Real-time lattice simulations of QCD in a semi-classical approximation

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Outline

The simulation

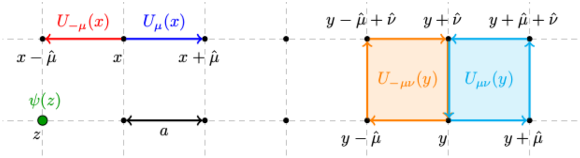
Lattice QCD

QCD Lagrangian with $N_f = 1$ quark flavor, in temporal gauge $A_0 = 0$

$$\mathcal{L}_{QCD} = \bar{\psi}\Big(\mathrm{i}\gamma^{\mu} D_{\mu} - m\Big)\psi - \frac{1}{2}\mathrm{tr}F^{\mu\nu}F_{\mu\nu}$$

Lattice discretization:

We keep real time! $\rightarrow$ Hypercubic lattice $N_x \times N_y \times N_z$ in Minkowski time with spacing a.



Lattice Wilson gauge action

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{tr}F^{\mu\nu}F_{\mu\nu} \quad \rightarrow \quad \mathcal{L}_{YM} = \mathrm{Re}\ \mathrm{tr}\left[E_i E_i - 2\sum_{i<j}\Big(1 - U_{ij}\Big)\right]$$

Chromo-electrical field $\qquad F_{0i}(x) = \partial_t A_i(x) = \frac{1}{ga^2}E_i(x).$

Lattice QCD

Lattice covariant derivative

$$D_i\psi(x) = \frac{1}{2a}\Big(U_i(x)\psi(x+\hat{\imath}) - U_i^\dagger(x-\hat{\imath})\psi(x-\hat{\imath})\Big).$$

„Conventional" discretization in time direction $a_t \ll a$

$$\partial_t\psi(x) = \frac{1}{2a_t}\Big(\psi(x+\hat{t}) - \psi(x-\hat{t})\Big).$$

Fermion doubling problem: Remove spacial doublers by adding the Wilson term

$$\mathcal{L}_W = \frac{r}{2a}\sum_i \bar{\psi}(x)\Big(U_i(x)\psi(x+\hat{\imath}) - 2\psi(x) + U_i^\dagger(x-\hat{\imath})\psi(x-\hat{\imath})\Big).$$

Full lattice Lagrangian:

$$g^2a^4\mathcal{L}_{QCD} = \mathrm{Re\ tr}\left[E_iE_i - 2\sum_{i<j}\left(1-U_{ij}\right)\right] + \frac{ig^2}{2a_t}\left[\bar{\psi}(x)\gamma^0\psi(x+\hat{t}) - \bar{\psi}(x)\gamma^0\psi(x-\hat{t})\right]$$

$$+\frac{ig^2}{2}\sum_{i=1}^{3}\left[\bar{\psi}(x)\left(\gamma^i - ir\right)U_i(x)\psi(x+\hat{\imath}) - \bar{\psi}(x)\left(\gamma^i + ir\right)U_i^\dagger(x-\hat{\imath})\psi(x-\hat{\imath})\right]$$

$$-g^2\left(m+3r\right)\bar{\psi}(x)\psi(x).$$

Color glass initial conditions

Physical system: very early, gluon dominated phase of a heavy ion collision
→ Effective description of densely packed gluons at very high energy densities.

- Color charge densities ρ_1 and ρ_2 of the incoming nuclei → frozen in due to time dilation → form a static current (MV-model[1])

$$J^{\mu a}(x) = \delta^{\mu +} \rho_1^a(x_\perp, x^-) + \delta^{\mu -} \rho_2^a(x_\perp, x^+),$$

  with $(x^\pm, x_\perp)$ lightcone coordinates and the Kronecker deltas $\delta^{\mu\pm}$ reside the current to the lightcone.

- Dynamical gauge fields A^μ → coupled to the static current via the **classical** Yang-Mills equation

$$D_\mu F^{\mu\nu} = J^\nu .$$

⇒ The gauge fields behave classical (at LO)!

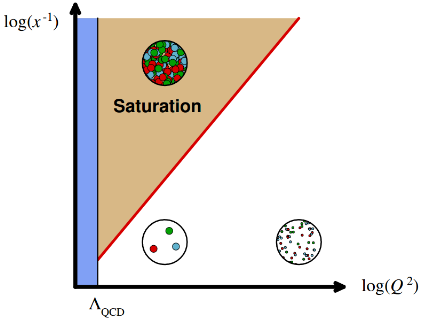
[1] Larry D. McLerran and Raju Venugopalan. "Computing quark and gluon distribution functions for very large nuclei". In: **Phys. Rev. D** 49 (1994), pp. 2233–2241.

- The distribution of color charge ρ_1^a/ρ_2^a is assumed to be Gaussian $\rightarrow$ Gaussian variance $g^4\mu^2$, where μ^2 is the color charge per unit area.
- The quantity μ^2 is obtained from the saturation scale[1] Q_s

$$Q_s \approx g^2\mu.$$

$\Rightarrow$ At a given momentum only a finite amount of gluons can be packed into a nucleon.

- Experiment: $Q_s \approx 2GeV^{\;2}$ for ultra-relativistic Au heavy-ion collisions at LHC.
- Conventional choice $g = 2$ ($\alpha_s \approx 0.3$).[2]



[1] T. Lappi. "Wilson line correlator in the MV model: Relating the glasma to deep inelastic scattering". In: **Eur. Phys. J.** C55 (2008), pp. 285–292.

[2] H. Fujii et al. "Initial energy density and gluon distribution from the Glasma in heavy-ion collisions". In: **Phys. Rev.** C79 (2009), p. 024909

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Stochastic low-cost fermions[2]

Solution of the free Dirac equation with canonical quantized fermions

$$\psi(x) = \int \sum_s \left(\hat{a}_s(\mathbf{p}) u_s(p) e^{-\mathrm{i}px} + \hat{b}_s^\dagger(\mathbf{p}) v_s(p) e^{\mathrm{i}px} \right) \frac{1}{\sqrt{2p_0}} \frac{d^3p}{(2\pi)^{\frac{3}{2}}}$$

- Replace the creation and annihilation operators by complex numbers $\xi, \eta \in \mathbb{C}$.
- Introduce an ensemble of size N_{ens} of "gendered" fermion fields

$$\psi_{M/F}(x) = \int \sum_s \left(\xi_s(\mathbf{p}) u_s(p) e^{-\mathrm{i}px} \pm \eta_s^*(\mathbf{p}) v_s(p) e^{\mathrm{i}px} \right) \frac{1}{\sqrt{2p_0}} \frac{d^3p}{(2\pi)^{\frac{3}{2}}}$$

- Sample the complex numbers according to

$$\langle \xi_r(\mathbf{p}) \xi_s^*(\mathbf{k}) \rangle_{N_{ens}} = (2\pi)^3 \delta(\mathbf{p} - \mathbf{k}) \delta_{rs}.$$

- The statistical propagator is given via an ensemble average

$$F(x,y) = \frac{1}{2} \langle [\psi(x), \bar{\psi}(y)] \rangle = \langle \psi_{M/F}(x) \bar{\psi}_{F/M}(y) \rangle_{N_{ens}}.$$

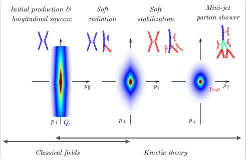
[2] S. Borsanyi and M. Hindmarsh. "Low-cost fermions in classical field simulations". In: **Phys. Rev. D79** (2009), p. 065010.

Running the simulation

1. Initialization[3]

   ⇒ Initialization of gauge links U_i and the chromo-electric field E_i via CGC.

   ⇒ Initialization of fermion fields as vacuum fields, using the low-cost method.



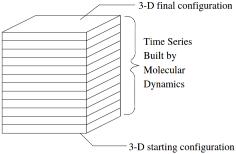
2. Time evolution of the fields via the EoM [4]
3. Measurement of observables and output.

⇒ Simulation in a static box and expanding box.

⇒ Main research questions

   ⇒ Pressure isotropization of longitudinal vs. transversal pressure $P_L/P_T \to 1$?

   ⇒ Validity of the approximation?



[4] Guy D. Moore. "Real time simulations in lattice gauge theory". In: **Nucl. Phys. B Proc. Suppl.** 83 (2000). Ed. by M. Campostrini et al., pp. 131–135

[3] Soeren Schlichting and Derek Teaney. "The First fm/c of Heavy-Ion Collisions". In: **Ann. Rev. Nucl. Part. Sci.** 69 (2019), pp. 447–476.

Simulating in a static box

Initialization and equation of motions

Initialization:

- Gauge links U_i and chromo-electric fields E_i from the MV-model, solving

$$D_\mu F^{\mu\nu} = J^\nu, \qquad J^{\nu a}(x) = \delta^{\nu +}\rho_1^a(x_\perp, x^-) + \delta^{\nu -}\rho_2^a(x_\perp, x^+).$$

- Fermions as stochastic vacuum fermions

$$\psi_{M/F}(x) = \int \sum_s \left(\xi_s(\mathbf{p})u_s(p)e^{-\mathrm{i}px} \pm \eta_s^*(\mathbf{p})v_s(p)e^{\mathrm{i}px}\right)\frac{1}{\sqrt{2p_0}}\frac{d^3p}{(2\pi)^{\frac{3}{2}}}$$

Static box equations of motion:

- Fermion field evolution via the Dirac equation

$$\psi_G(x+\hat{t}) = \psi_G(x-\hat{t}) + 2\mathrm{i}a_t\gamma^0(m+3r)\psi_G(x) - a_t\gamma^0\sum_{i=1}^{3}\left((\gamma^i - \mathrm{i}r)U_i(x)\psi_G(x+\hat{\imath}) - (\gamma^i + \mathrm{i}r)U_i^\dagger(x-\hat{\imath})\psi_G(x-\hat{\imath})\right).$$

- Gauge link evolution via the chromo-electric field

$$U_i(x+\hat{t}) = e^{\mathrm{i}ga_t aE_i(x)}U_i(x).$$

Chromo-electric field equation of motion

Derivation from the partition function

$$Z_{\mathcal{C}} = \int\int \rho(t_0) e^{\mathrm{i}\int_{\mathcal{C}}\int \mathcal{L}_{QCD}[A,\bar{\psi},\psi] dt d^3x} [dA][d\bar{\psi} d\psi],$$

with $\rho(t_0)$ the initial density matrix (initial conditions), $\mathcal{C}$ the time contour (Schwinger-Keldysh contour).

Ansatz: $$A_\mu(x) = \underbrace{\bar{A}_\mu(x)}_{\text{classical field}} + \frac{1}{2}\mathrm{sgn}_{\mathcal{C}} \underbrace{\tilde{A}_\mu(x)}_{\text{quantum correction}} .$$

Integrating out $\tilde{A}$ leads to the EoM:[4]

$$E_i^a(x+\hat{t}) = E_i^a(x) + 2 Z_R a_t \sum_{j\neq i} \mathrm{Imtr}\Big[T^a\Big(U_{ji}(x) + U_{-ji}(x)\Big)\Big]$$
$$+ g^2 a_t \mathrm{Retr}\Big[F(x+\hat{i},x)\big(\gamma^i - \mathrm{i}r\big) T^a U_i(x)\Big].$$

As well as a constraint: (Gauss constraint)

$$0 = g^2 \mathrm{Retr}\Big[F(x+\hat{t},x)\gamma^0 T^a\Big] - \frac{2}{a_t} Z_R \sum_i \mathrm{Imtr}\Big[T^a\Big(U_{i0}(x) + U_{-i0}(x)\Big)\Big].$$

[4] Valentin Kasper et al. "Fermion production from real-time lattice gauge theory in the classical-statistical regime". In: **Phys. Rev. D** 90.2 (2014), p. 025016.

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Static box observables

Derivation from the energy-momentum tensor $T^{\mu\nu}$.

- Energy density

$$\epsilon(x) = T^{00}(x) = Z\mathrm{ReTr}\Big[E_i E_i + 2\sum_{i<j}\left[1 - U_{ij}\right]\Big] - g^2(m+3r)\mathrm{tr}\Big[F(x,x)\Big]$$

$$+ g^2 \frac{\mathrm{i}}{2}\sum_i \Big(\mathrm{tr}\Big[F(x+\hat{\imath},x)(\gamma^i - \mathrm{i}r)U_i(x)\Big] - \mathrm{tr}\Big[F(x-\hat{\imath},x)(\gamma^i + \mathrm{i}r)U_i^\dagger(x-\hat{\imath})\Big]\Big).$$

- Longitudinal pressure

$$P_L(x) = T^{33}(x) = Z\mathrm{ReTr}\Big[E_1^2 + E_2^2 - E_3^2 + 2\left[1 + U_{12} - U_{13} - U_{23}\right]\Big]$$

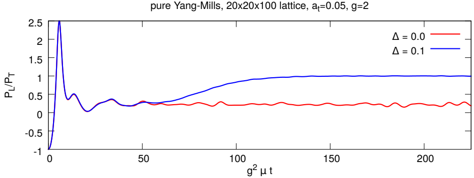
$$+ g^2\frac{\mathrm{i}}{2}\Big(\mathrm{tr}\Big[F(x+\hat{3},x)\gamma^3 U_3(x)\Big] - \mathrm{tr}\Big[F(x-\hat{3},x)\gamma^3 U_3^\dagger(x-\hat{3})\Big]\Big).$$

- Transversal pressure

$$P_T(x) = \frac{1}{2}\sum_{i=1}^{2} T^{ii}(x) = Z\mathrm{ReTr}\Big[E_3^2 + 2(1 - U_{12})\Big]$$

$$+ g^2\frac{\mathrm{i}}{4}\sum_{i=1}^{2}\Big(\mathrm{tr}\Big[F(x+\hat{\imath},x)\gamma^i U_i(x)\Big] - \mathrm{tr}\Big[F(x-\hat{\imath},x)\gamma^i U_i^\dagger(x-\hat{\imath})\Big]\Big).$$

Towards pressure isotropization



No isotropization in a static box in pure classical YM-theory with CGC initial conditions.
Additional quantum fluctuations necessary

$$E_i = g^3\mu^2[f(z-\Delta z) - f(z)]\xi^i, \quad f(z) = \Delta\cos(2\pi z/L_z),$$

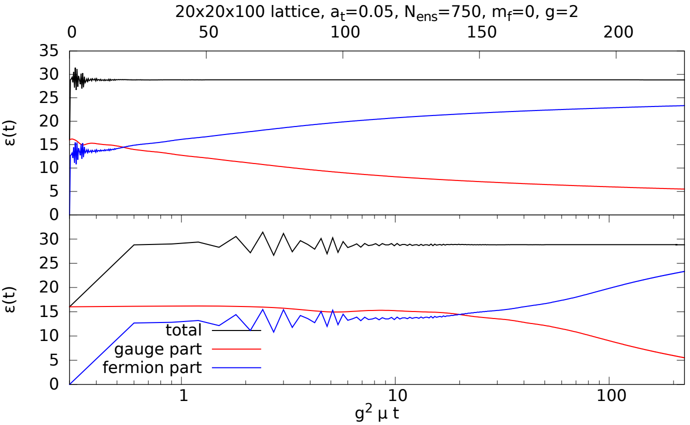
with Gaussian random numbers ξ_i and a parametrization $\Delta = 0.1$.[5]

$\Rightarrow$ Non isotropzation in a static box is rooted by the choice of initial conditions.[6]

[5] Paul Romatschke and Raju Venugopalan. "The Unstable Glasma". In: **Phys. Rev. D** 74 (2006), p. 045011.

[6] K. Fukushima. "Turbulent pattern formation and diffusion in the early-time dynamics in relativistic heavy-ion collisions". In: **Phys. Rev.** C89.2 (2014), p. 024907.

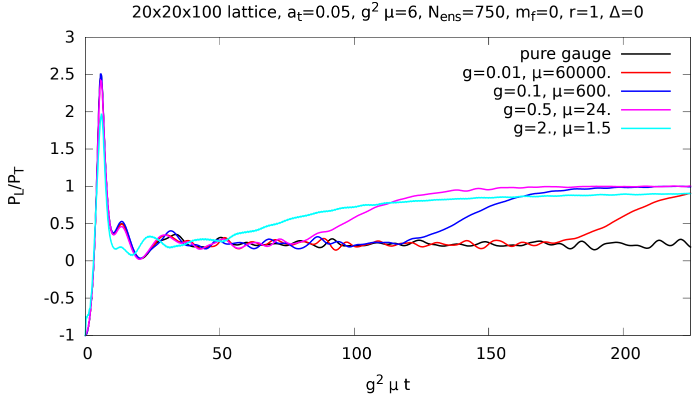
Energy density



⇒ Energy distributed in the gauge and fermion sector.

⇒ Quantum quench at $t \neq t_0$ when coupling both sectors.

Pressure isotropization



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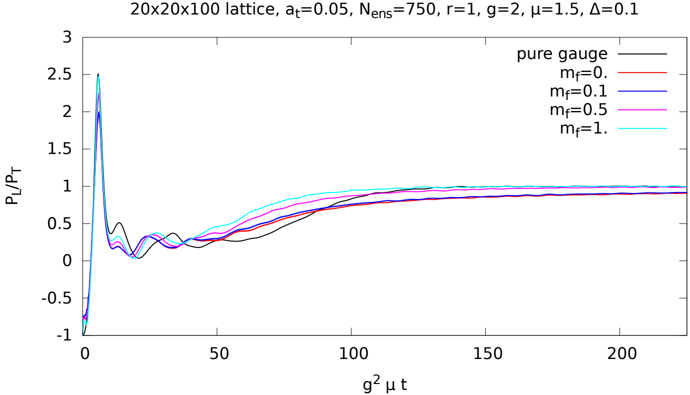
Mass dependence



Simulating in an expanding box

Milne coordinates

Heavy Ion collision as a longitudinally expanding system $\rightarrow$ Bjorken flow[7]

$\Rightarrow$ Best described in Milne coordinates (expansion in z-direction)

$$\tau = \sqrt{t^2 - z^2} \quad \text{(proper time)}, \qquad \eta = \frac{1}{2}\ln\left(\frac{t+z}{t-z}\right) \quad \text{(spacetime rapidity)}.$$

Non-trivial metric

$$g_{\mu\nu} = \mathrm{diag}\left(1, -1, -1, -\tau^2\right), \qquad g^{\mu\nu} = \mathrm{diag}\left(1, -1, -1, -\frac{1}{\tau^2}\right).$$

And Jacobian

$$\int d^4x \quad \rightarrow \quad \int \sqrt{-g}\, d\tau dx_\perp d\eta = \int \tau d\tau dx_\perp d\eta.$$

Especially note

$$\partial_0 = \cosh\eta\ \partial_\tau - \frac{1}{\tau}\sinh\eta\ \partial_\eta, \quad \partial_3 = -\sinh\eta\ \partial_\tau + \frac{1}{\tau}\cosh\eta\ \partial_\eta.$$

[7] J. D. Bjorken. "Highly Relativistic Nucleus-Nucleus Collisions: The Central Rapidity Region". In: **Phys. Rev. D** 27 (1983), pp. 140–151.

Equations of motion and observables

Derivation of the equation of motion and observables in analogy to the static box.

$\Rightarrow$ Longitudinal and transversal direction need some additional care.

Exemplary:

$$U_i(x+\hat{t}) = e^{\mathrm{i}ga_t aE_i(x)}U_i(x) \begin{cases} U_i(x+\hat{\tau}) & = \exp\left(\mathrm{i}\frac{a_\tau}{\tau}E_i(x)\right)U_i(x), \\ U_\eta(x+\hat{\tau}) & = \exp\left(\mathrm{i}a_\tau a_\eta \tau E_\eta(x)\right)U_\eta(x). \end{cases}$$

Energy density Yang-Mills part

$$\epsilon_{YM}(x) = Z_R \mathrm{Re}\ \mathrm{tr}\left[\frac{1}{\tau^2}E_i^2 + E_\eta^2 + 2(1-U_{12}) + \frac{2}{a_\eta^2\tau^2}\sum_{i=1}^{2}(1-U_{i\eta})\right]$$

Exception: Dirac equation and fermions!

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Expanding box fermions

Dirac equation is linear in ∂_0 and ∂_3 $\rightarrow$ significantly changed when introducing Milne coordinates.

One can show

$$i\gamma^0\partial_0 + i\gamma^3\partial_3 = i\gamma^0 e^{-\eta\gamma^0\gamma^3}\partial_\tau + \frac{i}{\tau}\gamma^3 e^{-\eta\gamma^0\gamma^3}\partial_\eta.$$

Defining an expanding box spinor[8]

$$\hat{\psi}(\tau, \mathbf{x}_\perp, \eta) = \sqrt{\tau} e^{-\frac{\eta}{2}\gamma^0\gamma^3}\psi(x),$$

leads to a much simpler Dirac equation in the expanding box

$$\left[i\gamma^0\partial_\tau + \frac{i}{\tau}\gamma^3\partial_\eta + i\gamma^i\partial_i - m\right]\hat{\psi}(\tau, \mathbf{x}_\perp, \eta) = 0.$$

$\Rightarrow$ Evolution equation of the spinors in the simulation.

$\Rightarrow$ Note: Nonlinear partial differential equation of motion (additional $1/\tau$)!

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[8] Francois Gelis and Naoto Tanji. "Quark production in heavy ion collisions: formalism and boost invariant fermionic light-cone mode functions". In: **JHEP** 02 (2016), p. 126.

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Initialization

CGC initial conditions in analogy to the static box.

Non-trivial vacuum spinor solution of the Dirac equation in an expanding box (nonlinear eq.).

Ansatz

$$\hat{\psi}(\tau, \mathbf{x}_\perp, \eta) = \int \sum_{s=1}^{2} \left(\hat{a}_s(\mathbf{k}_\perp, y_k) \hat{\psi}^+_{\mathbf{k}_\perp, y_k, s}(\tau, \mathbf{x}_\perp, \eta) + \hat{b}^\dagger_s(\mathbf{k}_\perp, y_k) \hat{\psi}^-_{\mathbf{k}_\perp, y_k, s}(\tau, \mathbf{x}_\perp, \eta) \right) \frac{d^2 k_\perp}{(2\pi)^2} dy_k,$$

with

$$\hat{\psi}^\pm_{\mathbf{k}_\perp, y_k, s}(\tau, \mathbf{x}_\perp, \eta) = \int \hat{\psi}^\pm_{\mathbf{k}_\perp, \nu, s}(\tau) e^{\mathrm{i}\nu\eta} e^{\pm \mathrm{i}\mathbf{k}_\perp \mathbf{x}_\perp} e^{-\mathrm{i}\nu y_k} d\nu$$

Solving the Dirac equation for the mode functions $\hat{\psi}^\pm_{\mathbf{k}_\perp, y_k, s}(\tau, \mathbf{x}_\perp, \eta)$ leads to

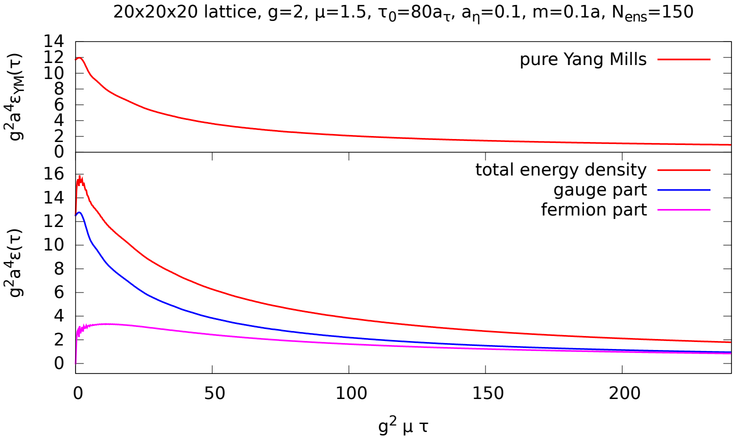
$$\hat{\psi}^+_{\mathbf{k}_\perp, \nu, s}(\tau, \mathbf{x}_\perp, \eta) = -\frac{\mathrm{i}}{2}\sqrt{\frac{\pi\tau}{2}} e^{\mathrm{i}\nu\eta + \mathrm{i}\mathbf{k}_\perp \mathbf{x}_\perp} e^{\frac{\nu\pi}{2}}$$
$$\times \left(e^{-\mathrm{i}\frac{\pi}{4}} H^{(2)}_{\mathrm{i}\nu + \frac{1}{2}}(M_{\mathbf{k}_\perp}\tau) P^+ + e^{\mathrm{i}\frac{\pi}{4}} H^{(2)}_{\mathrm{i}\nu - \frac{1}{2}}(M_{\mathbf{k}_\perp}\tau) P^- \right) u_s(\mathbf{k}_\perp, y_k = 0),$$

$P^\pm = \frac{1}{2}(1 \pm \gamma^0\gamma^3)$, $M_{\mathbf{k}_\perp} = \sqrt{k_\perp^2 + m^2}$ and the Hankel functions.

Finally: replace $\hat{a}, \hat{b}^\dagger$ by complex numbers and use stochastic fermions.

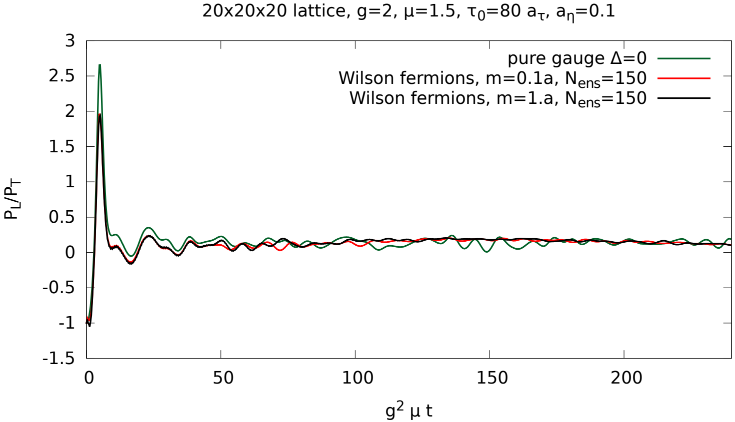
Energy density

Note: Necessity of a finite initial time since $\frac{1}{\tau}$ is singular for $\tau = 0$
$\rightarrow$ stable results for $\tau = 80a_\tau$.
Lattice spacing in longitudinal direction has to be set independently $a_\eta = 0.1$.



Pressure

Pressure ratio P_L/P_T in the expanding box, compared to the pure Yang-Mills simulation

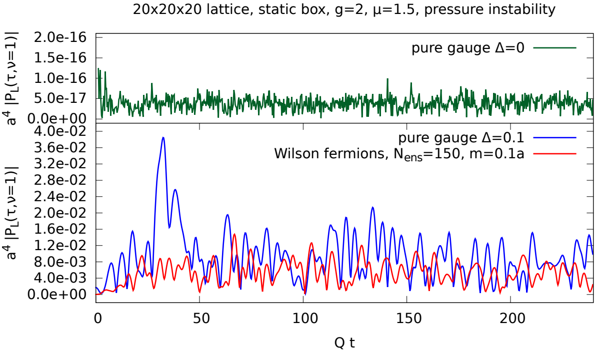


Pressure instability

Necessary to move towards isotropization: a pressure instability

$$P_L^{YM}(\tau, \nu = \nu_0) = \frac{1}{NxNy} \int \frac{1}{N_z} \int P_L^{YM}(\tau, \mathbf{x}_\perp, \eta) e^{i\eta\nu_0} d\eta dx_\perp$$

In a pure gauge simulation vanishes for $\nu_0 \neq 0$, because of the initial conditions.
$\Rightarrow$ Picks up a contribution for $\Delta \neq 0 \rightarrow$ pressure instability driving isotropization.
$\Rightarrow$ How about a semiclassical simulation including fermions?



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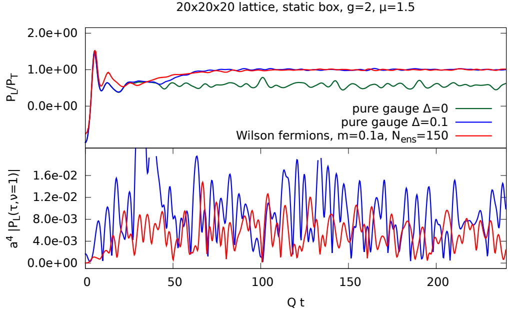
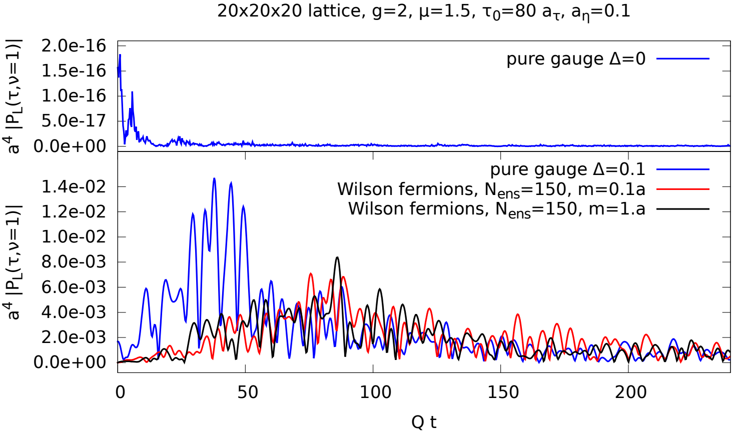


Pressure instability

In an expanding box:



Pressure instability

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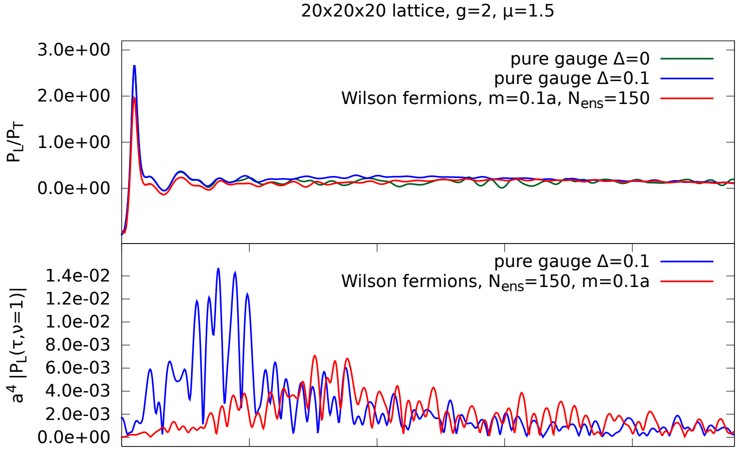

Conclusion

Conclusion

Static box:

- Pressure isotropization due to the coupling to fermions.
- Light fermions seem to slow down the isotropization.

Beware: Continuum limit not possible since the classical theory is UV-divergent (Rayleigh-Jeans divergence).

$\Rightarrow$ Possible Solution: Matching of the energy density fixing a and N_i.

Expanding box:

- No pressure isotropization observed at this point.
- Pressure instability occurs, but is not strong enough to drive the isotropization.

$\Rightarrow$ Scan the parameter space further: a, a_η, g, m dependence?

Perspectives:

- Occupation of modes $\rightarrow$ validity of the approximation.
- Systematic study of NLO quantum effects in the gauge sector.[9]

[9] Thomas Epelbaum and Francois Gelis. "Pressure isotropization in high energy heavy ion collisions". In: **Phys. Rev. Lett.** 111 (2013), p. 232301.

BACKUP

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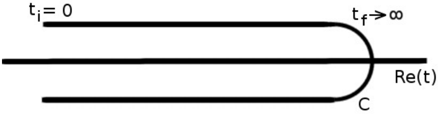
The Keldysh-Contour

Non-equilibrium real time calculations require a certain contour for the time integration referred to as Keldysh-Contour $\mathcal{C}$



⇒ t_i labels the initial time → initial condition.

⇒ Propagators become matrices that can either live on the upper- or lower-branch alone, denoting

$$\Delta^{11}/\Delta^{22},$$

⇒ or connecting branches, e.g. in the form of Wightman functions

$$\Delta^{>}/\Delta^{<}.$$

Derivation of the chromo-electric EoM

$$Z_{\mathcal{C}} = \int\int \rho(t_0) e^{\mathrm{i}S_{QCD}[A,\bar{\psi},\psi]}[dA][d\bar{\psi}d\psi] \to \int\int \rho_A(t_0) e^{\mathrm{i}\mathrm{tr}\log\left[\Delta_{\mathcal{C}}[A]^{-1}\right]+\mathrm{i}S_{YM}[\bar{A},\tilde{A}]}[d\tilde{A}][dA]$$

Taylor expanding with respect to $\tilde{A}$

$$\mathrm{tr}\log\left[\Delta_{\mathcal{C}}[A]^{-1}\right] \approx \mathrm{tr}\log\left[\Delta_{\mathcal{C}}[\bar{A}]\right] + \frac{\mathrm{i}g}{2}\mathrm{tr}\Big(\Delta_{\mathcal{C}}[\bar{A}]\mathrm{sgn}_{\mathcal{C}}\gamma^{\mu}\tilde{A}^a_{\mu}T^a\Big) + \mathcal{O}(\tilde{A}^2)$$

and rewriting (neglecting $\mathcal{O}(\tilde{A}^2)$)

$$S_{YM}[\bar{A},\tilde{A}] = \underbrace{S_0[\tilde{A},\bar{A}]}_{\text{no interactions}} + \underbrace{S_1[\tilde{A},\bar{A}]}_{\sim\mathcal{O}(\tilde{A})} + \underbrace{\cancel{S_2[\bar{A},\tilde{A}]}}_{\sim\mathcal{O}(\tilde{A}^3)}$$

$$\approx \int_{\mathcal{C}^+} \tilde{A}^a_{\nu}\left[\partial_{\mu}F^{\mu\nu,a}[\bar{A}] - gf^{abc}\bar{A}^b_{\mu}F^{\mu\nu,c}[\bar{A}]\right]dx.$$

Integrating out $\tilde{A}$ leads to

$$\partial_{\mu}F^{\mu\nu,a}(x) - gf^{abc}\bar{A}^b_{\mu}(x)F^{\mu\nu,c}(x) = -g\mathrm{tr}\left[F_{\bar{A}}(x,x)\gamma^{\nu}T^a\right]$$

$\Rightarrow$ Lattice discretization, including Wilson term and the renormalization $Z-$factor leads to the result given previously.

Renormalization

- $Z = 1 + \delta Z$-renormalization factor:
  - $\Rightarrow$ Quantum nature of the fermion fields $\rightarrow$ fermion loops allowed and renormalization (in principle) necessary.
  - $\Rightarrow$ Derivation as in Borsanyi, Hindmarsh via fermion loop corrections.
  - $\Rightarrow$ Counterterm for the Yang-Mills sector affects equations of motions and observables

$$\mathcal{L}_C = -\frac{Z-1}{4} F^{\mu\nu,a} F^a_{\mu\nu}.$$

  Observation: In analogy to Zong-Gang Mou et al.[10] only a small deviation δZ occurs.

- Vacuum contribution:
  - $\Rightarrow$ Fermion sector of observables has a negative vacuum contribution due to antiparticles (Dirac sea).
  - $\Rightarrow$ Subtraction of the vacuum contribution of the fermion sector at initial time t_0

$$\mathcal{O}^R(t) = \mathcal{O}(t) - \mathcal{O}(t_0).$$

[10] Zong-Gang Mou et al. "Ensemble fermions for electroweak dynamics and the fermion preheating temperature". In: JHEP 11 (2013), p. 097.

Renormalization Z-factor

Ansatz: $A_\mu = \tilde{A}^\pm_\mu + \bar{A}$ quantum and classical field.
Quantum field on the Keldysh-Contour $\mathcal{C}$ on upper- and lower-branch.

Effective linearized action:

$$S_{eff}[A^\pm_\mu, \psi^\pm, \bar{\psi}^\pm] = \int \mathcal{L}_{free}(\tilde{A}^\pm_\mu, \psi^\pm, \bar{\psi}^\pm) + \left(\tilde{A}^{a+}_\nu - \tilde{A}^{a-}_\nu\right)\left[\partial_\mu F^{\mu\nu,a}[\bar{A}] - gf^{abc}\bar{A}^b_\mu F^{\mu\nu,c}[\bar{A}]\right]$$

$$- g\bar{\psi}^+\gamma^\nu\left(\tilde{A}^{+,a}_\nu + \bar{A}^a_\nu\right)T^a\psi^+ + g\bar{\psi}^-\gamma^\nu\left(\tilde{A}^{-,a}_\nu + \bar{A}^a_\nu\right)T^a\psi^- d^4x.$$

Derivation of a „linearized“ EoM from the condition $\langle \tilde{A}^{d,+}_\alpha(y)\rangle = 0$[11]

$$0 = \partial_\mu F^{\mu\nu,a}[\bar{A}] - gf^{abc}\bar{A}^b_\mu F^{\mu\nu,c}[\bar{A}] - g\,\mathrm{tr}\left[S^{++}(x,x)\gamma^\nu T^a\right]$$

$$+ \mathrm{i}g^2\int \theta(x_0 - z_0)\underbrace{\mathrm{tr}\left[S^<(x,z)\gamma^\mu T^b S^>(z,x)\gamma^\nu T^a - S^>(x,z)\gamma^\mu T^b S^<(z,x)\gamma^\nu T^a\right]}_{\to\Sigma^{\mu\nu,ab}(x,z)}\bar{A}^b_\mu(z)d^4z$$

Z-factor from fermion self energy $\Sigma^{\mu\nu,ab}(x,z)$

$$\delta Z = -\int_0^\infty \frac{t^2}{2}\Sigma_{\mathbf{0}}(t)dt = g^2\int\frac{\mathbf{p}^2}{4E^5_{\mathbf{p}}}\frac{d^3p}{(2\pi)^3}$$

[11] J. Baacke et al. “Initial time singularities in nonequilibrium evolution of condensates and their resolution in the linearized approximation”. In: **Phys. Rev. D** 63 (2001), p. 045023.

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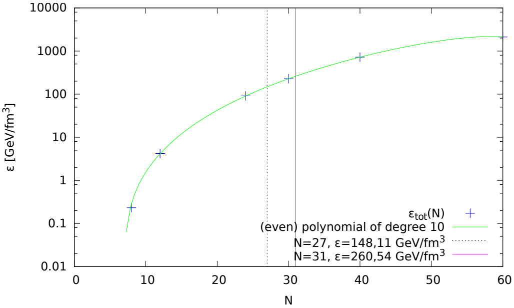
Matching the energy density

Estimate of the initial energy density at LHC ($\tau_0 = 0.1$ fm/c, $\sqrt{s_{NN}} = 2.76$ TeV)[12]

$$\epsilon_0 \approx 270 \frac{\text{GeV}}{\text{fm}^3}$$

Extract the lattice size $V = N \times N \times N$ from matching energy densities



[12] Giuliano Giacalone et al. "Hydrodynamic attractors, initial state energy and particle production in relativistic nuclear collisions". In: **Phys. Rev. Lett.** 123.26 (2019), p. 262301.