gamma}{\delta \phi(p)}\left[\bar{\phi}\right]\left(\phi(p) - \bar{\phi}\right) + \frac{1}{2}\int \mathrm{d}p \int \mathrm{d}q \, \frac{\delta^2 \Gamma}{\delta \phi(p) \delta \phi(q)}\left[\bar{\phi}\right]\left(\phi(p) - \bar{\phi}\right)\left(\phi(q) - \bar{\phi}\right)$$

- Choose $\bar{\phi}$ such that $\Gamma\left[\bar{\phi}\right]$ is minimal
- $\frac{\delta \Gamma}{\delta \phi(p \neq 0)}\left[\bar{\phi}\right] = 0$ because of translation symmetry
- Negative eigenvalue of $\frac{\delta^2 \Gamma}{\delta \phi(p) \delta \phi(q)}\left[\bar{\phi}\right]$ implies inhomogeneous phase

Mean field approximation

$$Z[\bar{\eta}, \eta, J_\sigma, J_\pi] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma\mathcal{D}\pi \exp\left(- S[\sigma, \pi, \bar{\psi}, \psi] + \text{sources}\right)$$

$\downarrow$ MFA

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp(-S\left[\langle\sigma\rangle, \langle\pi\rangle, \bar{\psi}, \psi\right])$$

- Use that S is quadratic in $\bar{\psi}$, ψ to evaluate fermionic path integral
- Find $\langle\sigma\rangle, \langle\pi\rangle$ by minimizing $-\log(Z) \approx$ free energy $\Omega \approx$ effective action Γ)

Functional renormalization group

CRC-TR 211

The functional renormalization group (FRG)

- is exact
- transforms path integral into ∞-dim. PDE
- requires/allows for uncontrolled/non-perturbative approximations

$$Z[J] = \int \mathcal{D}\phi \exp\left(-S[\phi] + \text{sources}\right)$$

$\downarrow$ FRG

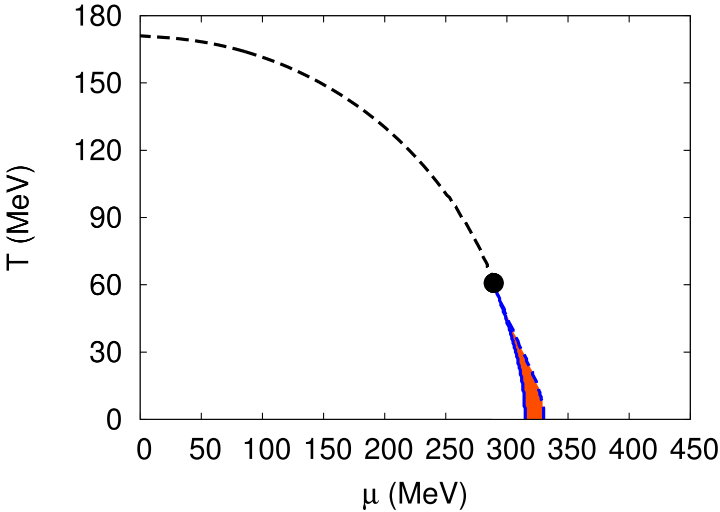
"UV" "IR"

$$S = \Gamma_\Lambda \xrightarrow[\text{Wetterich equation}]{\partial_k \Gamma_k[\phi] = \frac{1}{2} \mathrm{STr}\left(\left(\Gamma_k^{(2)}[\phi] + R_k^{\mathrm{T}}\right)^{-1} \partial_k R_k^{\mathrm{T}} - \text{norm.}\right)} \Gamma_0 = \Gamma$$

classical action quantum effective action

What has been done so far

Inhomogeneous phases in mean field

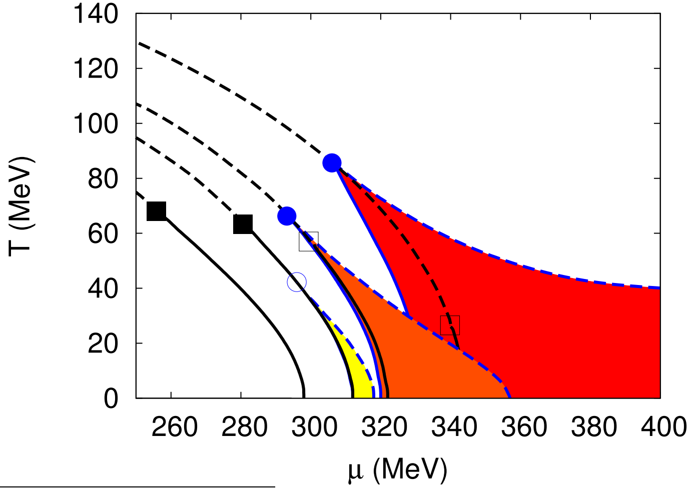


S. Carignano, M. Buballa, and B.-J. Schaefer, Phys.Rev. **D90**, 014033 (2014)

Inhomogeneous phases in mean field

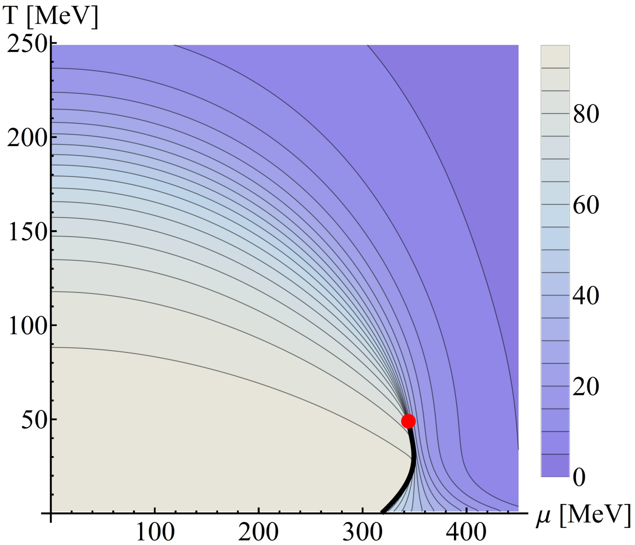
CRC–TR 211

$$m_\sigma = 550,\ 590,\ 610,\ 650\,\text{MeV}$$



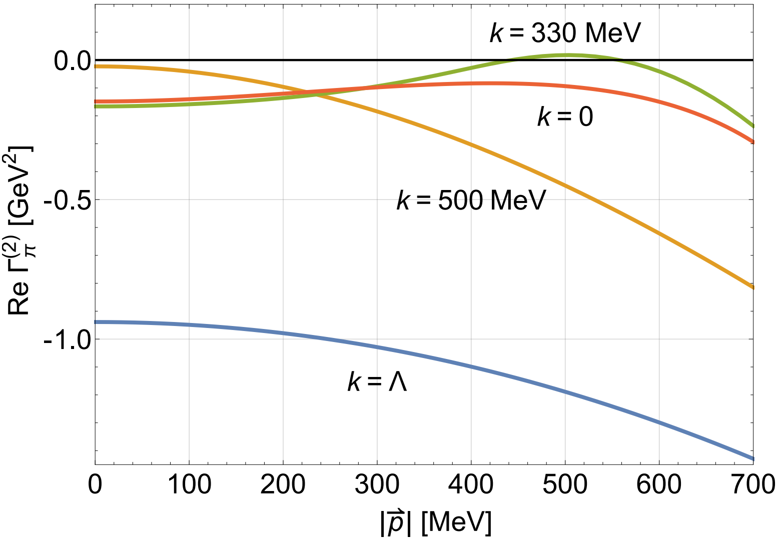
S. Carignano, M. Buballa, and B.-J. Schaefer, Phys.Rev. **D90**, 014033 (2014)

Homogeneous phases beyond mean field



R.-A. Tripolt, B.-J. Schaefer, L. von Smekal, and J. Wambach, Phys.Rev. **D97**, 034022 (2018)

Inhomogeneous phases beyond mean field



R.-A. Tripolt, B.-J. Schaefer, L. von Smekal, and J. Wambach, Phys.Rev. **D97**, 034022 (2018)

Reproducing mean field results with the functional renormalization group

Mean field approximation in FRG

$$\partial_k \Gamma_k[\phi] = \frac{1}{2}\,\mathrm{STr}\left(\left(\Gamma_k^{(2)}[\phi] + R_k^{\mathrm{T}}\right)^{-1}\partial_k R_k^{\mathrm{T}} - \mathrm{norm.}\right)$$

$$\Gamma^{(2)} = \begin{pmatrix} \frac{\delta\Gamma}{\delta\sigma\delta\sigma} & \frac{\delta\Gamma}{\delta\sigma\delta\pi} & \frac{\delta\Gamma}{\delta\sigma\delta\bar{\psi}} & \frac{\delta\Gamma}{\delta\sigma\delta\psi} \\ \frac{\delta\Gamma}{\delta\pi\delta\sigma} & \frac{\delta\Gamma}{\delta\pi\delta\pi} & \frac{\delta\Gamma}{\delta\pi\delta\bar{\psi}} & \frac{\delta\Gamma}{\delta\pi\delta\psi} \\ \frac{\delta\Gamma}{\delta\bar{\psi}\delta\sigma} & \frac{\delta\Gamma}{\delta\bar{\psi}\delta\pi} & \frac{\delta\Gamma}{\delta\bar{\psi}\delta\bar{\psi}} & \frac{\delta\Gamma}{\delta\bar{\psi}\delta\psi} \\ \frac{\delta\Gamma}{\delta\psi\delta\sigma} & \frac{\delta\Gamma}{\delta\psi\delta\pi} & \frac{\delta\Gamma}{\delta\psi\delta\bar{\psi}} & \frac{\delta\Gamma}{\delta\psi\delta\psi} \end{pmatrix} \rightarrow \begin{pmatrix} \frac{\delta\Gamma}{\delta\bar{\psi}\delta\bar{\psi}} & \frac{\delta\Gamma}{\delta\bar{\psi}\delta\psi} \\ \frac{\delta\Gamma}{\delta\psi\delta\bar{\psi}} & \frac{\delta\Gamma}{\delta\psi\delta\psi} \end{pmatrix}$$

$$R = \begin{pmatrix} R^\sigma & 0 & 0 & 0 \\ 0 & R^\pi & 0 & 0 \\ 0 & 0 & 0 & R^q \\ 0 & 0 & -\left(R^q\right)^{\mathrm{T}} & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & R^q \\ -\left(R^q\right)^{\mathrm{T}} & 0 \end{pmatrix}$$

Mean field approximation in FRG: 3D regulator

- In MFA no ansatz for Γ_k is needed
- Choose 3D Litim regulator

$$R_k^{\mathsf{B}} = Z^{\mathsf{B}}(k^2 - \vec{p}^2)\,\Theta(k^2 - \vec{p}^2)$$

$$R_k^{\mathsf{F}} = Z^{\mathsf{F}}\vec{\gamma} \cdot \vec{p}\left(\frac{k}{|\vec{p}|} - 1\right)\Theta(k^2 - \vec{p}^2)$$

- Exactly reproduces renormalized homogeneous mean field results
- But:
  - Meson propagator not Lorentz invariant
  - Wrong results for inhomogeneous phase (no matter how you project/fit)

Ad hoc solution

Allow for breaking of Lorentz symmetry in the UV action

Mean field approximation in FRG: 3D regulator

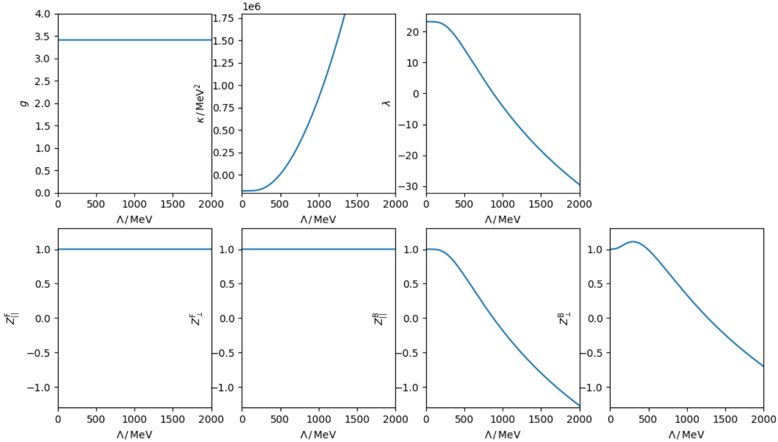
$$S = \int \mathrm{d}^4x \Big(\bar\psi (Z^{\mathrm{F}}_{||}\gamma^0\partial_0 + Z^{\mathrm{F}}_{\perp}\vec\gamma\cdot\vec\nabla + g\sigma + g\mathrm{i}\gamma_5\tau\cdot\pi)\psi + \frac{Z^{\mathrm{B}}_{||}}{2}(\partial_0\sigma)(\partial_0\sigma) + \frac{Z^{\mathrm{B}}_{\perp}}{2}(\vec\nabla\sigma)\cdot(\vec\nabla\sigma) + \frac{Z^{\mathrm{B}}_{||}}{2}(\partial_0\pi)\cdot(\partial_0\pi) + \frac{Z^{\mathrm{B}}_{\perp}}{2}(\vec\nabla\pi)\cdot(\vec\nabla\pi) + \frac{\kappa}{2}(\sigma^2+\pi^2) + \frac{\lambda}{4}(\sigma^2+\pi^2)^2 \Big)$$

- Splitting of wave function renormalization constants

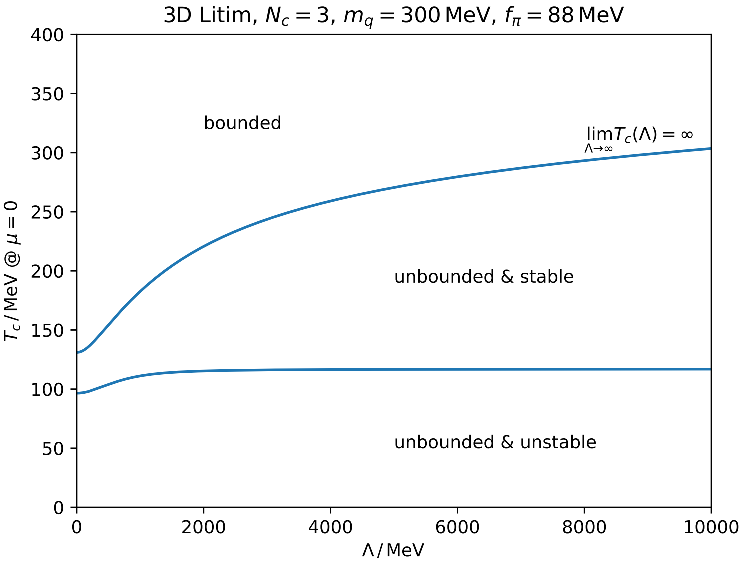
$$R^{\mathsf{B}}_k = Z^{\mathsf{B}}_{\perp}(k^2 - \vec p^{\,2})\,\Theta(k^2 - \vec p^{\,2})$$
$$R^{\mathsf{F}}_k = Z^{\mathsf{F}}_{\perp}\vec\gamma\cdot\vec p\left(\frac{k}{|\vec p|} - 1\right)\Theta(k^2 - \vec p^{\,2})$$

- Unambiguous parameter fitting
- Exactly reproduces all renormalized mean field results (a posteriori justification)

3D Litim, $N_c = 3$, $m_q = 300\,\mathrm{MeV}$, $f_\pi = 88\,\mathrm{MeV}$

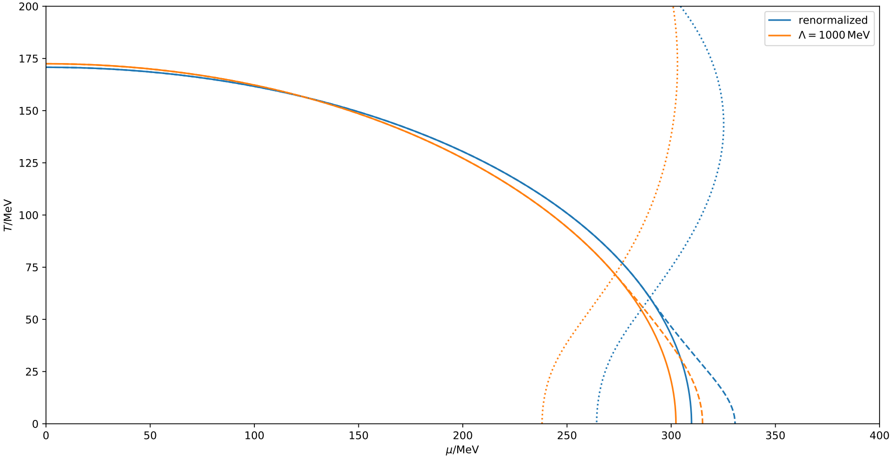


Possible observables



Phase diagram

$$N_{\mathrm{c}} = 3,\ m_q = 300\,\mathrm{MeV},\ m_\sigma = 600\,\mathrm{MeV},\ f_\pi = 88\,\mathrm{MeV}$$



Local potential ansatz in pion-quark truncation

Local potential approximation

- To include the meson contributions we need an ansatz

$$\Gamma^{\text{ansatz}} = \int \mathrm{d}^4x \bigg(\bar{\psi}(Z_{||}^{\mathrm{F}}\gamma^0\partial_0 + Z_{\perp}^{\mathrm{F}}\vec{\gamma}\cdot\vec{\nabla} + g\sigma + g\mathrm{i}\gamma_5\tau\cdot\pi)\psi \\ + \frac{Z_{||}^{\mathrm{B}}}{2}(\partial_0\sigma)(\partial_0\sigma) + \frac{Z_{\perp}^{\mathrm{B}}}{2}(\vec{\nabla}\sigma)\cdot(\vec{\nabla}\sigma) + \frac{Z_{||}^{\mathrm{B}}}{2}(\partial_0\pi)\cdot(\partial_0\pi) \\ + \frac{Z_{\perp}^{\mathrm{B}}}{2}(\vec{\nabla}\pi)\cdot(\vec{\nabla}\pi) + U_k\big(\sigma^2+\pi^2\big)\bigg)$$

- Only U can change during the flow
- Also choose as background homogeneous field configurations

Propagators in this approximation are of tree-level form
$\rightarrow$ Inhomogeneous phases excluded by construction
$\rightarrow$ Not "beyond mean field"

Local potential approximation

$$\Gamma^{\text{ansatz}} \rightarrow \text{Wetterich equation}$$

$$\downarrow$$

$$\partial_k \Gamma_k^{\text{ansatz}}[\sigma] = \frac{1}{2}\,\mathrm{STr}\left(\left(\Gamma_k^{\text{ansatz}\,(2)}[\sigma] + R_k^{\mathrm{T}}\right)^{-1}\partial_k R_k^{\mathrm{T}} - \mathrm{norm.}\right)$$

$$\sigma = \text{const}, \pi = 0 \quad \downarrow \quad \bar{\psi} = 0, \psi = 0$$

$$\partial_k U = \frac{Z_\perp^{\mathrm{B}} k^4}{12\pi^2\sqrt{Z_{||}^{\mathrm{B}}\left(Z_\perp^{\mathrm{B}} k^2 + \partial_\sigma^2 U\right)}} + \frac{Z_\perp^{\mathrm{B}} k^4}{4\pi^2\sqrt{Z_{||}^{\mathrm{B}}\left(Z_\perp^{\mathrm{B}} k^2 + \frac{1}{\sigma}\partial_\sigma U\right)}} - \frac{2N_{\mathrm{c}}(Z_\perp^{\mathrm{F}})^2 k^4}{3\pi^2 Z_{||}^{\mathrm{F}}\sqrt{(Z_\perp^{\mathrm{F}} k)^2 + g^2\sigma^2}} + \mathrm{norm.}$$

- Non-linear 2-dim. scalar 2nd order partial differential equation

Pion-quark truncation

$$\partial_k \Gamma_k[\phi] = \frac{1}{2}\,\mathrm{STr}\left(\left(\Gamma_k^{(2)}[\phi] + R_k^{\mathrm{T}}\right)^{-1}\partial_k R_k^{\mathrm{T}} - \mathrm{norm.}\right)$$

$$\Gamma^{(2)} = \begin{pmatrix} \frac{\delta\Gamma}{\delta\sigma\delta\sigma} & \frac{\delta\Gamma}{\delta\sigma\delta\pi} & \frac{\delta\Gamma}{\delta\sigma\delta\bar\psi} & \frac{\delta\Gamma}{\delta\sigma\delta\psi} \\ \frac{\delta\Gamma}{\delta\pi\delta\sigma} & \frac{\delta\Gamma}{\delta\pi\delta\pi} & \frac{\delta\Gamma}{\delta\pi\delta\bar\psi} & \frac{\delta\Gamma}{\delta\pi\delta\psi} \\ \frac{\delta\Gamma}{\delta\bar\psi\delta\sigma} & \frac{\delta\Gamma}{\delta\bar\psi\delta\pi} & \frac{\delta\Gamma}{\delta\bar\psi\delta\bar\psi} & \frac{\delta\Gamma}{\delta\bar\psi\delta\psi} \\ \frac{\delta\Gamma}{\delta\psi\delta\sigma} & \frac{\delta\Gamma}{\delta\psi\delta\pi} & \frac{\delta\Gamma}{\delta\psi\delta\bar\psi} & \frac{\delta\Gamma}{\delta\psi\delta\psi} \end{pmatrix} \to \begin{pmatrix} \frac{\delta\Gamma}{\delta\pi\delta\pi} & \frac{\delta\Gamma}{\delta\pi\delta\bar\psi} & \frac{\delta\Gamma}{\delta\pi\delta\psi} \\ \frac{\delta\Gamma}{\delta\bar\psi\delta\pi} & \frac{\delta\Gamma}{\delta\bar\psi\delta\bar\psi} & \frac{\delta\Gamma}{\delta\bar\psi\delta\psi} \\ \frac{\delta\Gamma}{\delta\psi\delta\pi} & \frac{\delta\Gamma}{\delta\psi\delta\bar\psi} & \frac{\delta\Gamma}{\delta\psi\delta\psi} \end{pmatrix}$$

$$R = \begin{pmatrix} R^\sigma & 0 & 0 & 0 \\ 0 & R^\pi & 0 & 0 \\ 0 & 0 & 0 & R^q \\ 0 & 0 & -\left(R^q\right)^{\mathrm{T}} & 0 \end{pmatrix} \to \begin{pmatrix} R^\pi & 0 & 0 \\ 0 & 0 & R^q \\ 0 & -\left(R^q\right)^{\mathrm{T}} & 0 \end{pmatrix}$$

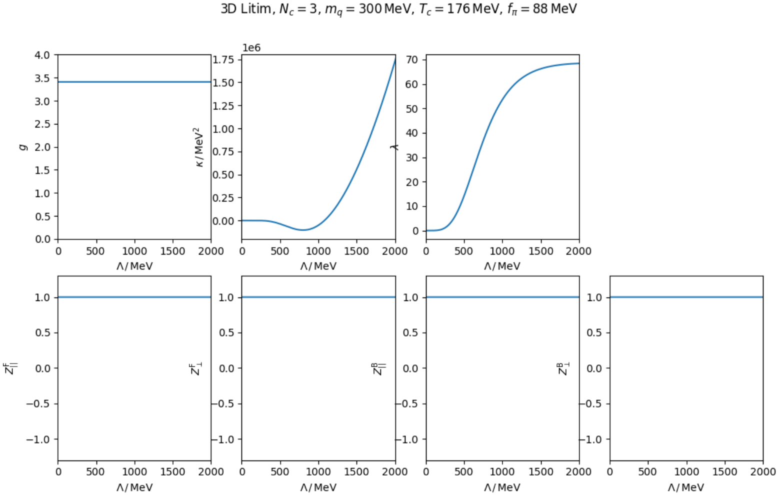
Pion quark truncation

- Inserting the local potential ansatz in the pion-quark truncation yields

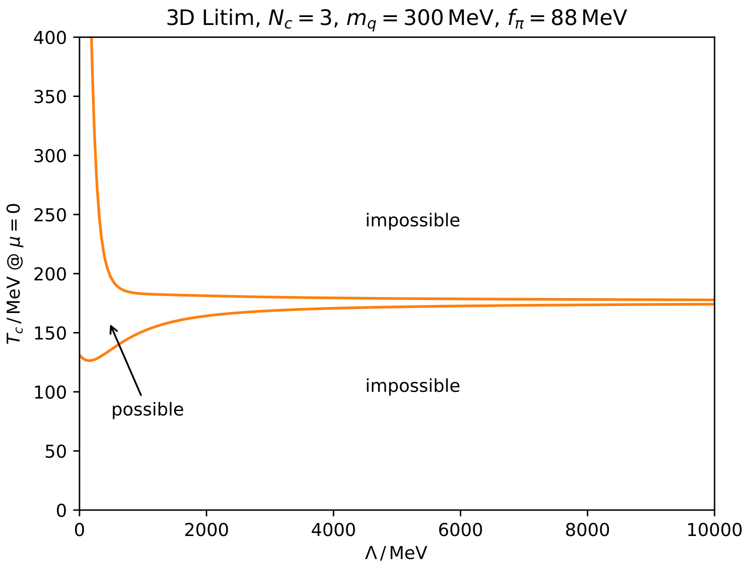
$$\partial_k U = \frac{Z_\perp^{\mathrm{B}} k^4}{4\pi^2 \sqrt{Z_{||}^{\mathrm{B}}\left(Z_\perp^{\mathrm{B}} k^2 + \frac{1}{\sigma}\partial_\sigma U\right)}} - \frac{2N_{\mathrm{c}}(Z_\perp^{\mathrm{F}})^2 k^4}{3\pi^2 Z_{||}^{\mathrm{F}} \sqrt{(Z_\perp^{\mathrm{F}} k)^2 + g^2\sigma^2}} + \mathrm{norm.}$$

- Non-linear 2-dim. scalar 1st order partial differential equation
- Convert to ODE system via method of characteristics
- Result still chirally symmetric by construction
- But: In π-q truncation $m_\sigma = 0$
  (compare $O(N)$ model in large N limit)

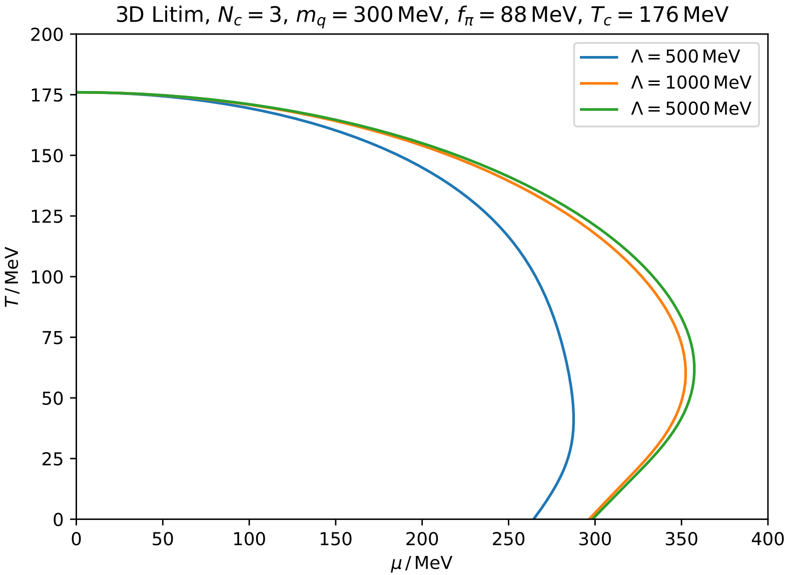
Couplings


3D Litim, $N_c = 3$, $m_q = 300$ MeV, $T_c = 176$ MeV, $f_\pi = 88$ MeV
g
κ / MeV^2
1e6
λ
$Z^F_{\parallel}$
$Z^F_{\perp}$
$Z^B_{\parallel}$
$Z^B_{\perp}$
Λ / MeV

Possible observables



Phase diagram



Local potential ansatz in pion-quark truncation with iteration

Iteration of Wetterich equation

Ad hoc solution

Iterate the Wetterich equation to get non-trivial momentum structure

- Step 1: Solve Wetterich equation in some approximation

$$\partial_k \Gamma_k^{\text{ansatz}}[\phi] = \frac{1}{2}\,\mathrm{STr}\left(\left(\Gamma_k^{\text{ansatz}\,(2)}[\phi] + R_k^{\mathrm{T}}\right)^{-1}\partial_k R_k^{\mathrm{T}} - \text{norm.}\right)\Bigg|_{\text{truncation}}$$

- Step 2: Integrate right hand side of untruncated Wetterich equation for result of step 1

$$\Gamma_k^{\text{baditer}}[\phi] = S[\phi] + \int_\Lambda^k \mathrm{d}s\,\frac{1}{2}\,\mathrm{STr}\left(\left(\Gamma_s^{\text{ansatz}\,(2)}[\phi] + R_s^{\mathrm{T}}\right)^{-1}\partial_s R_s^{\mathrm{T}} - \text{norm.}\right)$$

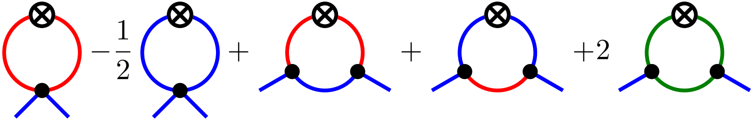
- Step 3: Do not iterate the potential to keep symmetry breaking

$$\Gamma_k^{\text{iterated}}[\phi] = \Gamma_k^{\text{baditer}}[\phi] - \int \mathrm{d}^4x\, U_{\text{baditer}} + \int \mathrm{d}^4x\, U_{\text{ansatz}}$$

Iteration of Wetterich equation

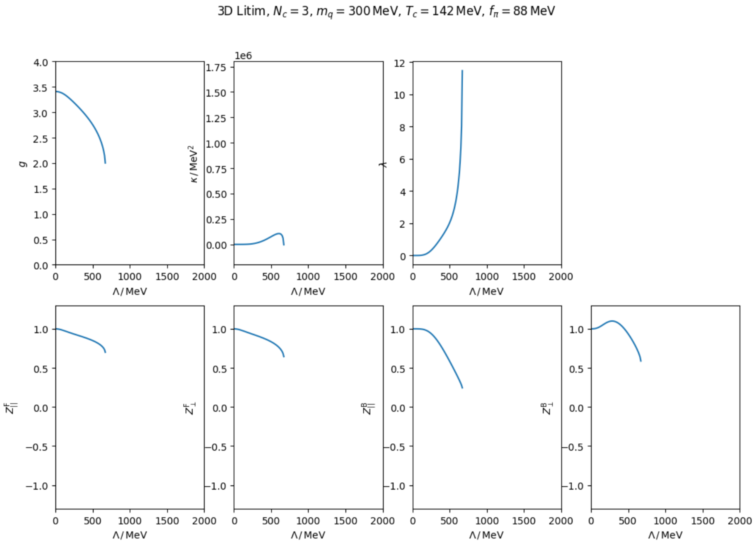
- Iteration leaves true solution unchanged
- No truncation in step 1 $\rightarrow$ step 3 does nothing
- I would call iterated LPA "beyond mean field"
  (even in π-q trunc.)
- Momentum structure of meson propagator in restored phase is the same as in MFA

$\partial_k K^\pi = -\frac{1}{2}$



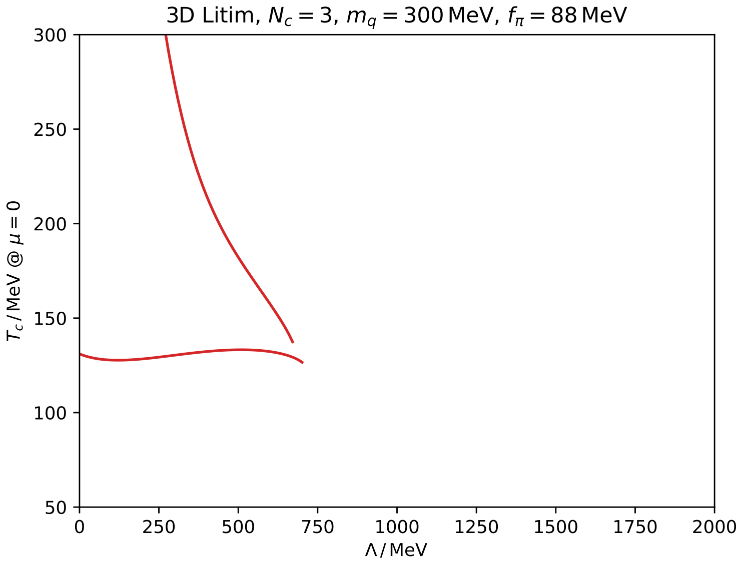
- But: Non-trivial momentum structure does not enter differential equation
  - Flow can not compensate for these contributions
  - Complications if ansatz is too simple

Couplings

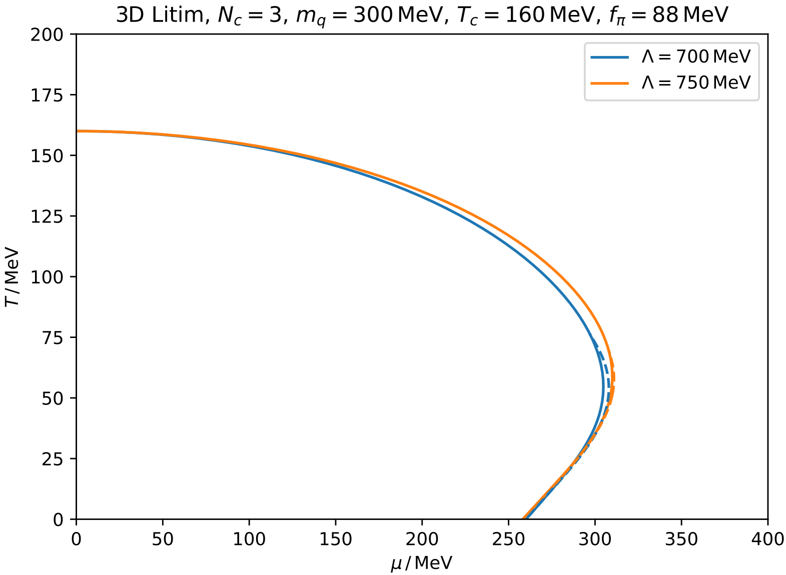


3D Litim, $N_c = 3$, $m_q = 300$ MeV, $T_c = 142$ MeV, $f_\pi = 88$ MeV
g
κ / MeV²
1e6
λ
$Z^F_\parallel$
$Z^F_\perp$
$Z^B_\parallel$
$Z^B_\perp$
Λ / MeV

Possible observables



Phase diagram



The end

Summary

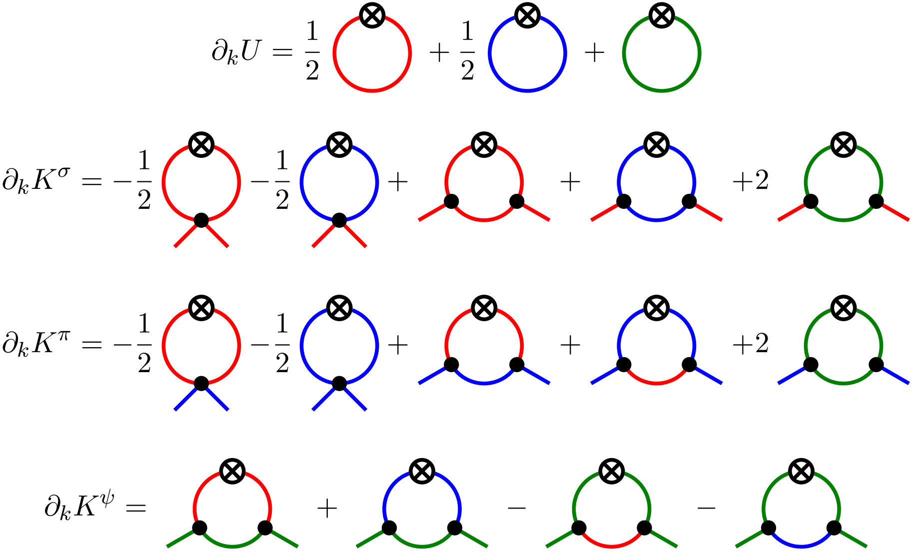
- Renormalized mean field can be exactly reproduced in FRG
- $\Lambda \to \infty$ seems possible for "pure" LPA (in π-q trunc.)
- Iteration of flow equation $+$ too simple ansatz $=$ small maximal Λ
- Inhomogeneous phases beyond mean field in quark-meson model still neither confirmed nor ruled out.

Outlook

- More general ansätze
- Include sigma contribution/diffusion term
- Calculate spectral functions to fit real time quantities
- Include explicit symmetry breaking

Appendix

Flow equation diagrams

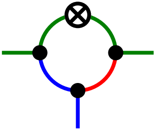
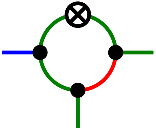
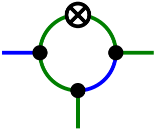


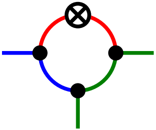
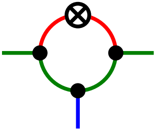
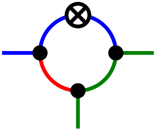
Flow equation diagrams

$$\partial_k R^{\sigma||} = \frac{\partial}{\partial q_0^2}\Big(\partial_k K^{\sigma}\Big), \quad \partial_k R^{\sigma\perp} = \frac{\partial}{\partial \vec{q}^{\,2}}\Big(\partial_k K^{\sigma}\Big)$$

$$\partial_k R^{\pi||} = \frac{\partial}{\partial q_0^2}\Big(\partial_k K^{\pi}\Big), \quad \partial_k R^{\pi\perp} = \frac{\partial}{\partial \vec{q}^{\,2}}\Big(\partial_k K^{\pi}\Big)$$

$$\partial_k R^{\psi||} = \frac{\partial}{\partial (\mathrm{i}\gamma^0 q_0)}\Big(\partial_k K^{\psi}\Big), \quad \partial_k R^{\psi\perp} = \frac{\partial}{\partial (\mathrm{i}\vec{\gamma}\cdot\vec{q})}\Big(\partial_k K^{\psi}\Big)$$

$\partial_k V^{\bar{\psi}\psi\pi} = 2$  $+\,2$  $+\,2$ 

$-\,2$  $-$  $-\,2$  $-$