QCD phases at nonzero chemical potential

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TU Darmstadt

HFHF Theory Retreat
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Introduction

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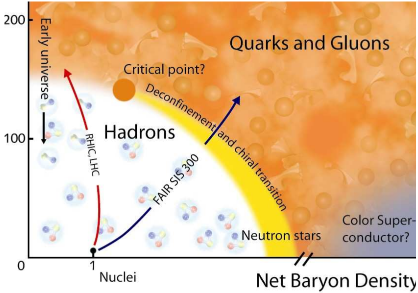
Focus: Moderate temperature and (not asymptotically) high density

- theoretically hard:
  - non-perturbative
  - sign problem on the lattice
- phenomenologically interesting:
  - neutron stars and neutron-star mergers
  - CBM physics at FAIR
- regions of special interest:
  - critical point
  - color superconducting phases



Aim of this talk:

Pedagogical introduction, laying the ground for Lennart's and Hosein's talks.

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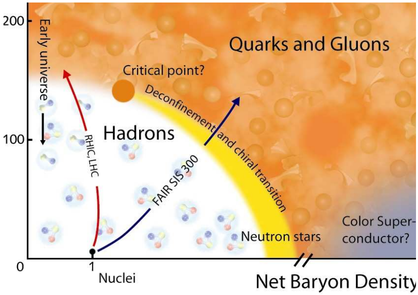

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COLOR SUPERCONDUCTIVITY

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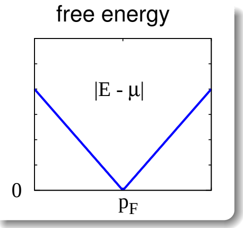

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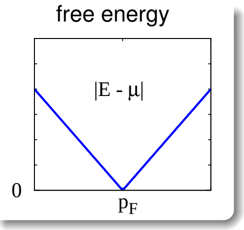

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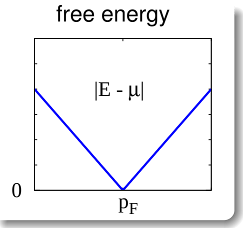

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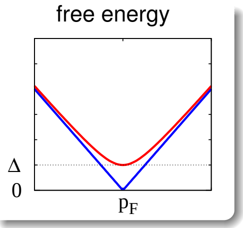

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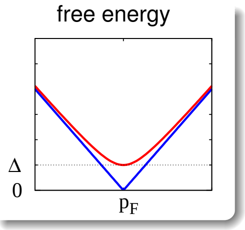

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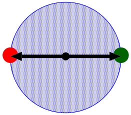
  - pairs with vanishing total momentum: $\vec{p}^{(1)} = -\vec{p}^{(2)}$
  - each partner close to the Fermi surface

    → works only if $p_F^{(1)} \approx p_F^{(2)}$



Diquark condensates

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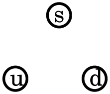
  - $\rightarrow$ pairing between different flavors
- example: $(\uparrow\downarrow - \downarrow\uparrow) \otimes (r\,g - g\,r) \otimes (ud - du)$

Three-flavor systems

- Pairing patterns in flavor space:

  no pairing: “normal quark matter” (NQ)



Three-flavor systems

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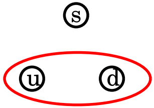
- Pairing patterns in flavor space:

  two-flavor superconducting (2SC) phase

  (+ two analogous phases with *us* or *ds* pairing)



Three-flavor systems

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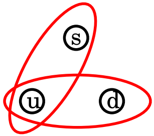
- Pairing patterns in flavor space:

  uSC phase

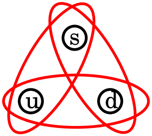
  (similar: dSC phase, sSC)



Three-flavor systems

- Pairing patterns in flavor space:

  color-flavor locked (CFL) phase



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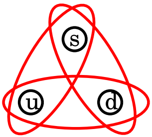
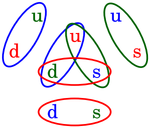


- CFL pairing (more explicitly):

$$(\uparrow\downarrow - \downarrow\uparrow) \otimes \Big((ud - du) \otimes (r\,g - g\,r) + (ds - sd) \otimes (g\,b - b\,g) + (su - us) \otimes (b\,r - r\,b) \Big)$$



(More) formal definition of the phases

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- Diquark condensates:

$$(\uparrow\downarrow - \downarrow\uparrow) \otimes (ud - du) \otimes (r\,g - g\,r) \leftrightarrow \langle q^T\, C\gamma_5\, \tau_2\, \lambda_2\, q\rangle \;\sim:\; \Delta_2$$

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$$(\uparrow\downarrow - \downarrow\uparrow) \otimes (su - us) \otimes (b\,r - r\,b) \leftrightarrow \langle q^T\, C\gamma_5\, \tau_7\, \lambda_7\, q\rangle \;\sim:\; \Delta_7$$

$C = i\gamma^2\gamma^0$ charge conjugation matrix, $C\gamma_5 \rightarrow J^P = 0^+$

τ_A: antisymmetric Gell-Mann matrices in flavor space

λ_A: antisymmetric Gell-Mann matrices in color space

- Phases:
  - NQ: $\Delta_2 = \Delta_5 = \Delta_7 = 0$
  - 2SC: $\Delta_2 \neq 0$, $\Delta_5 = \Delta_7 = 0$
  - CFL: $\Delta_2 = \Delta_5 = \Delta_7 \neq 0$ (ideal case; realistic: $\Delta_2 \approx \Delta_5 \approx \Delta_7 \neq 0$)
  - ...

Symmetries of the 2SC phase

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$$\Delta_2 = \langle q^T \, C\gamma_5 \, \tau_2 \, \lambda_2 \, q \rangle$$

- gauge symmetries:
  - **color:** $q \to e^{i\theta_a \frac{\lambda^a}{2}} q$ blue quarks unpaired $\Rightarrow$ $SU(3)_c \to SU(2)_c$

    → 5 of the 8 gluons get a nonzero Meissner mass.
  - **electromagnetism:** $q \to e^{i\alpha Q} q, \quad Q = \text{diag}_f(\frac{2}{3}, -\frac{1}{3})$ broken

    But there is an unbroken $U(1)$ gauge symmetry with charge $\tilde{Q} = Q - \frac{1}{2\sqrt{3}}\lambda_8$.

→ color superconductor but not electromagnetic superconductor

- global symmetries:
  - **baryon number:** $q \to e^{i\alpha} q \;\Rightarrow\; \Delta_2 \to e^{2i\alpha}\Delta_2$ broken

    But there is an unbroken “modified baryon number” $q \to e^{i\alpha(1-\sqrt{3}\lambda_8)} q$
  - $SU(2)_L \times SU(2)_R$ **chiral symmetry:** conserved

→ same global symmetries as 2-flavor restored phase, no Goldstone bosons

Symmetries of the (ideal) CFL phase

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$$\langle q^T C\gamma_5 \tau_2 \lambda_2 q \rangle = \langle q^T C\gamma_5 \tau_2 \lambda_2 q \rangle = \langle q^T C\gamma_5 \tau_2 \lambda_2 q \rangle = \Delta$$

- **color:** $SU(3)_c$ broken completely
- **chiral symmetry:** $SU(3)_L \times SU(3)_R$ broken completely

  but:

  residual $SU(3)$ under **combined color-flavor** rotations: $q \to e^{i\theta_a(\tau_a - \lambda_a^T)} q$

  → "color-flavor locking": $SU(3)_c \times SU(3)_L \times SU(3)_R \to SU(3)_{V+c}$

  → 8 massive gluons + 8 pseudoscalar Goldstone bosons (chiral limit)
- **baryon number:** $U(1)$ broken → 1 scalar Goldstone boson
- **electromagnetism:**

  unbroken $U(1)$ gauge symmetry with charge $\tilde{Q} = Q - \frac{1}{2}\lambda_3 - \frac{1}{2\sqrt{3}}\lambda_8$

→ color but not electromagnetic superconductor, baryon number superfluid

Which phase is favored?
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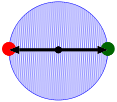

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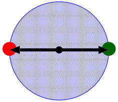

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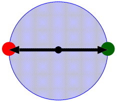

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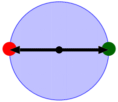

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- Expected phase structure:
  - $\mu \gg M_s \Rightarrow p_F^{(s)} \approx p_F^{(u,d)}$ → CFL
  - $\mu \lesssim M_s \Rightarrow p_F^{(s)} \ll p_F^{(u,d)}$ → 2SC

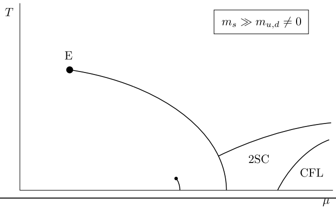

Figure: educated guess [Rajagopal (1999)]

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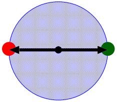

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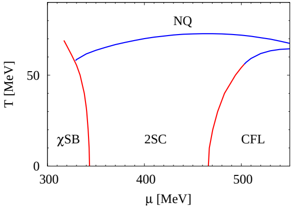

Figure: NJL [M. Oertel, MB (2002); MB (2005)]

Role of the strange quark mass

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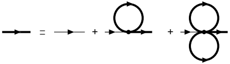
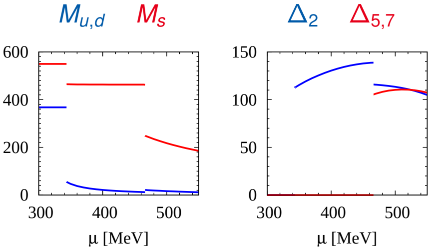


$$M_s = m_s - 4G\langle\bar{s}s\rangle + 2K\langle\bar{u}u\rangle\langle\bar{d}d\rangle$$

  - → M_s large in the 2SC phase
  - → stabilizes the 2SC phase



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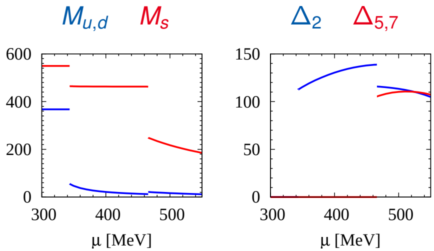

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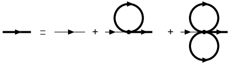

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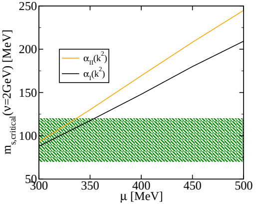
- Dyson-Schwinger QCD studies

[Nickel, Alkofer, Wambach (2006)]



  - → gluons screened by light quarks
  - → M_s smaller in the 2SC phase
  - → CFL phase favored much earlier

Compact star conditions

- constraints in compact stars:
  - color neutrality: $n_r = n_g = n_b$
  - electric neutrality: $n_Q = \frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s - n_e = 0$
  - β equilibrium: $\mu_e = \mu_d - \mu_u \quad \Rightarrow \quad n_e \ll n_{u,d}$

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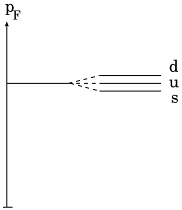
- Expansion in small M_s [Alford, Rajagopal (2002)]
  - → equidistant splitting
  - → no 2SC phase in compact stars



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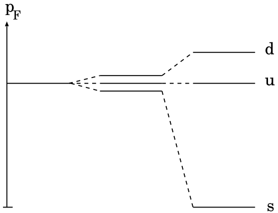
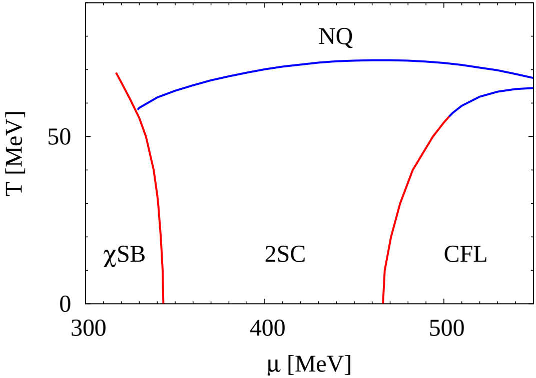
- Large M_s
  - → $n_s \approx 0,\; n_d \approx 2 n_u \;\Rightarrow\; p_F^{(d)} \approx 2^{1/3} p_F^{(u)} \approx 1.26\, p_F^{(u)}$
  - → 2SC pairing possible for strong couplings



NJL model results

Phase diagram without neutrality constraints

[M. Oertel, MB (2002)]



NJL model results

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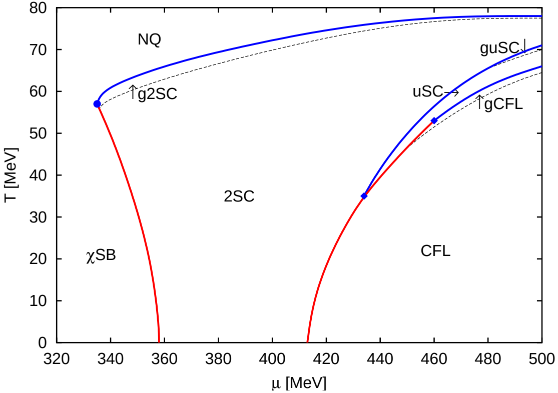
Phase diagram with neutrality constraints: "strong" qq coupling ($H = G$)

[Rüster, Werth, MB, Shovkovy, Rischke, (2005)]



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Phase diagram with neutrality constraints: "intermediate" *qq* coupling ($H = 0.75\ G$)

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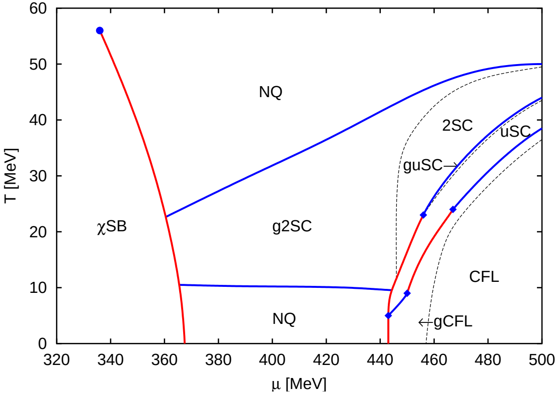

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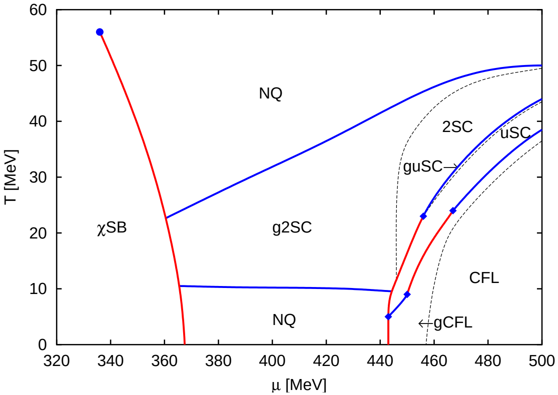
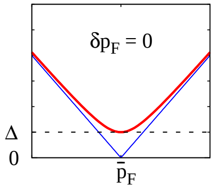


→ strong parameter dependence

Gapless color superconductors

- unequal Fermi momenta: $p_F^{a,b} = \bar{p}_F \pm \delta p_F$

free energy



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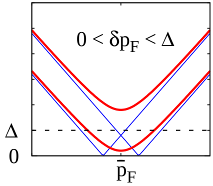

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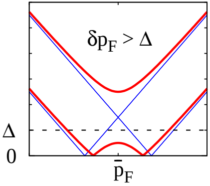

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  - unstable solution (maximum) at fixed μ_e
  - can be most favored neutral homogeneous solution

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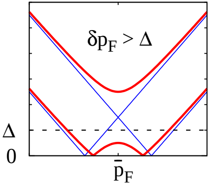

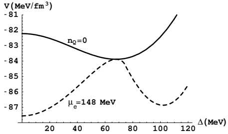

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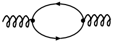

$$m^2_{M,a} = -\frac{1}{2} \lim_{\vec{p}\rightarrow 0} \left(g_{ij} + \frac{p_i p_j}{p^2}\right) \Pi^{ij}_{aa}(0, \vec{p})$$

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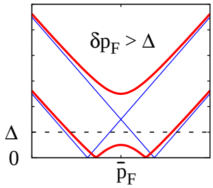

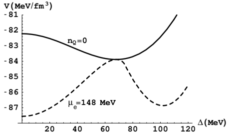

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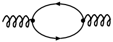

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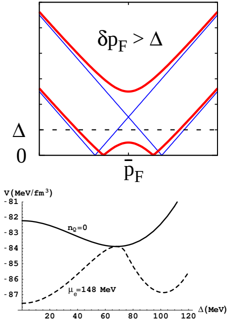
- chromomagnetic instability: $m^2_{M,a} < 0$ for $\delta p_F > \begin{cases} \frac{\Delta}{\sqrt{2}} & a = 4, \dots, 7 \\ \Delta & a = 8 \end{cases}$

free energy



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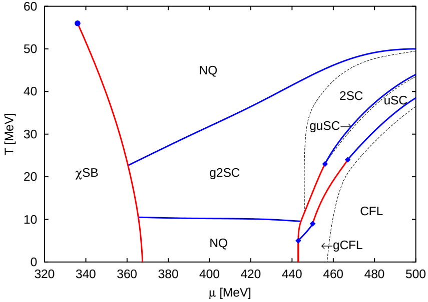
Main issues



- strong parameter dependence
- unstable phases

Getting rid of the parameter dependence

Getting rid of the parameter dependence

1. Theoretical approaches: starting from QCD

Getting rid of the parameter dependence

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- Dyson-Schwinger equations:

  [Nickel, Alkofer, Wambach (2006, 2008),
  Müller, MB, Wambach (2013, 2016)]

  - phase diagram without neutrality constraints
  - no cutoff artifacts at large μ
  - determination of the pressure difficult
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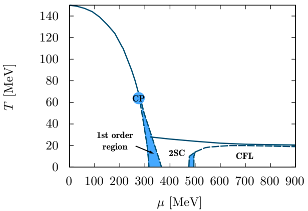

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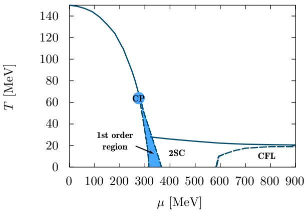

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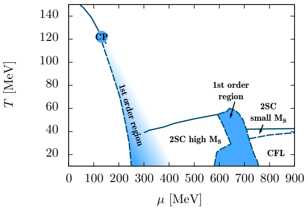

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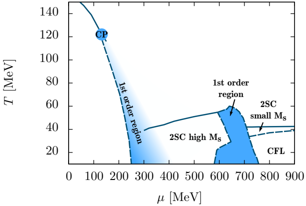
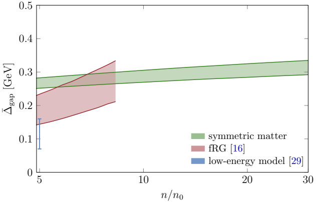


[Müller et al. (2016)]

- **Functional renormalization group:**

  [Braun, Schallmo (2022)]

  - study 2SC pairing at $T = 0$ by solving QCD flow equations at large μ ➞ very large gaps!



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Getting rid of the parameter dependence

2. Using empirical information

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TECHNISCHE UNIVERSITÄT DARMSTADT

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- Fitting NJL parameters to astrophysical constraints and heavy-ion data:

  [Klähn, Blaschke, ... (2006, 2007, 2013, ...)]

  - purely hadronic matter inconsistent

    (see also [Annala et al. (2020)])
  - vector repulsion to be stiff enough
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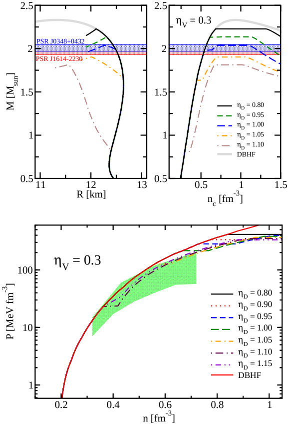

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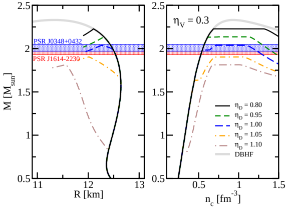
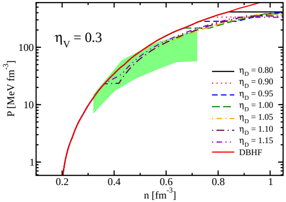
- Signals of CSC in the gravitational-wave spectrum from neutron-star mergers?
  - part of project B09 in the CRC-TR 211
    ↔ Hosein's thesis project
    (see his talk for preparatory work)





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CSC phases in neutron-star mergers (propaganda plots)

Overlay of unrelated calculations:

Points from merger simulations with purely hadronic EoS [E. Most, L. Rezzolla, priv. comm.] and NJL phase diagrams [Rüster et al. (2005, 2006)]

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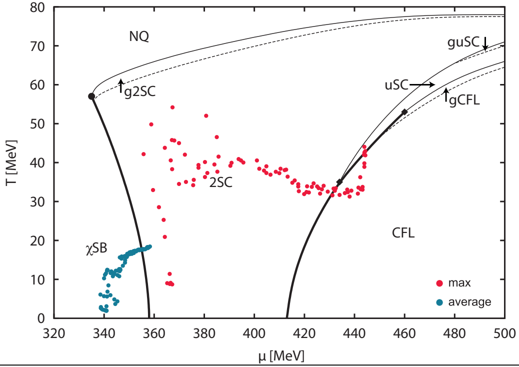

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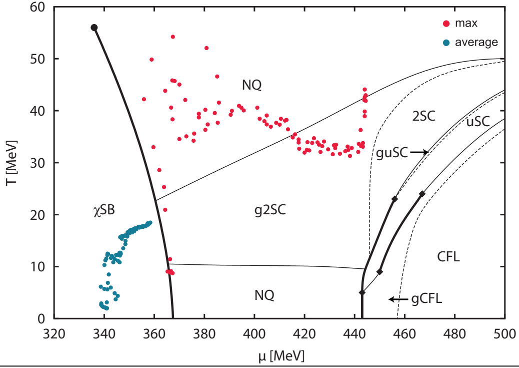

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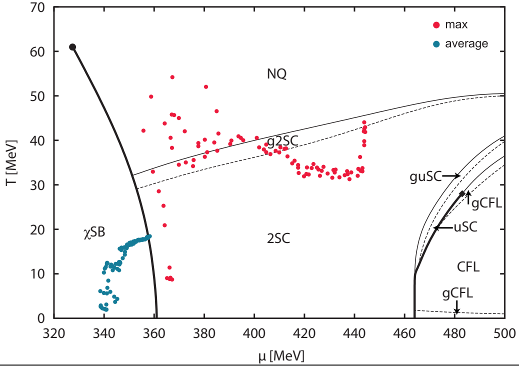
"intermediate" *qq* coupling + neutrino chemical potential μ_ν = 200 MeV



Trapped neutrinos

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- Proto-neutron stars: neutrinos trapped during the first few seconds

  → lepton number conserved

  → more electrons:

  $$\mu_e = \mu_d - \mu_u + \mu_\nu$$

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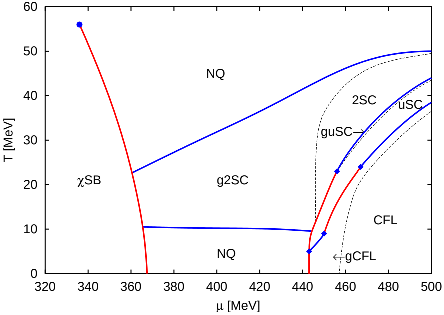

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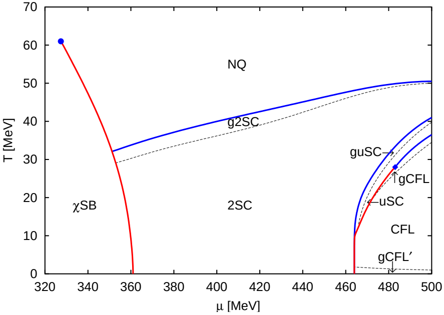

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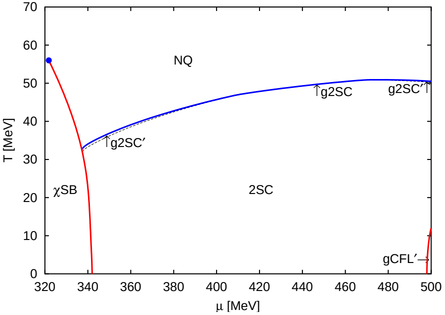

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- also relevant for neutron-star mergers!

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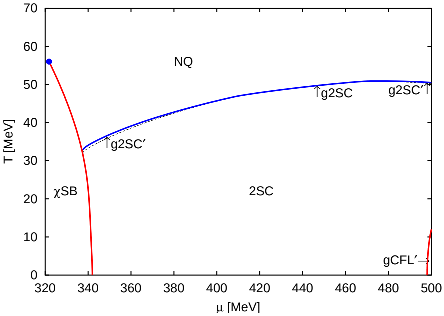

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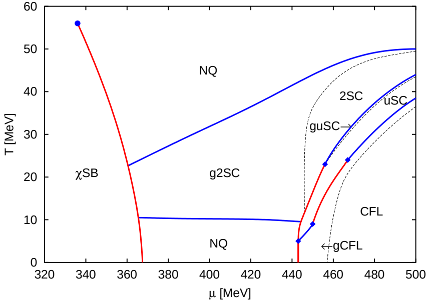
Main issues



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- unstable phases

Kaon condensation in the CFL phase

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- CFL: chiral symmetry broken → Goldstone bosons $\sim \mathcal{O}(10\text{ MeV})$
  [Son, Stephanov, PRD (2000)]
- $\mu_s^{eff} \simeq \frac{m_s^2 - m_u^2}{2\mu}$ → K^0 condensation [T. Schäfer, PRL (2000); Bedaque, Schäfer, NPA (2002)]
- NJL model: include pseudoscalar diquark conds. [M.B., PLB (2005); M.M. Forbes, PRD (2005)]

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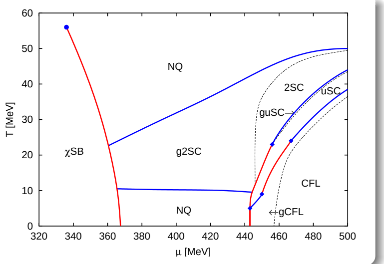

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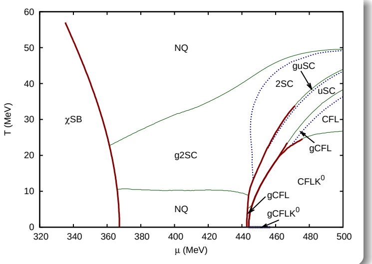

[H. Basler, M.B., PRD (2010); H. Warringa (2006)]

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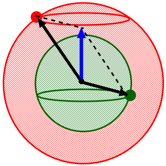

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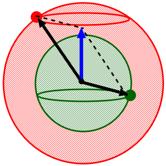

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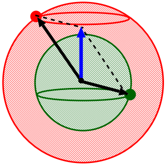

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- LOFF in CSC (→ [Anglani et al., Rev. Mod. Phys. (2014)])

  Indications:

  chromomagnetic instabilities $\leftrightarrow$ instabilities towards LOFF phases

  [Giannakis, Ren; Giannakis, Hou, Ren, PLB (2005)]

NJL-model results

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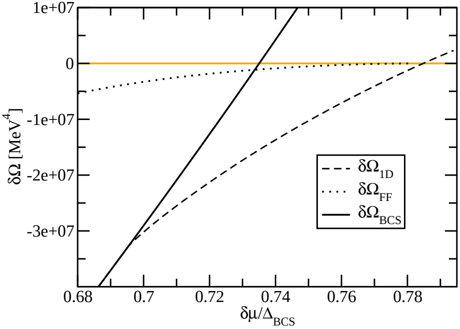
free-energy gain:



[D. Nickel, MB, PRD (2008)]

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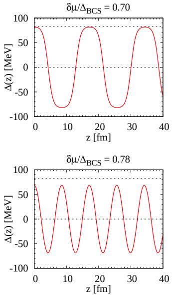

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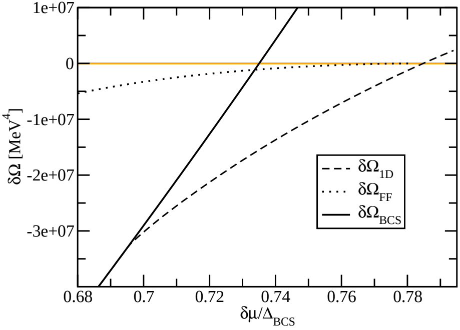

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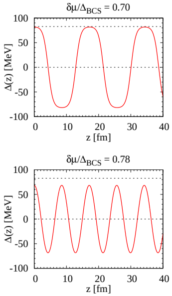

- still missing: comprehensive calculation of neutral phase diagram with LOFF phases

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INHOMOGENEOUS CHIRAL PHASES

Stressed chiral-symmetry breaking

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- $\langle \bar{q}q \rangle = \langle \bar{q}_L q_R \rangle + \langle \bar{q}_R q_L \rangle$
- chiral-symmetry breaking in vacuum:
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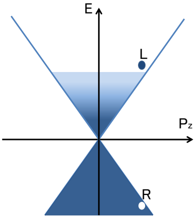

[Kojo et al. (2010)]

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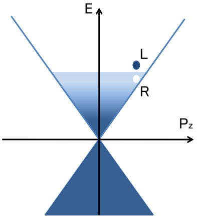

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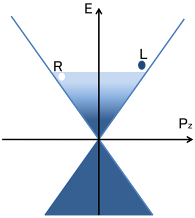

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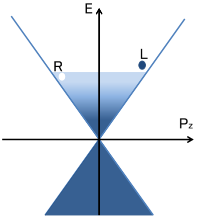
    $\rightarrow$ inhomogeneous chiral condensates!



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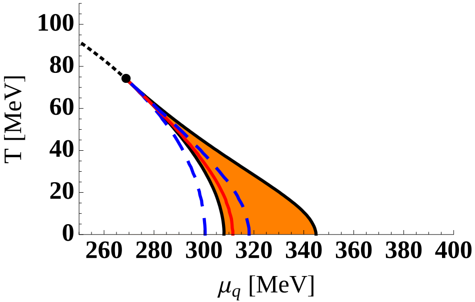
Phase diagram

NJL [D. Nickel (2009)]



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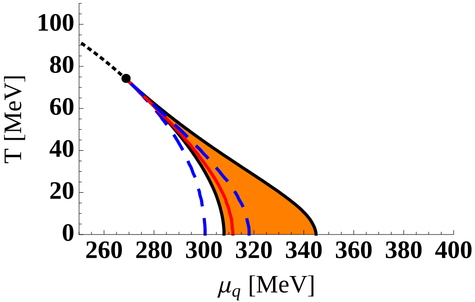
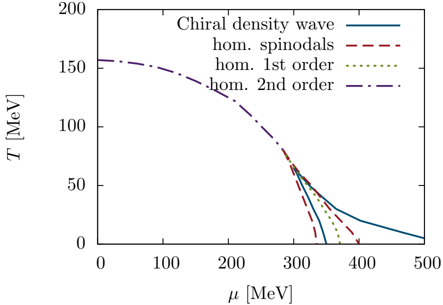


DSE [D. Müller et al. (2013)]



- NJL: inhomogeneous phase covers homogeneous first-order line
- DSE: phase-transition region qualitatively similar

Methods to investigate inhomogeneous phases

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- Main difficulty: Functional minimization of the free energy
  w.r.t. to arbitrarily shaped condensate functions $\phi(\vec{x})$

e.g., $\Omega[\phi(\vec{x})] = \frac{T}{V}\mathbf{Tr}\log\frac{S^{-1}[\phi(\vec{x})]}{T} + \frac{1}{V}\int_V d^3x\ \left\{\frac{1}{2}(\boldsymbol{\nabla}\phi(\vec{x}))^2 + U(\phi(\vec{x}))\right\}$

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- strategy 3: stability analysis

  split $\phi(\vec{x}) = \bar{\phi} + \delta\phi(\vec{x})$, $\bar{\phi}$ = homogeneous minimum

  and expand in powers of the fluctuations $\Omega = \sum_n \Omega^{(n)}$, $\Omega^{(n)} = \mathcal{O}(\delta\phi^n)$

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$\Omega^{(2)} < 0 \longrightarrow$ instability towards an inhomogeneous phase (sufficient condition)

Inhomogeneous chiral phases: current state

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- seen in NJL and Quark-Meson model in (extended) mean-field approximation, QCD with DSEs; indications from FRG ("moat regime") [Fu et al. (2020)]
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  - explicit symmetry breaking
- open questions:
  - Are the inhomogeneous phases stabie against fluctuations beyond mean field?
    → FRG approach, see Lennart's talk (next)
  - What are the favored condensate shapes?
  - What is the role of the cutoff? (see Laurin's talk on Friday)