Non-perturbative studies of Polyakov-loop effective theories

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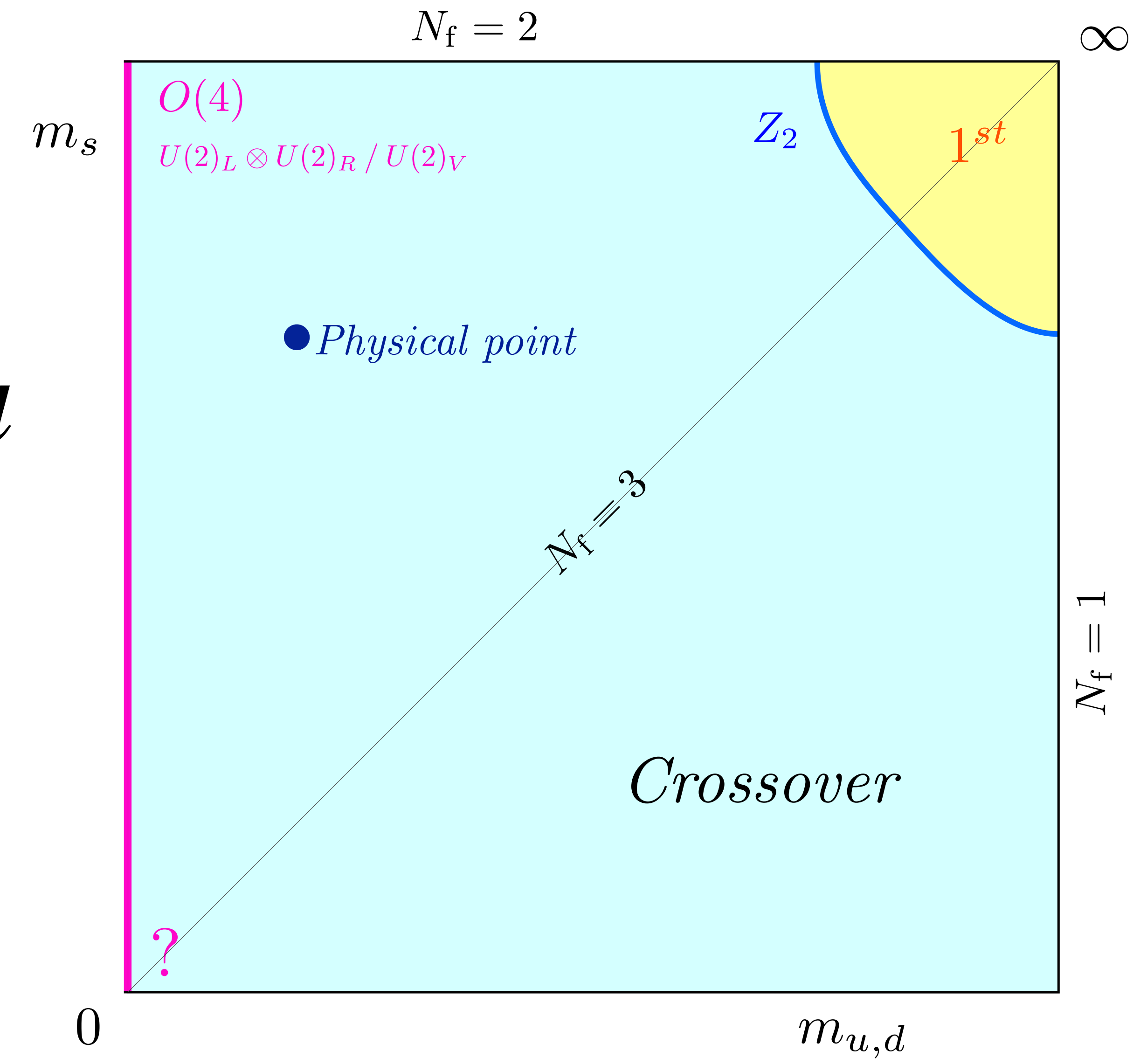
Outline

1. Motivation
2. Polyakov loop effective theories
3. Addition of heavy quarks
4. Mean-field theories
5. Determination of couplings
6. Finite-cluster method
7. Conclusion

Motivation

- Want to understand the phase-diagram of QCD
- The sign problem inhibits progress with direct simulations at nonzero μ
- Thimbles, Langevin, etc



Cuteri et al., JHEP (2021)

Alternatively…

- 3D effective theories derived in strong-coupling
- Milder sign problem at nonzero baryon chemical potential (many d.o.f. integrated out)
- Amenable to both analytical as well as numerical approaches i.e. MFT and series expansions
- BUT: Addition of light quarks hard and large number of effective couplings!

Effective theory for heavy quarks

- Historical roots: Z(N) symmetry + Yang-Mills

Svetitsky and Yaffe, Nucl. Phys. B (1982)

Polonyi and Szachlanyi, PLB (1982)

- Step 1: split the integration of the temporal and spatial link integrations

$$Z_{\rm QCD} = \int \mathcal{D}U_0 \mathcal{D}U_i e^{-S_{\rm QCD}} = \int \mathcal{D}U_0 e^{-S_{\rm eff}[U_0]} = \int dL e^{-S_{\rm eff}[L]}$$

Integration over spatial links after a dual expansion in $\beta = \frac{1}{g^2}$ and $\kappa = \frac{1}{2m+8}$; has finite radius of convergence R!

- Step 2: evaluate effective theory; mild sign problem → MC simulation

- Step 3: analytically evaluate theory using series expansion methods (weak coupling)

Proof of Principle: Z(3) Spin Model

$$S = -\sum_x \left[\tau \sum_k \left(L_x L^*_{x+\hat{i}} + L^*_x L_{x+\hat{i}} \right) + \eta L_x + \bar{\eta} L^*_x \right]$$

$$\eta(\mu) = \bar{\eta}(-\mu) = \kappa e^{\mu}$$

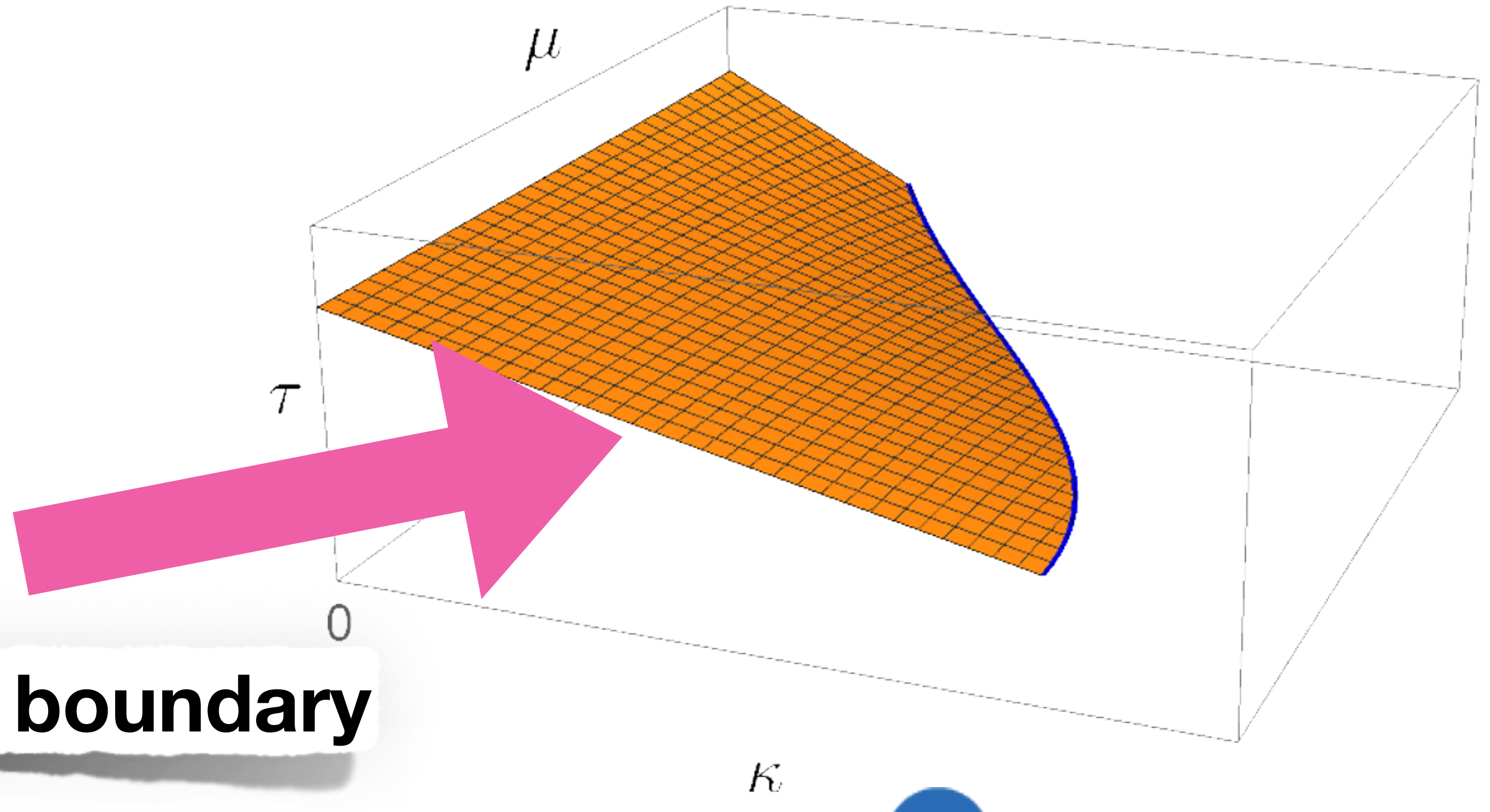
- Studied using variety of methods: flux rep, complex Langevin — Gattringer and Mercado, Nucl. Phys. B (2011); Karsch and Wyld, PRL (1985)
- Linked cluster expansion for perturbative series up to $O(\tau^{14}, \kappa^{30})$ — Kim et al, JHEP (2020)

Free energy

$$f = -\frac{\log Z}{V} = \sum_n a_n(\kappa, \mu)\tau^n$$

"Interaction Measure" → EOS

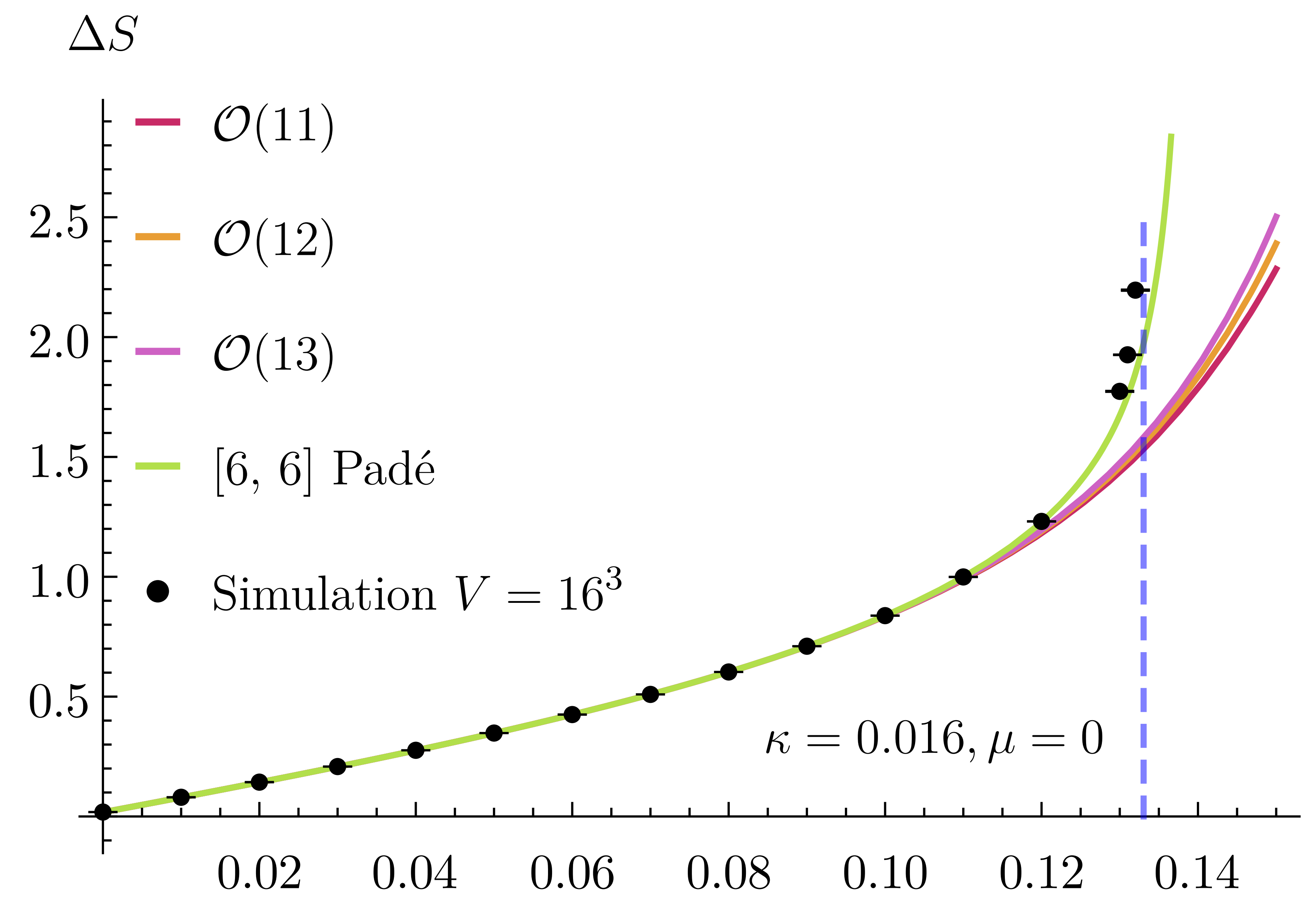
$$\Delta S = -\frac{\partial f}{\partial \tau} - \frac{\partial f}{\partial \eta}$$



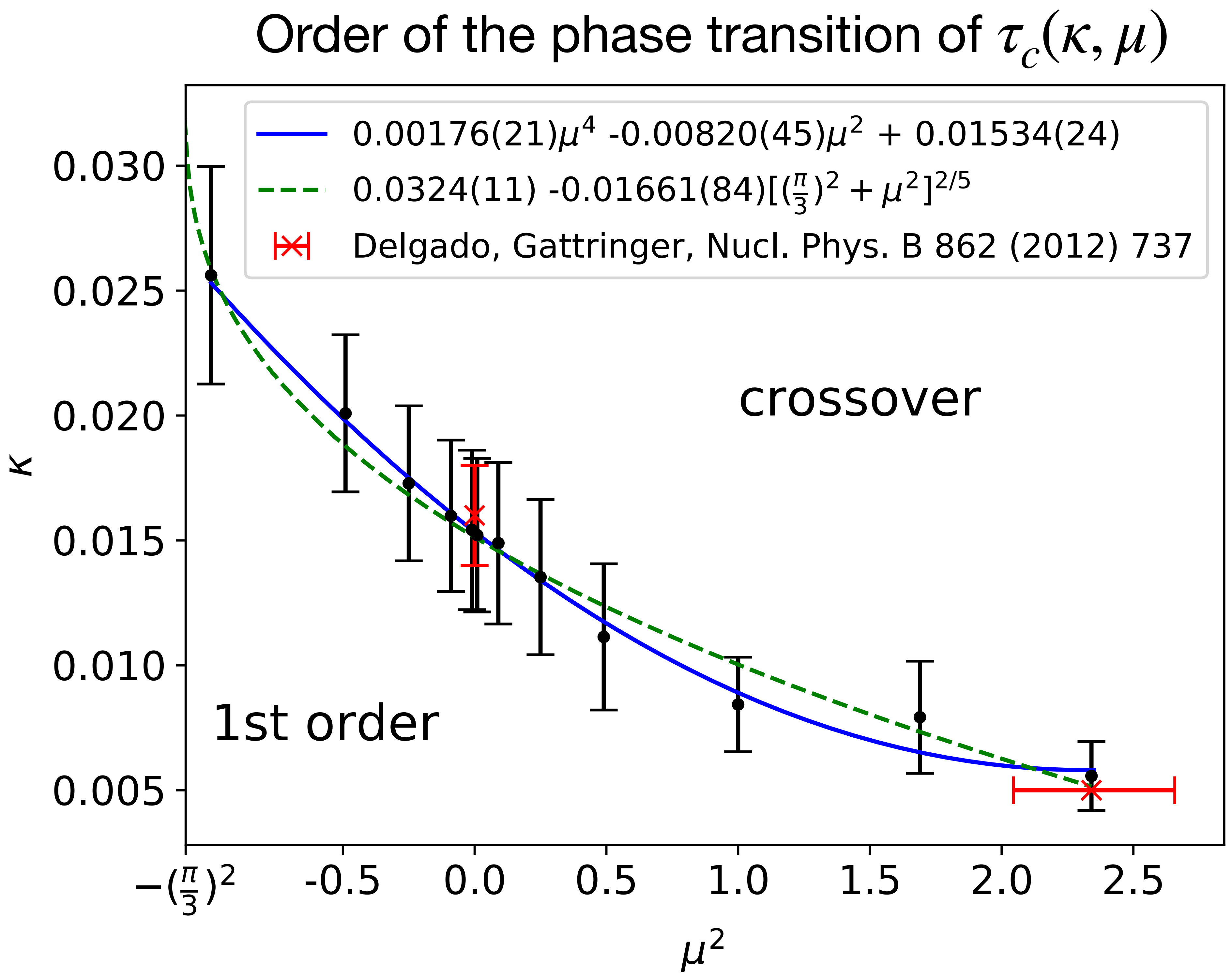
first-order surface terminating in Z(2) boundary

Z(3) Spin Model (continued)

- Results compatible with numerical "solutions"



Pade approximants needed for phase transition



phase transition computed with susceptibilities

Background

- Direct application of strong-coupling expansion for free energy

$$S_W = \frac{\beta}{2}\sum_{\mathrm{p}} \Re\,\mathrm{Tr}\; U_p$$

$$Z = \int [dU][d\psi d\bar{\psi}]e^{-(S_G+S_F)} = \int [dU_0][dU_i]\det Q\; e^{-S_G}$$

$$S_F = \sum_n \left\{ \bar{\psi}(n)\psi(n) - \sum_\mu \left(\bar{\psi}(n)\kappa(1-\gamma_\mu)U_\mu(n)\psi(n+\hat{\mu}) + \bar{\psi}(n)(1+\gamma_\mu)U_\mu(n)\psi(n)\right)\right\}$$

- Resummation of gauge action accomplished by character expansion of gauge action

$$e^{-S_G} = c_0^{N_p}\prod_p \left(1+\sum_{\mathbf{r}} d_{\mathbf{r}} a_{\mathbf{r}}(\beta)\chi_{\mathbf{r}}(U_p)\right)$$

$$c_{\mathbf{r}} = \int dU \chi^*_{\mathbf{r}}(U) e^{-S_G(U)}$$

expansion coefficients

Langelage, Münster, Philipsen, JHEP (2008)

Langelage and Philipsen, JHEP (2010)

- Subtraction of zero-temperature ($N_\tau \to \infty$) graphs
- Accurate determination of deconfinement transition in $SU(2)$ and $SU(3)$ theory

Polyakov Loop Effective Theories

- Originally both spatial and temporal gauge links integrated out

$$Z = c_0^{N_p} \sum_G \Phi(G)$$ **where** $$\Phi(G) = \int [dU] \prod_{p\in G} d_{\mathbf{r}_p} a_{\mathbf{r}_p} \chi_{\mathbf{r}_p}(U) = \prod_i \Phi(X_i)$$

disjoint "polymers"

- Apply moment-cumulant formalism when computing the free energy: $f = -\frac{1}{V}\log Z$

- Integrate over just spatial links and obtain a dimensionally-reduced effective theory solely in terms of Polyakov loops

$$-S_{\text{eff}} = \ln \int [dU_i] \prod_p \left[1 + \sum_{r\neq 0} d_r a_r(\beta)\chi_r(U_p)\right]$$

$$\lambda_1 S_1 + \lambda_2 S_2 + \ldots$$

effective couplings depend on β and N_τ!

Polyakov Loop Effective Theories

- Effective couplings λ_i represent couplings between Polyakov loops in the effective theory

$$S_1 = \lambda_1 \sum_{\langle \mathbf{x},\mathbf{y}\rangle} \left(L_{f,\mathbf{x}} L^*_{f,\mathbf{y}} + \text{c.c.}\right)$$

nearest-neighbors coupling

$$S_2 = \lambda_2 \sum_{[\mathbf{x},\mathbf{y}]} \left(L_{f,\mathbf{x}} L^*_{f,\mathbf{y}} + \text{c.c.}\right)$$

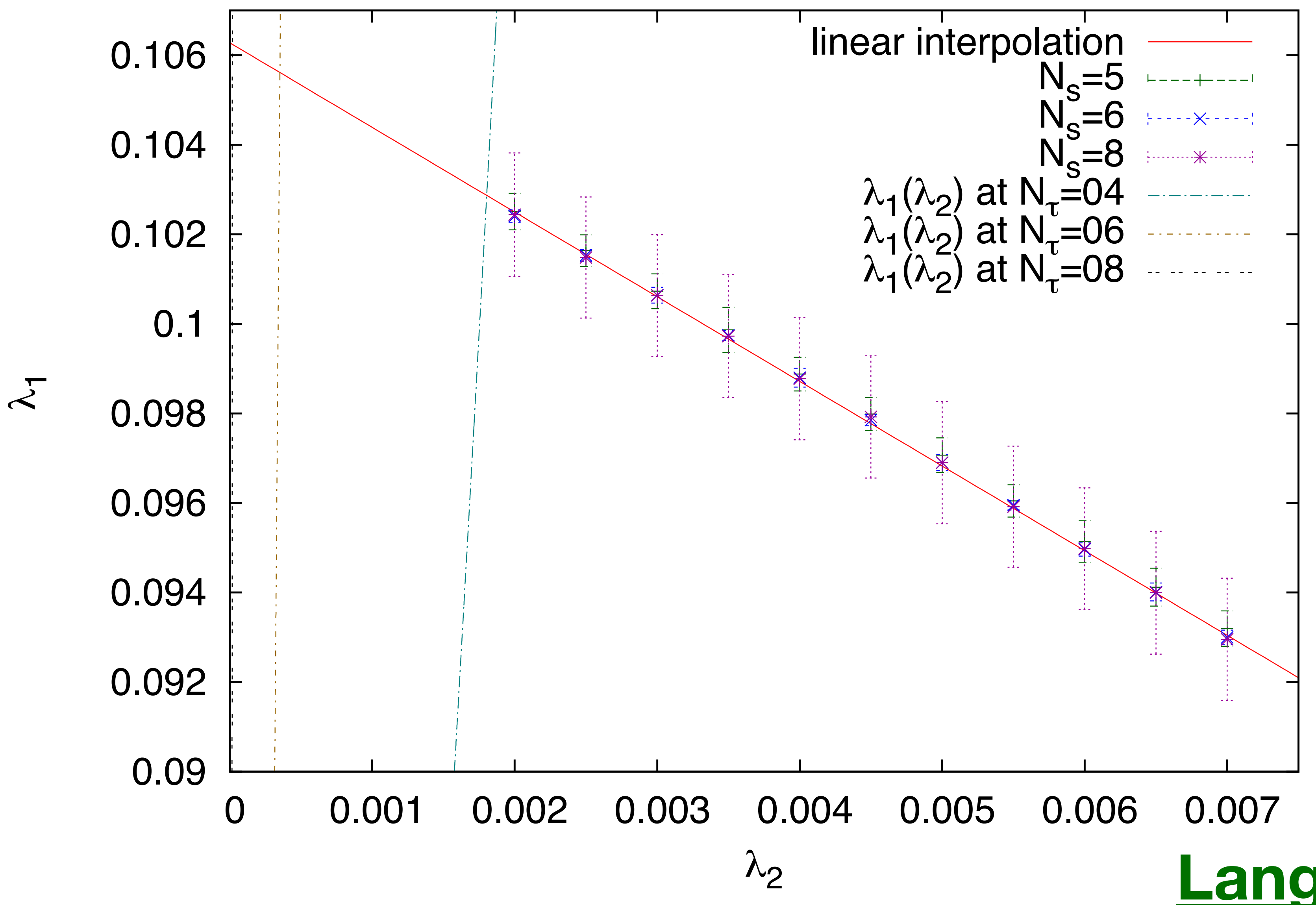
next-nearest-neighbors coupling

resummation!

$$S_1 \to \sum_{\langle \mathbf{x},\mathbf{y}\rangle} \log\left(1 + \lambda_1 \left(L_{f,\mathbf{x}} L^*_{f,\mathbf{y}} + \text{c.c.}\right)\right)$$

log-action

- Simulations of two-coupling model give good agreement with YM simulations



Map back to β_c for fixed N_τ using strong-coupling expressions for couplings

$$\lambda_1 = u^{N_\tau} e^{N_\tau P(u;N_\tau)}$$

$$\lambda_2 = N_\tau \left(N_\tau - 1\right) u^{2N_\tau + 2}$$

- Effects of longer-range interactions and interactions of "higher" representations are suppressed (small β)
- Near β_c, more couplings become important!

Langelage et al., JHEP 2010

Including heavy quarks

- Terms generated by gauge action invariant under global $Z(3)$ center symmetry

$$S_{\text{symm}} = \sum_{\mathbf{x},\mathbf{r}} \sum_{n} \sum_{\{\mathbf{x}_i,\mathbf{r}_i\}}' c^{\mathbf{r}}_{\{\mathbf{x}_i,\mathbf{r}_i\}} \chi_{\mathbf{r}}(W(\mathbf{x})) \prod_i^n \chi_{\mathbf{r}_i}(W(\mathbf{x}+\mathbf{x}_i))$$

terms $Z(3)$ and cubic symmetry

- Introduction of quarks explicitly breaks this symmetry $S_{\text{eff}} = S_{\text{symm}} + S'$

explicit breaking terms

$$S' = \sum_{f=1}^{N_f} \sum_i S'_i[W, W^\dagger]$$

- Hopping parameter expansion of quark determinant

$$\det Q = \det Q_{\text{stat}} \det Q_{\text{kin}}$$

"static" **"kinetic"**

$$\det Q_{\text{kin}} = e^{\text{Tr}\,\log Q_{\text{kin}}}$$

expanded as series in κ^2

- One hopes that low-orders are sufficient

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Mean-Field Theory (General)

Kogut et al., Nucl. Phys. B (1982) Greensite and Splittorf, PRD (2012)

- Applied to Z(3) spin-model and lattice chiral models (T, μ) phase diagram
- Express field by its average plus fluctuations: $L_x \to \bar{L} + \delta L_x$, $L_x^* = \bar{L}^* + \delta L_x^*$
- Expand the action around vanishing fluctuations
- Neglect higher-order non-local fluctuations: $O(\delta L_x \delta L_y)$
- Expectation values factorize as everything is **local**

$$\langle L_x L_y \rangle_{\rm mf} = \langle L_x \rangle_{\rm mf} \langle L_y \rangle_{\rm mf}$$

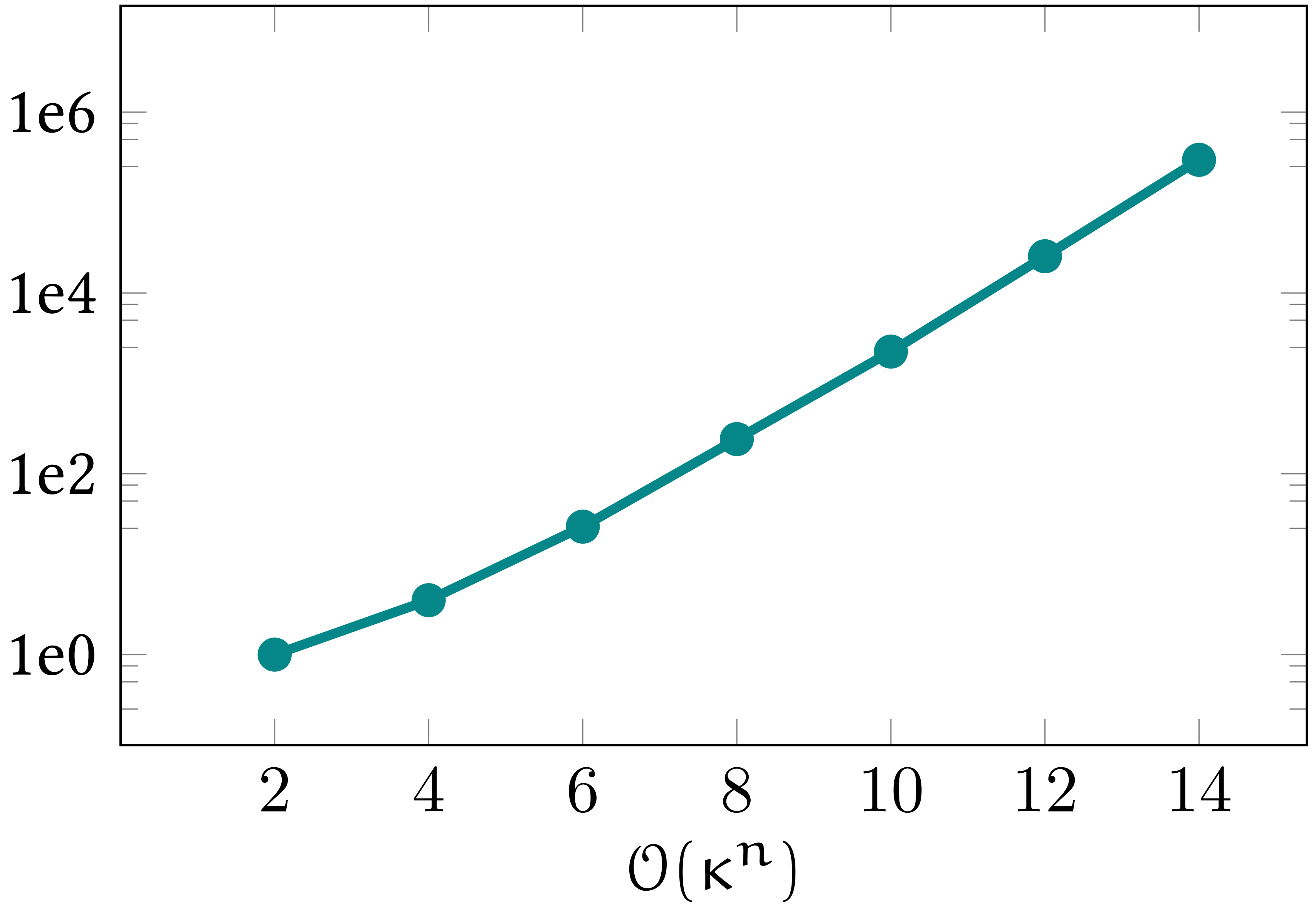
- Modification for log-action: each power of the field receives its own mean -field
- Resummation: keep all local fluctuations: $\delta L_x^n \delta L_x^{*m}$

$$-S_{\rm eff} = 6 \sum_x \log\left[1 + \lambda_1 \left(L_x \bar{L}^* + L_x^* \bar{L}\right)\right] + \dots$$

- Why would this work? Mild sign problem, early studies had success with deconfinement transition

Mean-Field Theory in PET

- Exponentially increasing number of terms describing interactions between Polyakov-loops
- Distance between interaction terms grows with increasing order in κ^2



Glessan, 2016

$$\mathrm{Det} D = \prod_{l_0} \prod_{\{C_{l_0}\}} \det_{c,d} \left(\mathbb{1} - \kappa^{l_0} M_{C_{l_0}} \right)$$

determinant expressed in terms of closed loops

product of hops (spatial/temporal) from Dirac matrix

$$M_{C_{l_0}} = H_{x_1,x_2} \ldots H_{x_{l_0},x_1}$$

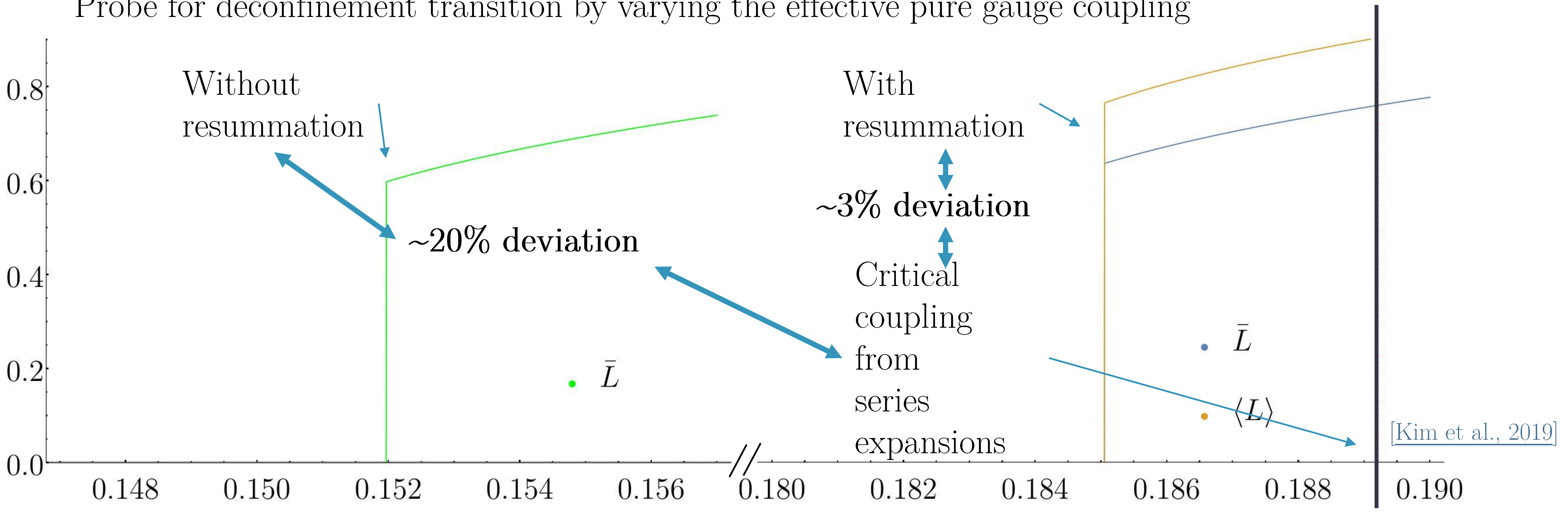
- Effort to compute all these terms will overtake MC evaluation of full determinant
- One is forced to truncate hopping parameter expansion at desired order

- HOWEVER: coordination number effectively increases as corrections are included
- Mean-field theory exact at infinite coordination number

Mean-Field Theory (Results) [Konrad, 2022]

- Pure gauge mean-field results compared with high-order series expansion

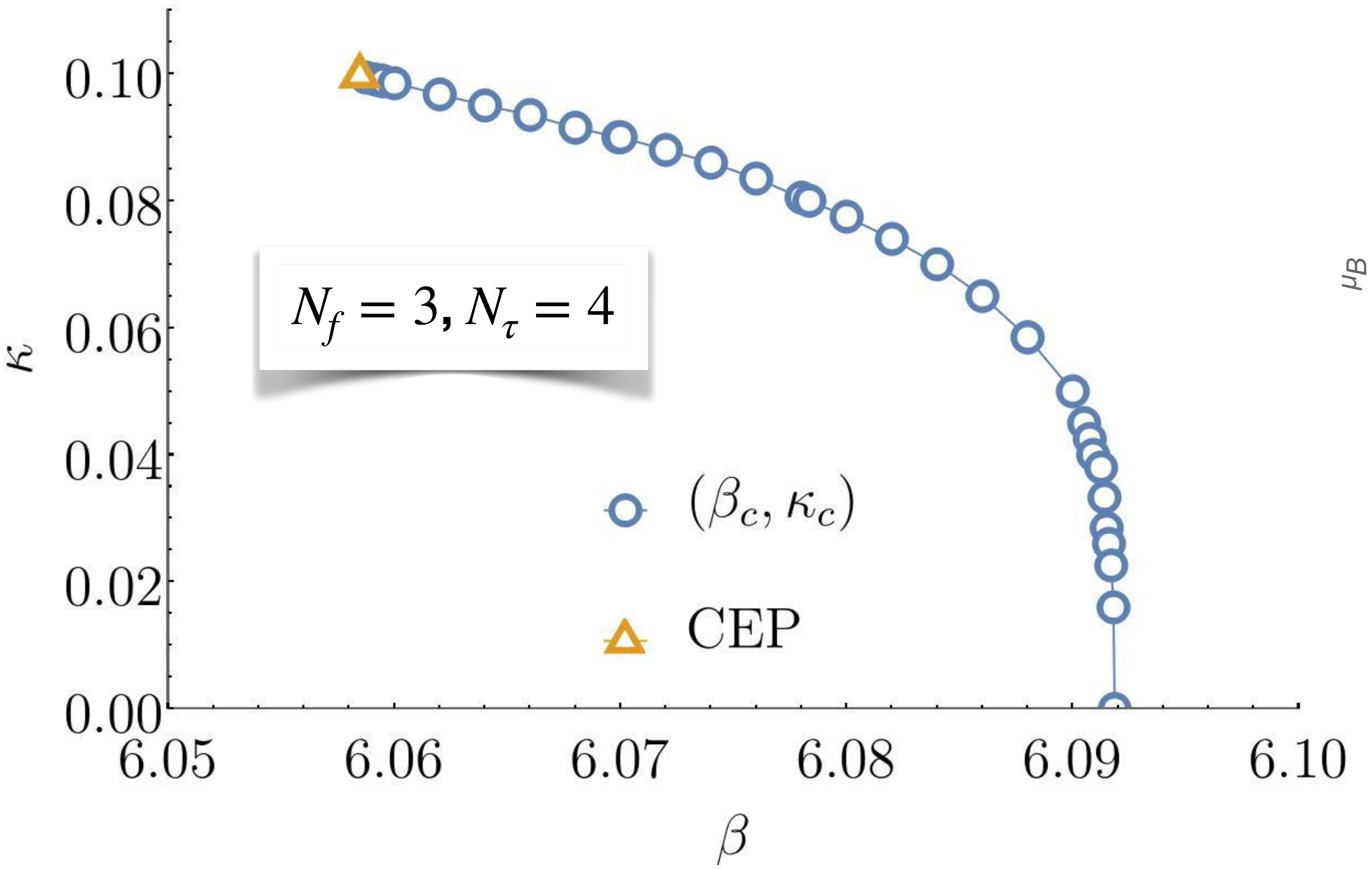
Probe for deconfinement transition by varying the effective pure gauge coupling

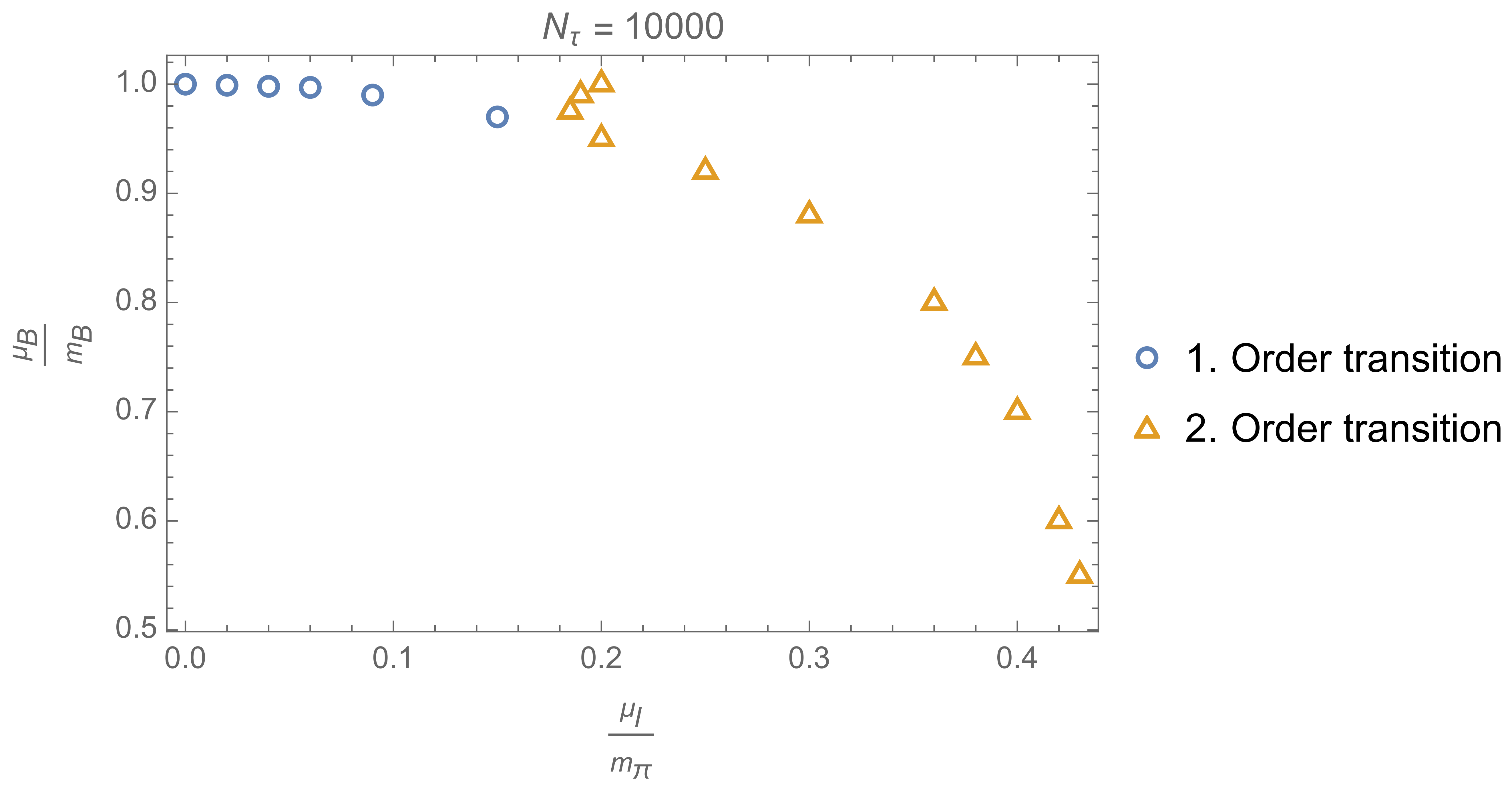


- Resummation works and gives good agreement (~3%)
- Self-consistency vs. variational approach

Mean-Field Theory (Results) Konrad, 2022 Chabane, 2022

- Include terms from the hopping interaction to $O(\kappa^4)$
- Extend to nonzero baryon μ_B and isospin chemical potential: $\mu_I = \mu_u = -\mu_d$





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Determining couplings non-perturbatively

- Effective action can be expanded and powers of characters at given site can be reexpressed
- as linear combination of characters

in practice, number of terms truncated

$$e^{-S_{\text{symm}}} = \tilde{\mathcal{N}}\left(1 + \sum_{\mathbf{x},\mathbf{r}} \sum_{n} \sum_{\{\mathbf{x}_i,\mathbf{r}_i\}}' \tilde{\lambda}^{\mathbf{r}}_{\{\mathbf{x}_i,\mathbf{r}_i\}} \chi_{\mathbf{r}}(W(\mathbf{x})) \prod_i^n \chi_{\mathbf{r}_i}(W(\mathbf{x}+\mathbf{x}_i))\right)$$

no correlation at distances larger than largest separation of terms in effective action

- Better representation which includes long-range correlations is log action

$$e^{-S_{\text{symm}}} = \mathcal{N}_0 \prod_{\mathbf{x},\mathbf{r},n} \prod_{\{\mathbf{r}_i,\mathbf{x}_i\}}' \left[1 + \lambda^{\mathbf{r}}_{\{\mathbf{x}_i,\mathbf{r}_i\}} \left(\chi_{\mathbf{r}}(W(\mathbf{x})) \prod_i^n \chi_{\mathbf{r}_i}(W(\mathbf{x}+\mathbf{x}_i)) + \text{c.c.}\right)\right]$$

Bergner et al., JHEP 2015

- Observable calculated in full effective theory (no truncation) should match full QCD

$$\tilde{\lambda}^{\mathbf{r}}_{\{\mathbf{x}_i,\mathbf{r}_i\}} \propto \langle \chi_{\mathbf{r}}(W(\mathbf{x})) \prod_i^n \chi_{\mathbf{r}_i}(W(\mathbf{x}+\mathbf{x}_i))\rangle_{\text{eff}} = \langle \chi_{\mathbf{r}}(W(\mathbf{x})) \prod_i^n \chi_{\mathbf{r}_i}(W(\mathbf{x}+\mathbf{x}_i))\rangle_{\text{QCD}}$$

- Express correlators in QCD as a perturbative series in couplings of log-action

"Inverse" Monte Carlo method Wozar et al., PRD (2007)

Determining couplings non-perturbatively

- Expansion guided by knowledge of couplings from strong-coupling
- Expressions for correlators can be inverted to obtain $\lambda_i(\beta,\kappa,N_\tau), h_i(\beta,\kappa,N_\tau)$

$$\lambda_3 \equiv \lambda^f_{(2,0,0),\bar{f}} \propto u^{2N_\tau+6}$$

$$\lambda_4 \equiv \lambda^f_{(1,1,1),\bar{f}} \propto u^{3N_\tau+4}$$

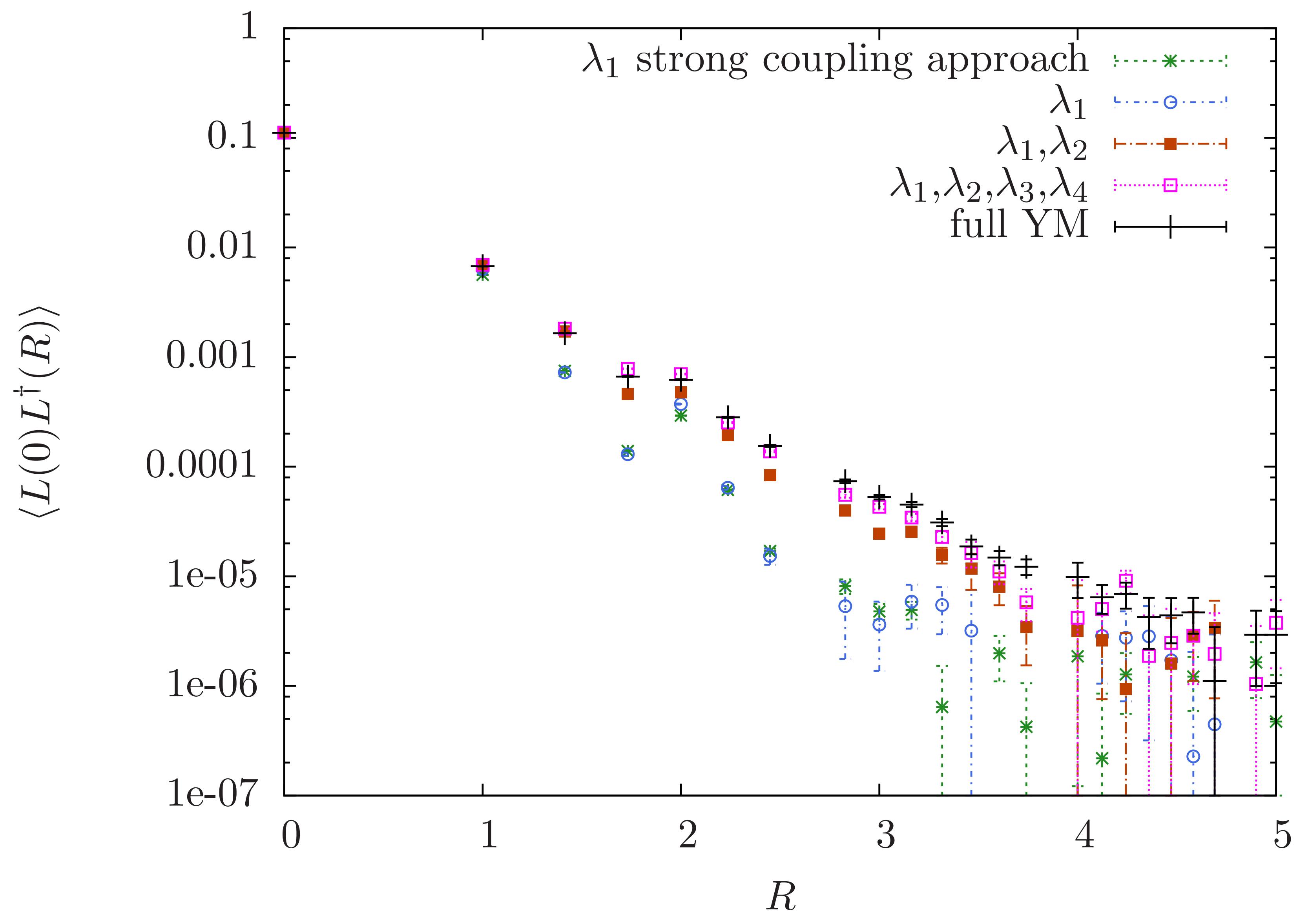
$$\lambda_5 \equiv \lambda^f_{(3,0,0),\bar{f}} \propto u^{3N_\tau+4}$$

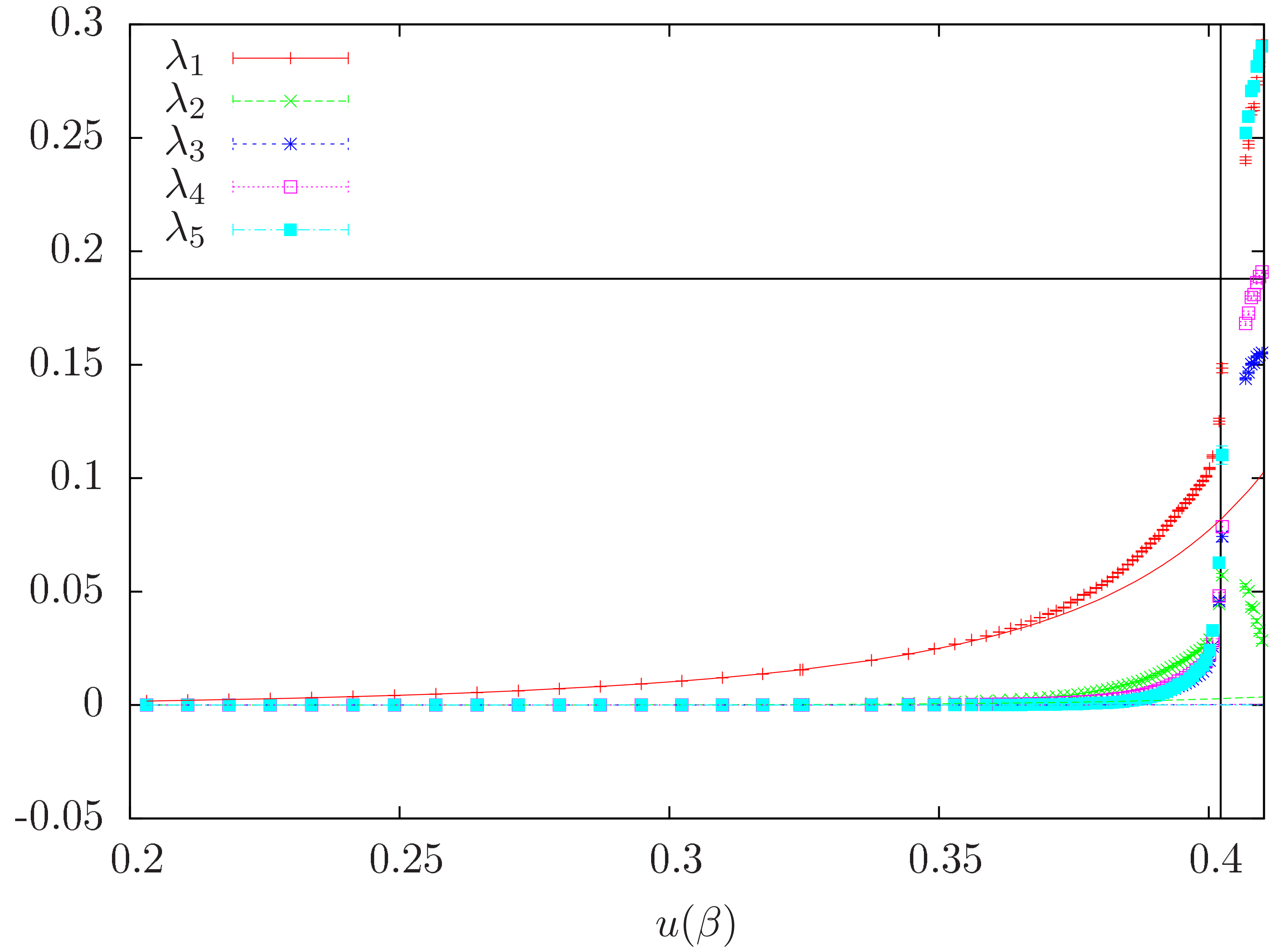
$$\lambda_{\text{adj}} = \lambda^{(1,1)}_{(1,0,0),(1,1)} \propto v^{N_\tau} \propto u^{2N_\tau}$$

$$\lambda_{\text{sextet}} = \lambda^{(2,0)}_{(1,0,0),(0,2)} \propto w^{N_\tau} \propto u^{2N_\tau}$$

$$\tilde{\lambda}_a = \sum_{\{n_i\}} \sum_{\{m_i,\bar{m}_i\}} c^{(a)}_{n_1,\ldots,n_N;m_1,\bar{m}_1,\ldots,m_M,\bar{m}_M} \prod_{i=1}^{N} \lambda_i^{n_i} \prod_{i=1}^{M} h_i^{m_i} \bar{h}_i^{\bar{m}_i}$$

$$\tilde{h}_a = \sum_{\{n_i\}} \sum_{\{m_i,\bar{m}_i\}} d^{(a)}_{n_1,\ldots,n_N;m_1,\bar{m}_1,\ldots,m_M,\bar{m}_M} \prod_{i=1}^{N} \lambda_i^{n_i} \prod_{i=1}^{M} h_i^{m_i} \bar{h}_i^{\bar{m}_i}$$





Bergner et al., JHEP 2015

couplings become large at β_c

Pure YM, $N_\tau = 4{,}6$; good agreement

Evaluation of log-action

- Given a set of couplings $\{\lambda_i, h_i\}$, how to efficiently evaluate Z (and it's derivatives)?

$$\tilde{Z}(G) = \frac{1}{Z_0(G)} \int \prod_{v \in V(G)} dL_v \det Q_{\mathrm{stat},v} \prod_{i=\{\mathrm{NN},\dots 5\mathrm{NN}\}} \prod_{l \in E_i(G)} \prod_{\mathbf{r}(l)} \left[1 + \lambda_{i'} (L_{\mathbf{r}(l),v_1(l)} L_{\bar{\mathbf{r}}(l),v_2(l)} + \mathrm{c.c.})\right] \prod_j \Delta_i^{(j)}(l,\kappa)$$

$$\tilde{Z}(G) = 1 + \sum_{g \in G \backslash \emptyset} \tilde{\phi}(g)$$

cast as sum over subgraphs

- Give a subgraph g, weight can be determined by performing site integrals

$$I_{n,m} = \int dL \; L^n \, (L^*)^m$$

$$\tilde{\phi}(g) = \frac{1}{z_0^{|V(g)|}} \int \prod_{v \in V(g)} dL_v \det Q_{\mathrm{stat},v} \prod_{l \in E(g)} \prod_{\mathrm{r}(l)} \lambda_i(l) \left(L_{\mathbf{r}(l),v_1(l)} L_{\bar{\mathbf{r}}(l),v_2(l)} + \mathrm{c.c.}\right) \prod_j \Delta_i^{(j)}(l,\kappa)$$

terms generated by hopping expansion

$$z_0 = \int dL \det Q_{\mathrm{stat}}[L, L*]$$

- Brute-force evaluation on thermodynamically large system difficult (disconnected graphs)

Finite-cluster method

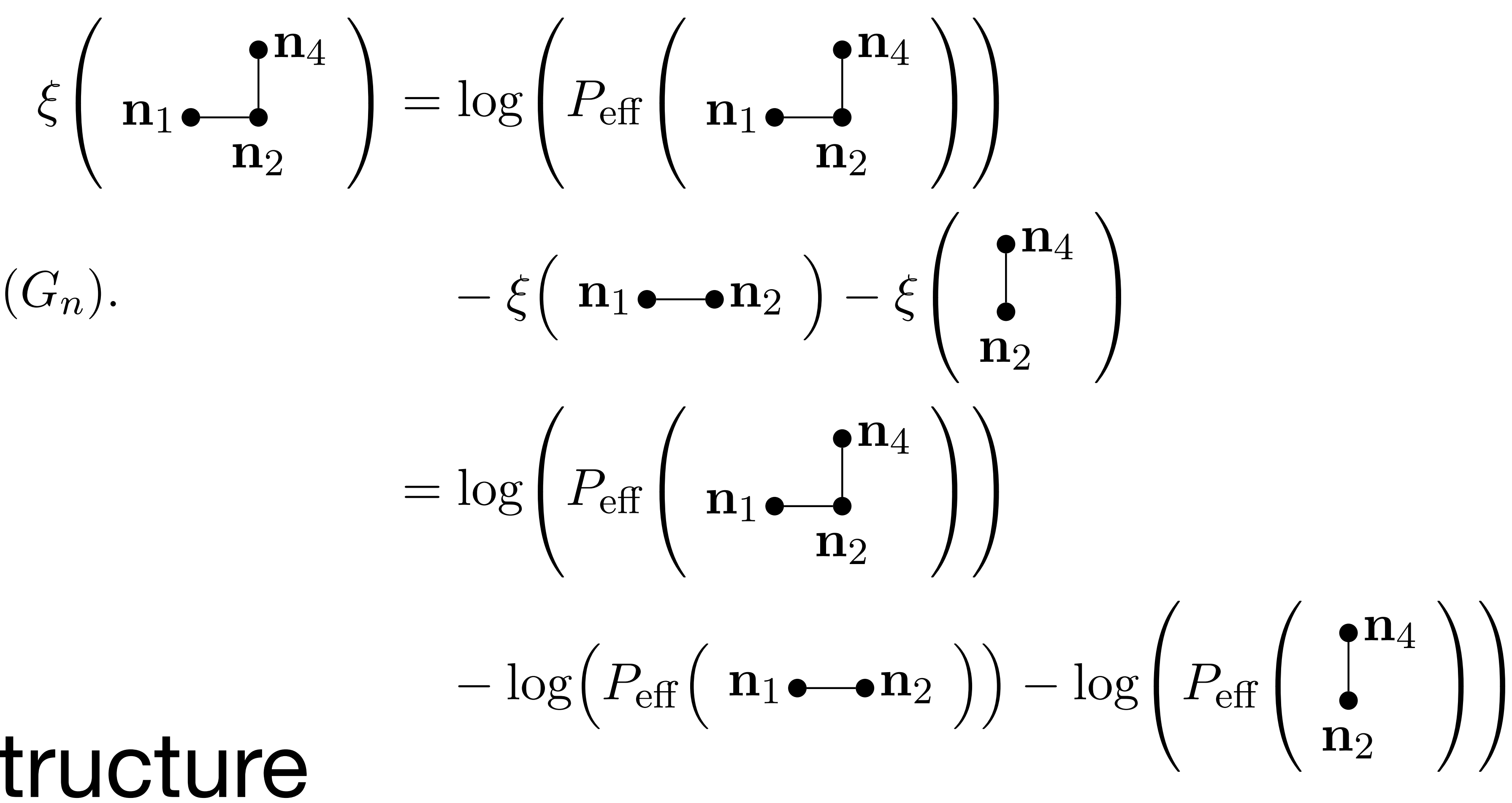
- Each graph weight $\tilde{\phi}(g)$ depends only on the topology of g and NOT the underlying lattice
- Derivation of effective action or evaluation of $\log Z$ could have worked on arbitrary
- embedding graph

$$\xi(G) = \log \tilde{Z}(G) - \sum_{g \in \mathcal{G}_c(G) \setminus G} \xi(g)$$

Scheunert, 2021

$$-S_{\text{eff}} = \log \det Q_{\text{stat}} + \log \left[1 + \sum_{G \in \mathcal{G}(G_{\Lambda_s})} \phi(g) \right]$$

$$P_{\text{eff}}(\mathcal{G}_c(G_{\Lambda_s})) := 1 + \sum_{n=1}^{|\mathcal{G}_c(G_{\Lambda_s})|} \sum_{\{G_1,\ldots,G_n\} \in \mathcal{D}_n(\mathcal{G}_c(G_{\Lambda_s}))} \varphi(G_1) \cdots \varphi(G_n).$$



- Direct evaluation of weights on small clusters
- Avoids embedding of disconnected graphs and preserves log-structure
- Ideal for evaluation of series expansion for correlators in the effective theory

Finite-cluster method

- Direct application to $\log Z$ for effective theory

$$\frac{\log \tilde{Z}}{V} = \sum_{l=1}^{N_{\mathrm{max,MD}}} \sum_{g\in\{\mathcal{G}_c(l)\}} \sum_{p\in\mathcal{P}} \frac{W(G;p)}{S(G)} \xi(G_{\Lambda_s}^{(p)})$$

- Completely generalizable to arb. reps and n-point correlation functions

$$\xi(G) \to \xi(G; J_{\mathbf{r},\mathbf{x}})$$

$$\xi^{(n)} \equiv \frac{\partial^n \xi(G)}{\partial J_{\mathbf{r},\mathbf{x}} \dots \partial J_{\mathbf{r}_{n-1},\mathbf{x}_{n-1}}}\bigg|_{J=0}$$

introduces rooted/colored vertices at canonical positions

$$\langle L_{\mathbf{r}}(\mathbf{x}) \dots L_{\mathbf{r}_{n-1}}(\mathbf{x}_{n-1})\rangle = \sum_{l=1}^{N_{\mathrm{max,MD}}} \sum_{g\in\{\mathcal{G}_{c,n}(l)\}} \sum_{p\in\mathcal{P}} \frac{W^{(n)}(G;p)}{S(G)} \xi^{(n)}(G_{\Lambda_s}^{(p,n)})$$

- Main cost in computing generalized weak embedding numbers of colored graphs
- Publically available software for graph isomorphism (canoncalization) problem: Nauty
- Incorporation of higher-order in κ^2 terms relatively straightforward

modification of $\tilde{\phi}(g)$

Conclusion and Outlook

- Matching between correlators in QCD and PEFT
- Mean-field studies at non-zero μ_B, μ_I
- Can in principle do better then strong-couplings expressions for couplings
- Efficient method for calculating series expression for arbitrary correlators in PEFT
- $N_f = 2$ dynamical Wilson simulations in order to obtain both gauge and fermion couplings
- Perform determination of couplings at imaginary μ and use analytic continuation
- Ultimate goal: exploring chiral region with PEFT

Backup

$O\left(\kappa^2\right)$ **contribution to log-action**

$$\prod_{\langle \mathbf{n},\mathbf{m}\rangle}\left(1+2\frac{\kappa^2 N_\tau}{N_c}(W_{1100}(\mathbf{n})-W_{0011}(\mathbf{n}))(W_{1100}(\mathbf{n})-W_{0011}(\mathbf{n}))\right)$$

$$W^{(f)}_{n_1 m_1 n_2 m_2}(\mathbf{n}) := \operatorname{tr}\left(\frac{\left(h_1^{(f)}W(\mathbf{n})\right)^{m_1}}{\left(1+h_1^{(f)}W(\mathbf{n})\right)^{n_1}}\frac{\left(\bar{h}_1^{(f)}W(\mathbf{n})^\dagger\right)^{m_2}}{\left(1+\bar{h}_1^{(f)}W(\mathbf{n})^\dagger\right)^{n_2}}\right).$$