Gapless Hartree–Fock Resummation Scheme for the $O(N)$ Model

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A modified selfconsistent Hartree–Fock approximation to the $\lambda\phi^4$ theory with spontaneously broken $O(N)$ symmetry is proposed. It preserves all the desirable features, like conservation laws and thermodynamic consistency, of the selfconsistent Dyson scheme generated from a 2PI functional, also known as the Φ-derivable scheme, while simultaneously respecting the Nambu–Goldstone theorem in the chiral-symmetry broken phase. Various approximate resummation schemes are discussed.

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I. INTRODUCTION

The $O(N)$ model with the spontaneously broken symmetry is a traditional touchstone for new theoretical approaches with applications to a variety of physical phenomena, such as the chiral phase transition in nuclear matter [1, 2], the formation of Bose–Einstein condensates in atomic gases [3, 4], or inflationary cosmology scenario [5, 6]. However, even for such a simple model, conceptual problems arise for the dynamic and even thermodynamic treatment of the system. To be precise, the present discussion concerns non-perturbative approaches based on partial resummtion schemes of various kinds: the Φ-derivable approximation of Baym [7] or CJT formalism [8] as it is referred in the field theory, $1/N$ expansions constructed on top of the CJT formalism [9, 10], gapless approximations[33] [11], approximations of mixed type like the variational two-particle-point-irreducible (2PPI) concepts [12, 13] or the Baym–Grinstein approximation [14].

Approaches of Φ-derivable type which lead to a truncated Dyson resummation scheme are preferable for dynamical treatments. They provide selfconsistent Dyson equations of motion which respect conservation laws of Noether currents, such as energy-momentum and charge, at least on the expectation value level [7, 15–18]. Moreover, the Φ-derivable scheme also guarantees the thermodynamic consistency of the approximation [7], which is advantageous for thermodynamic calculations. However, beyond the one-point level such kind of approximations violate Ward–Takahashi identities, which, in particular, leads to a violation of the Nambu–Goldstone (NG) theorem [11, 14, 19] in case of spontaneously broken symmetry. On the other hand, so called "gapless" approximations [11] respect the NG theorem (which is referred to as the Hugenholtz–Pines theorem in the physics of Bose–Einstein condensation), yet violate conservation laws and thermodynamic consistency. The standard remedy is to construct, via functional variation, higher order vertex equations, e.g. the Bethe–Salperter equation, which is to be solved with frozen propagators of the Dyson scheme, cf. [19]. Thus constructed vertex functions respect the corresponding Ward–Takahashi identities and hence the NG theorem. However, they are not a result of a selfconsistent scheme in the sense that all dynamical quantities follow from one set of selfconsistent equations. Rather the higher order vertex functions do not enter any selfconsistent scheme. Thus, the resulting modification e.g. of the propagator suffers from violating conservation laws and thermodynamic consistency. Hence, we return back to the problems addressed.

In this respect, the particular statement in ref. [20] and some subsequent publications that "the NG theorem is always satisfied at any finite N in the Hartree–Fock approximation" of the Φ-functional (or CJT) formalism, is certainly misleading if not incorrect. This statement is based on the identification of the pion and sigma masses with the curvatures of the effective potential at its minimum rather than with the pole positions of the Green functions. What however matters in selfconsistent treatments, both transport and thermodynamic, are the pole masses of Green functions. This fact was realized long ago in the theory of Bose–Einstein condensation [3, 11]. As a consequence, kinetic descriptions of condensed systems were based on the gapless approximations [3, 11] rather than the Φ-functional, accepting the lack of

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conservation laws as a minor defect compared to the else poorly reproduced Goldstone boson.

The above statement can also formulated in formal terms. Strictly speaking, the NG theorem [21] claims that the inverse propagator of the pion vanishes as p^2 as $p^2 \to 0$. If the Green function is calculated as the second variation of the effective action over fields (like in the "gapless scheme"), then the zero curvature of the effective potential at its minimum in the "pion direction" is equivalent to the the NG theorem. However, this is precisely the problem that in the Φ-functional (or CJT) formalism the Green function is associated with stationary point of the effective action with respect to the variation over the Green function itself rather than the fields. In this case, the fact that the curvature of the effective potential at its minimum is zero is irrelevant to the NG theorem. This is why the recovery of the of the "gapless scheme" was the main idea behind the restoration of the NG theorem, reported in Ref. [19]. In fact, similar arguments on the validity of the NG theorem have been already put forward in Ref. [22].

From this point of view, the two-particle-point-irreducible[34] (2PPI) approach [13] still violates the NG theorem. In leading (zero) order in the $1/N$ expansion the $\lambda\phi^4$ model is indeed free of the above-discussed problems. However, with $N = 2$ for Bose–Einstein condensation and $N = 4$ for the chiral phase transition in nuclear matter this zero order scheme is not quite adequate. Already terms of next-to-leading order in $1/N$ in the self energies violate the NG theorem in the above sense, i.e. due to the finite pole mass of the pion, contrary to the statement of ref. [10].

This is clearly seen in the thermodynamic limit of this formalism [22].

First attempts to find a compromise between Φ-derivable and gapless schemes were undertaken by Baym and Grinstein [14]. In fact, their modified Hartree–Fock approximation is a simplified gapless scheme, which still respects the NG theorem. Therefore it was accepted as a main tool for studies of disoriented chiral condensates [23–25]. Baym and Grinstein [14] also found severe problems with the renormalization of partial resummation schemes. Great progress in the proper renormalization of such schemes was recently achieved in refs. [19, 26–29]. However, we will not touch the question of renormalization in this article, since it leads to extra complications which deserve a special discussion.

In this paper we discuss the standard example of a spontaneously broken $O(N)$ model in Φ-derivable Hartree–Fock approximation[35]. The considerations are confined to the thermodynamic equilibrium, since the discussed problems reveal themselves already at that level. Our goal is to construct a modified Hartree–Fock approximation, which is still Φ-derivable, and therefore automatically conserving and thermodynamically consistent, and at the same time respects the NG theorem. The present treatment is based on a naive renormalization, where all divergent terms are simply omitted.

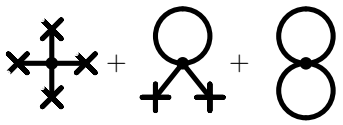
II. CONVENTIONAL HARTREE–FOCK APPROXIMATION

We consider the $O(N)$-model Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_a)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4N}(\phi^2)^2 + H \cdot \phi, \tag{2.1}$$

where $\phi = (\phi_1, \phi_2, ..., \phi_N)$ is an N-component scalar field, $\phi^2 = \phi_a\phi_a$ with summations over a implied. For $H = 0$ this Lagrangian is invariant under $O(N)$ rotations of the fields. If $H = 0$ and $m^2 < 0$, the symmetry of the ground state is spontaneously broken down to $O(N-1)$, with $N-1$ Goldstone bosons (pions). The external field $H \cdot \phi = H_a\phi_a$ is a term which explicitly breaks the $O(N)$ symmetry. It is introduced to give the physical value of 140 MeV to the pion mass.

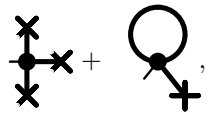
The conventional Hartree–Fock approximation to the $O(N)$ model is defined by the Φ functional[36]

$\Phi_{\rm HF} =$  , (2.2)

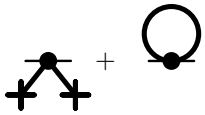
where the crosses denote the classical field ϕ. Within the Φ-derivable scheme the r.h.s. of the equations of motion for the classical field (J) and the Green function (self energy Σ) follow from the functional variation of $\Phi_{\rm HF}$ with respect to the

classical field ϕ and Green function G, respectively

$$\Box\phi + m^2\phi = J = \frac{\delta\Phi_{\rm HF}}{\delta\phi}$$

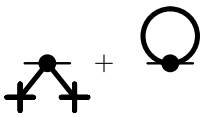


$$G^{-1}_{\rm HF} - D^{-1} = \Sigma_{\rm HF} = 2\frac{\delta\Phi_{\rm HF}}{\delta G} \qquad (2.3)$$

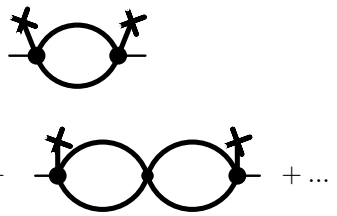


where D is the free propagator. In the spontaneously broken-symmetry phase, i.e. for $H = 0$ and $m^2 < 0$, the solutions of equation set (2.3) violate the NG theorem, as it has been demonstrated in numerous papers (see e.g. Refs. [14, 30, 31]). A detailed analysis of these equations in a notation similar to ours has been given in ref. [30].

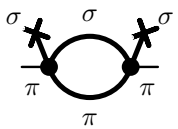
It is possible to remedy the defect of the Hartree–Fock approximation by correcting the self energy by standard random phase approximation (RPA) technique as follows [19, 31]

$\Sigma_{\rm gapless} =$  $+ \Delta\Sigma^{\rm RPA}$, (2.4)

where

$\Delta\Sigma^{\rm RPA} =$  (2.5)

This $\Delta\Sigma^{\rm RPA}$ is constructed in terms of Green functions $G_{\rm HF}$ obtained from solving the selfconsistent Hartree–Fock equations (2.3) and in fact represents the RPA series [31]. Actually, this correction converts the scheme into that of gapless rather than of Φ-derivable type. This results from the loss of the selfconsistency, since the RPA self energies do not enter the Dyson equation (2.3). The most important ingredient of this scheme is the pion self energy

$\Delta\Sigma^{\rm RPA}_\pi =$  (2.6)

$$= \left(\frac{\lambda}{N}\right)^2 \sigma^2 \frac{({\rm i}G_\pi {\rm i}G_\sigma)}{1 - \frac{2\lambda}{N}({\rm i}G_\pi {\rm i}G_\sigma)}$$

which provides the NG theorem. Here

$$({\rm i}G_\pi {\rm i}G_\sigma) \equiv \int_\beta d^4q {\rm i}G_\pi(k+q/2){\rm i}G_\sigma(k-q/2), \qquad (2.7)$$

where the Matsubara integration

$$\int_\beta d^4q f(q) \equiv T\sum_{n=-\infty}^{\infty}\int\frac{d^3q}{(2\pi)^3} f(2\pi {\rm i}nT, \vec{q}), \qquad (2.8)$$

is implied with T being a finite temperature. At $H = 0$, by means of the Hartree–Fock equations of motion (2.3), $\Delta\Sigma^{\rm RPA}_\pi(k=0)$ is reduced to [31]

$$\Delta\Sigma^{\rm RPA}_\pi(k=0) = -\frac{2\lambda}{N}(Q_\pi - Q_\sigma), \qquad (2.9)$$

where

$$Q_a \equiv \int_\beta d^4k G_a(k) \qquad (2.10)$$

is the tadpole diagram of the Green function

$$G_a(k) = \frac{1}{k^2 + M_a^2}, \qquad (2.11)$$

which is the solution of the Hartree–Fock equations of motion (2.3) with M_a being the effective mass of the particle. Upon substitution to (2.4), this $\Delta\Sigma^{\rm RPA}_\pi$ provides $G^{-1}_{\pi({\rm gapless})}(k=0) = 0$ or $M^2_{\pi({\rm gapless})} = M^2_\pi + \Delta\Sigma^{\rm RPA}_\pi(k=0) = 0$ in the broken-symmetry phase. This is precisely the requirement of the NG theorem.

However, the recovery of the NG theorem is achieved here on the expense of loss of selfconsistency (i.e. Φ-derivable nature) of the approximation. It means that the dynamic treatment based on scheme (2.4) will not possess proper conservation laws and thermodynamic consistency is lost. The latter means in particular, that the numbers of particles calculated as a derivative of the thermodynamic potential Ω

$$\frac{1}{T}\Omega\{\phi, G\} = I_0(\phi) + \frac{1}{2}{\rm Tr}\left(\ln G^{-1}\right) - \frac{1}{2}{\rm Tr}\left(\Sigma G\right) + \Phi\{\phi, G\} \qquad (2.12)$$

with $I_0(\phi)$ being the free classical action of ϕ field, over chemical potential[37], on the one side, and as the Green function G integrated over momentum, on the other side, will be different. Moreover, the spectral function of the sigma meson calculated

within this approach has the wrong threshold behavior [19], which however does not affect the validity of the NG theorem. It is not gapless because of the still nonzero pole mass of the pion Green function (2.11). Moreover, the chiral phase transition still proves to be of the first rather than of the second order, as required by the universality class of the $O(N)$ model.

III. GAPLESS HARTREE–FOCK (GHF) APPROXIMATION

For the exact theory, i.e. when all diagrams of the Φ functional are taken into account, both the gapless and Φ-derivable schemes become identical. It is desirable to have them identical also at a certain approximation level such as the Hartree–Fock approximation. If not completely identical, then at least identical in the Goldstone-boson sector, which is of prime importance. The latter means that corrections to the pion self energy (2.6) primarily need to be incorporated in a Φ-derivable way. In fact, we need far less to satisfy the NG theorem, namely only that the pion self energy vanishes at vanishing four-momentum p.

For this purpose one can introduce *a phenomenological symmetry-restoring correction to the Φ functional* of Eq. (2.2)

$$\Delta\Phi = -\frac{(N-1)\lambda}{2N}(Q_\pi - Q_\sigma)^2 \tag{3.1}$$

which precisely produces the required correction term, namely the RPA term $\Delta\Sigma_\pi^{\rm RPA}(k=0)$ to the pion self energy, see Eq. (2.9). This $\Delta\Phi$ can be presented in a manifestly $O(N)$-symmetric form

$$\Delta\Phi = -\frac{\lambda}{2N}\left(NQ_{ab}Q_{ba} - Q_{aa}Q_{bb}\right), \tag{3.2}$$

where summations over a and b are implied and

$$Q_{ab} = \int_\beta d^4k G_{ab}(k) \tag{3.3}$$

is the tadpole diagram of the Green function G_{ab}. Thus, the Φ functional taken as a functional of the classical field ϕ and the Green functions G_{ab} for the gapless Hartree–Fock approximation is defined as

$$\Phi_{\rm gHF}[\phi, G_{ab}] = \Phi_{\rm HF}[\phi, G_{ab}] + \Delta\Phi[G_{ab}] \tag{3.4}$$

In the broken-symmetry phase with non-vanishing classical field $\phi = \{\phi_a\}$ and in an arbitrary, not necessarily diagonal, representation the Green functions, G_{ab} takes the form

$$G_{ab} = \frac{\phi_a\phi_b}{\phi^2}G_\sigma + \left(\delta_{ab} - \frac{\phi_a\phi_b}{\phi^2}\right)G_\pi \tag{3.5}$$

in terms of the pion and sigma propagators, cf. e.g.[13].

The advantage of the $\Phi_{\rm gHF}$ is that the resulting Φ-derivable approximation obeys the NG theorem, at the same time keeping all the pleasant features of the Φ-derivable approach. Moreover, $\Phi_{\rm gHF}$ does not change the Hartree–Fock equation for the classical field ϕ (2.3), as $\Delta\Phi$ does not depend on ϕ, cf. Eq. (2.3). This is a desirable feature, since the classical field equation coincides in both the Φ-derivable and the gapless schemes, and therefore it is reasonable to keep this unchanged also in the gapless Hartree–Fock approximation. Another advantage of $\Delta\Phi$ is that it does not change the results in the $O(N)$-symmetry restored phase, where $M_\sigma^2 = M_\pi^2$ and hence $Q_\pi = Q_\sigma$ and $\Delta\Sigma_a = 0$. Indeed, this phase is described quite reasonably within conventional Hartree–Fock approximation and hence requires no modifications. For the further dicussion of this modified Φ-derivable approximation we use the notion in terms of CJT effective potential, see e.g. [8, 20, 30], in order to comply with numerous previous considerations in the literature.

The manifestly symmetric form of the CJT effective potential in the conventional Hartree–Fock approximation reads

$$\begin{aligned} V_{\rm HF}(\phi, G) \;=\; & \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4N}(\phi^2)^2 - H\cdot\phi + \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\ln\det G^{-1}(k) \\ & + \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left\{\left[\left(k^2+m^2\right)\delta_{ab} + \frac{\lambda}{N}\left(\phi^2\delta_{ab} + 2\phi_a\phi_b\right)\right]G_{ba}(k) - 1\right\} \\ & + \frac{\lambda}{4N}\left(Q_{aa}Q_{bb} + 2Q_{ab}Q_{ba}\right), \end{aligned} \tag{3.6}$$

e.g., cf. [20]. The phenomenological symmetry-restoring correction, corresponding to $\Delta\Phi$ of Eq. (3.1), is

$$\Delta V = -\frac{\lambda}{2N}\left(NQ_{ab}Q_{ba} - Q_{aa}Q_{bb}\right), \qquad (3.7)$$

where

$$G^{-1}_{ab}(k) = k^2\delta_{ab} + M^2_{ab} \qquad (3.8)$$

is the Green function, and Q_{ab} is the corresponding tadpole, cf. (3.3). Here and below, summation over repeated indices $a, b, c, ...$ is implied, if it is not pointed out otherwise. All the quantities are symmetric with respect to permutations of these indices.

The equations for G_{cd}, i.e. for the corresponding tadpoles Q_{ab}, and the fields ϕ_c result from variations of $V_{\rm gHF} = V_{\rm HF} + \Delta V$ over G_{cd} and ϕ_c, respectively,

$$M^2_{cd} = m^2\delta_{cd} + \frac{\lambda}{N}\left[\phi^2\delta_{cd} + 2\phi_c\phi_d + 3Q_{aa}\delta_{cd} + 2(1-N)Q_{cd}\right], \qquad (3.9)$$

$$H_c = m^2\phi_c + \frac{\lambda}{N}\left[\phi^2\phi_c + Q_{aa}\phi_c + 2Q_{cd}\phi_d\right]. \qquad (3.10)$$

These equations are in an arbitrary nondiagonal representation. Introducing projectors on π and σ states

$$\Pi^{\pi}_{cd} = \frac{1}{N-1}\left(\delta_{cd} - \phi_c\phi_d/\phi^2\right), \qquad (3.11)$$

$$\Pi^{\sigma}_{cd} = \phi_c\phi_d/\phi^2, \qquad (3.12)$$

one arrives at

$$M^2_{\pi} =: \Pi^{\pi}_{dc}M^2_{cd} = m^2 + \frac{\lambda}{N}\left[\phi^2 + Q_{aa} + 2\left(\phi_a\phi_b/\phi^2\right)Q_{ba}\right], \qquad (3.13)$$

$$M^2_{\sigma} =: \Pi^{\sigma}_{dc}M^2_{cd} = m^2 + \frac{\lambda}{N}\left[3\phi^2 + 3Q_{aa} + 2(1-N)\left(\phi_a\phi_b/\phi^2\right)Q_{ba}\right] \qquad (3.14)$$

from Eq. (3.9). In order to project the mean-field equation (3.10) on the σ-direction, we just multiply it by ϕ_c

$$\phi^2\left\{m^2 + \frac{\lambda}{N}\left[\phi^2 + Q_{aa} + 2\left(\phi_a\phi_b/\phi^2\right)Q_{ba}\right]\right\} = H_c\phi_c. \qquad (3.15)$$

In the diagonal representation ($\phi_\sigma \neq 0$, $H_\sigma = H$ and $H_\pi = \phi_\pi = 0$) these equations take the following form

$$M^2_{\sigma} = m^2 + \frac{\lambda}{N}\left[3\phi^2 + (5-2N)Q_{\sigma} + 3(N-1)Q_{\pi}\right]$$
$$= M^2_{\pi} + \frac{\lambda}{N}\left[2\phi^2 + 2(N-1)(Q_{\pi} - Q_{\sigma})\right], \qquad (3.16)$$

$$M^2_{\pi} = m^2 + \frac{\lambda}{N}\left[\phi^2 + 3Q_{\sigma} + (N-1)Q_{\pi}\right], \qquad (3.17)$$

$$H = \phi\left[m^2 + \frac{\lambda}{N}\left(\phi^2 + 3Q_{\sigma} + (N-1)Q_{\pi}\right)\right]. \qquad (3.18)$$

Here we used $Q_{\sigma} = Q_{\sigma\sigma}$ and $Q_{\pi} = Q_{\pi\pi}$ in terms of definition (3.3). The tadpoles consist of two parts: $Q_a = Q^T_a + Q^{(\rm div)}_a$, one of them

$$Q^{(\rm div)}_a = \int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2 + M^2_a)^{1/2}}\frac{1}{2}. \qquad (3.19)$$

is divergent. Here we adopt a naive renormalization, which consists in omitting this divergent contribution. Therefore, the tadpoles are defined as follows

$$Q_a \equiv Q^T_a = \int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2 + M^2_a)^{1/2}}\frac{1}{\exp[(k^2 + M^2_a)^{1/2}/T] - 1}. \qquad (3.20)$$

From these equations it is evident that the NG theorem is fulfilled. Indeed, in the broken-symmetry phase ($H = 0$) the square-bracketed term of the field equation (3.18) equals zero, which precisely determines the pion mass, cf. Eq. (3.17).

A. Vacuum ($T = 0$)

At $T = 0$, the quantities under investigation are "experimentally" known[38]: $M_\pi(T = 0) = m_\pi = 139$ MeV, $M_\sigma(T = 0) = m_\sigma = 600$ MeV, and the pion decay constant $\phi_0 = f_\pi = 93$ MeV. These known quantities should satisfy Eqs. (3.16)–(3.18) at $T = 0$, i.e. with $Q_\sigma = Q_\pi = 0$. In order to make this set of equations consistent, we should put

$$H = m_\pi^2 f_\pi, \tag{3.21}$$

i.e. precisely the same as at the tree level. Then equations (3.17) and (3.18) become identical. Resolving Eqs. (3.16)–(3.18) with respect to m^2 and λ in terms of "experimental" quantities m_π, m_σ and f_π, we arrive at

$$\lambda = \frac{N\left(m_\sigma^2 - m_\pi^2\right)}{2f_\pi^2}, \tag{3.22}$$

$$-m^2 = -m_\pi^2 + \frac{\lambda}{N} f_\pi^2. \tag{3.23}$$

B. Symmetry Restoration Point T_1 (at $H = 0$)

If $H = 0$, above some temperature the $O(N)$ symmetry is restored. First, we apply the strong condition

$$M_\sigma^2 = M_\pi^2 = \phi^2 = 0. \tag{3.24}$$

These equations determine the symmetry restoration temperature T_1. In this case all three equations (3.16)–(3.18) reduce to the same single one with the solution

$$T_1^2 = \frac{12}{(N+2)} f_\pi^2. \tag{3.25}$$

Let us consider the region in the vicinity of T_1, assuming $|T - T_1| \ll T_1$, $\phi \ll T_1$ and $M_a \ll T_1$. Here we use brief notation

$$\tilde{m}^2 = -m^2/T^2 > 0, \quad s = \phi/T, \quad \tilde{M}_a = M_a/T,$$

$$R = -\tilde{m}^2 + \frac{\lambda}{N}(N+2)\frac{1}{12} = \frac{\lambda}{N}\frac{1}{12}(N+2)\frac{T^2 - T_1^2}{T^2}.$$

Keeping only linear in $\tilde{M}_a$ terms in Eqs (3.16)–(3.18) accordingly to (A.3), we arrive at the following set of equations

$$0 = \frac{\lambda}{N}\left[2s^2 + \frac{2(N-1)}{4\pi}\left(\tilde{M}_\sigma - \tilde{M}_\pi\right)\right], \tag{3.26}$$

$$0 = R + \frac{\lambda}{N}\left(s^2 - \frac{1}{4\pi}\left[3\tilde{M}_\sigma + (N-1)\tilde{M}_\pi\right]\right), \tag{3.27}$$

$$0 = s\left[R + \frac{\lambda}{N}\left(s^2 - \frac{1}{4\pi}\left[3\tilde{M}_\sigma + (N-1)\tilde{M}_\pi\right]\right)\right]. \tag{3.28}$$

The solution to this set with $s = 0$ is trivial

$$s = 0, \quad \tilde{M}_\sigma = \tilde{M}_\pi = \frac{N}{\lambda}\frac{4\pi}{(N+2)}R = \frac{\pi}{3}\frac{T^2 - T_1^2}{T^2} \tag{3.29}$$

and exists only at $T > T_1$. This is the phase of restored symmetry. If $s \neq 0$, then $\tilde{M}_\pi = 0$. Hence we arrive at the solution

$$s^2 = -\frac{N}{\lambda}\frac{2(N-1)}{2N+1}R = -\frac{2(N-1)}{2N+1}\frac{1}{12}(N+2)\frac{T^2 - T_1^2}{T^2}, \tag{3.30}$$

$$\tilde{M}_\sigma = \frac{N}{\lambda}\frac{8\pi}{2N+1}R = \frac{8\pi}{2N+1}\frac{1}{12}(N+2)\frac{T^2 - T_1^2}{T^2}, \tag{3.31}$$

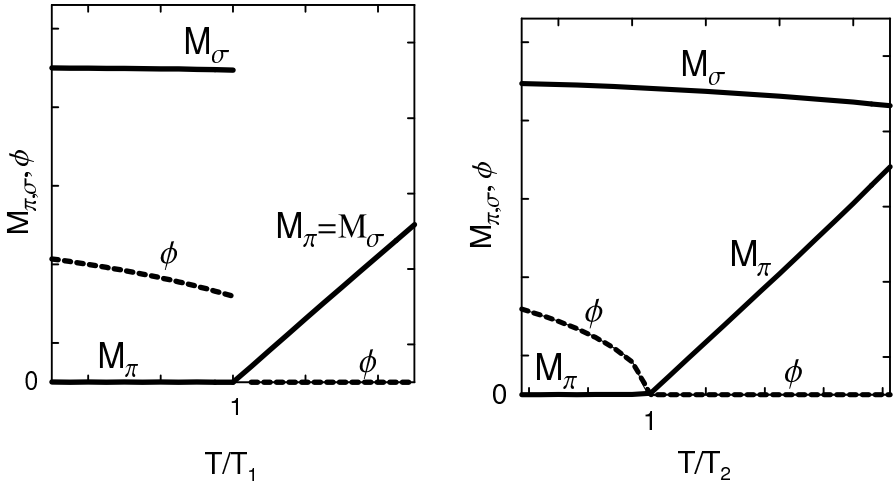


FIG. 1: Schematic behavior of the solution in the vicinity of the T_1 (left panel) and T_2 (right panel) points.

implying that there are no solutions with $s^2 > 0$ and $\tilde{M}_\sigma > 0$ simultaneously. In fact, this solution exists only at $T > T_1$, since $\tilde{M}_\sigma$ should be positive, as required by definition of Q_σ^T. However, at $T > T_1$, the solution is unphysical, since s^2 becomes negative: $s^2 < 0$. Of course, there exists a finite solution with $\phi > 0$ and $M_\sigma > 0$ at $T < T_1$, as it is numerically demonstrated in the next section. This behavior of the solution in the vicinity of $T = T_1$ is schematically demonstrated in Fig. 1. It implies that the phase transition at T_1, if it ever occurs, is of the first order, since the physical quantities reveal jumps at this point.

C. Partial Symmetry Restoration Point T_2 (at $H = 0$)

The transition point T_2 is determined by the weak condition

$$M_\pi^2 = \phi^2 = 0, \quad M_\sigma^2 \neq 0. \tag{3.32}$$

The symmetry is not completely restored at this point, since the meson masses still remain different. A calculation very similar to that above gives

$$T_2^2 = \frac{12 f_\pi^2}{(N+2)} \left(1 + \frac{3}{(N-1)} \frac{M_\sigma^2(T_2)}{m_\sigma^2}\right) \tag{3.33}$$

Strictly speaking, this is not a solution for T_2, since the r.h.s. of (3.33) still depends on T_2 through $M_\sigma^2(T_2)$. However, it numerically occurs that $M_\sigma(T_2) \approx m_\sigma$, which almost removes this dependence, and hence

$$T_2^2 \simeq \frac{12}{(N-1)} f_\pi^2 \tag{3.34}$$

The behavior of the solution in the vicinity of the T_2 is schematically demonstrated in Fig. 1. Such a behavior implies that the phase transition at T_2, if it ever occurs, is of the second order, since quantities reveal no jumps and only their derivatives are discontinues at this point.

IV. RESULTS FOR $N = 4$

The actual structure of the solution to Eqs (3.16)–(3.18) turns out to be even more involved than qualitatively discussed above. This is demonstrated for the case of exact $O(4)$ symmetry, i.e. $H = 0$, in Fig. 2.

First of all, there are two different solutions to the set of Eqs. (3.16)–(3.18). To decide which branch is more stable, we should compare the corresponding values of the CJT effective potential, cf. Eqs (3.6) and (3.7), which in explicit form reads

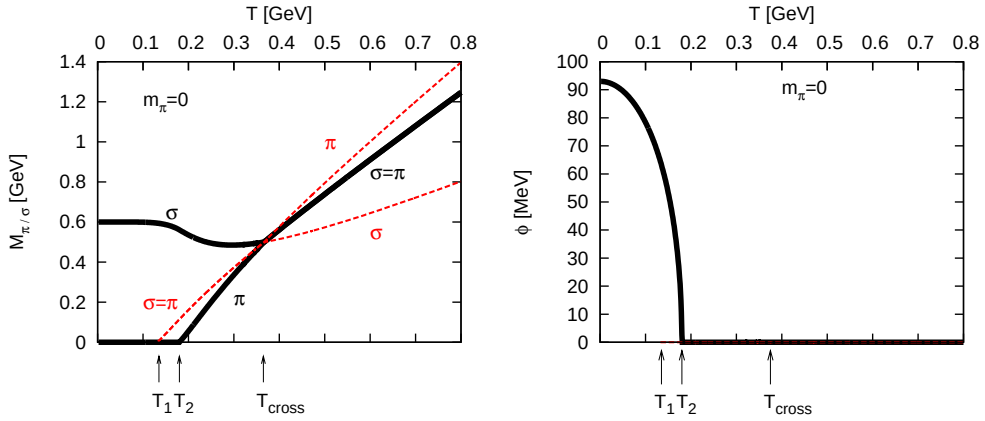


FIG. 2: Meson masses (left panel) and ϕ (right panel) as functions of temperature for the $m_\pi = 0$ case. Stable and less stable branches are presented by solid and dashed lines, respectively.

$$
\begin{aligned}
V_{\rm gHF} &= \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4N}\phi^4 - H\phi + L_\sigma + (N-1)L_\pi \\
&+ \frac{1}{2}\left(m^2 + \frac{3\lambda}{N}\phi^2 - M_\sigma^2\right)Q_\sigma + \frac{1}{2}(N-1)\left(m^2 + \frac{\lambda}{N}\phi^2 - M_\pi^2\right)Q_\pi \\
&+ \frac{\lambda}{4N}\left[(5-2N)Q_\sigma^2 + (N-1)^2Q_\pi^2 + 6(N-1)Q_\sigma Q_\pi\right], \qquad (4.1)
\end{aligned}
$$

where

$$
L_a = \frac{T}{(2\pi)^3}\int d^3k \ln\left(1 - \exp\frac{(k^2 + M_a^2)^{1/2}}{T}\right) \qquad (4.2)
$$

and Q_a are given by Eq. (3.20) with M_a being solutions of equations of motion. Note also that this effective potential is nothing else but the negative pressure

$$
P = -V_{\rm gHF}. \qquad (4.3)
$$

A stable branch should correspond to a minimum of the effective potential with respect to the field variation, i.e. the second derivative $d^2V_{\rm gHF}(\phi, G(\phi))/d\phi^2$ should be positive. As it is seen from Eqs (B.3)–(B.5), all branches with positive masses M_σ and M_π correspond to minima. This is precisely the case for all branches discussed here. This minimum is only very shallow, when $M_\pi = 0$. Therefore, we can only choose between more stable and less stable branches. The most stable branch should have minimal absolute value of $V_{\rm gHF}$, i.e. maximal pressure.

A. Exact $O(4)$ Symmetry

The results for stable and less stable branches in the case of exact $O(4)$ symmetry, i.e. $H = 0$, are displayed in Figs. 2 and 3. In fact, it is impossible to resolve these branches near their crossing point at $T_{\rm cross} \simeq 360$ MeV. However, these branches can be distinctly resolved for the case of the partially violated symmetry, see Figs. 4 and 5, which confirms our present identification.

The stable branch exists at all temperatures. It starts at $T = 0$ from the physical vacuum values of the masses and classical field and crosses the less stable branch at $T_{\rm cross}$. In terms of the pressure, they are touching rather than crossing (see left panel of Fig. 3). Therefore, no transition from one branch to another occurs at $T_{\rm cross}$. In the broken-symmetry phase, the stable branch behaves precisely in accordance with our expectations – the pion mass equals zero. Then a phase transition of the second order occurs at $T_2 \simeq 180$

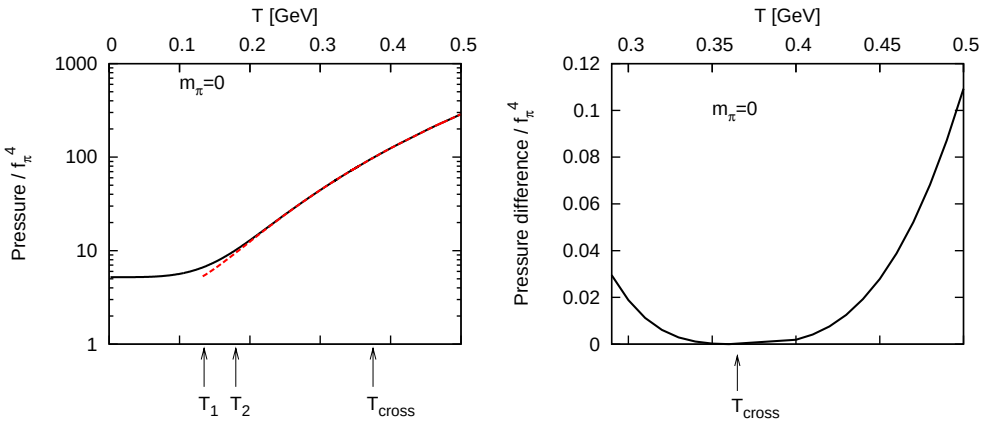


FIG. 3: Pressure (left panel) and pressure difference between stable and less stable branches (right panel) as functions of temperature for $m_\pi = 0$ case. The pressure for the stable and less stable branches are presented by solid and dashed lines, respectively.

MeV, at which the field becomes zero. However, the π and σ masses still differ beyond this transition point. They become equal only after the second phase transition also of second order at $T_{\rm cross}$. Note that the equal-mass solution above $T_{\rm cross}$ is precisely the same as in the conventional Hartree–Fock approximation, since the gapless modification term (3.1) vanishes in this case.

The T_1 point proves to be irrelevant for the stable branch. Rather it is the starting point for the less stable branch, which in the range of $T_1 < T < T_{\rm cross}$ precisely coincides with the solution of the conventional Hartree–Fock approximation. The existence of this branch above $T_{\rm cross}$ is an artifact of the introduced NG-theorem restoring term $\Delta\Phi$. The corresponding field always vanishes for this branch.

Summarizing, in the spontaneously broken-symmetry phase the NG theorem is indeed fulfilled. Thus, we achieved the goal of the proposed modification of the Hartree–Fock approximation, however partially, since the phase transition now proceeds through a sequence of two second-order phase transitions rather than a single one.

B. Approximate $O(4)$ Symmetry

For the explicitly broken symmetry case (finite pion mass) the results for both branches are displayed in Figs. 4 and 5. In this case these two branches are well resolved in the entire range of temperatures, since they do not even touch each other, see Fig. 5 (right panel). As it was expected, the sequence of two phase transitions is transformed here into smooth cross-over transition.

V. CONCLUSION

A modified Φ-derivable Hartree–Fock approximation to the $O(N)$-model is proposed, which simultaneously preserves all the desirable features of Φ-derivable scheme (i.e. conservations and thermodynamic consistency) while respecting the NG theorem in the broken-symmetry phase. This is achieved by adding a correction $\Delta\Phi$ to the conventional Φ functional. With this correction term, the chiral phase transition proceeds through a sequence of two second-order phase transitions rather than a single one. In the first transition the field disappears but the meson masses still remain different. In the second transition also the masses become equal, and the $O(N)$ symmetry is completely restored.

The nature of this correction can be understood as follows. For the full theory, i.e. when all diagrams in Φ functional are taken into account, the gapless and Φ-derivable schemes are identical and both respect the NG theorem. The conventional Φ-derivable Hartree–Fock approximation omits an infinite set of diagrams which are necessary to restore its equivalence with the gapless scheme. The $\Delta\Phi$ correction to the Hartree–Fock approximation takes into account a part of those omitted diagrams (at the level of actual approximation), and thus restores this equivalence in the pion sector.

This $\Delta\Phi$ correction is unambiguously determined proceeding from the following requirements: (i) it restores the NG theorem in the broken-symmetry phase, (ii) it does not change the

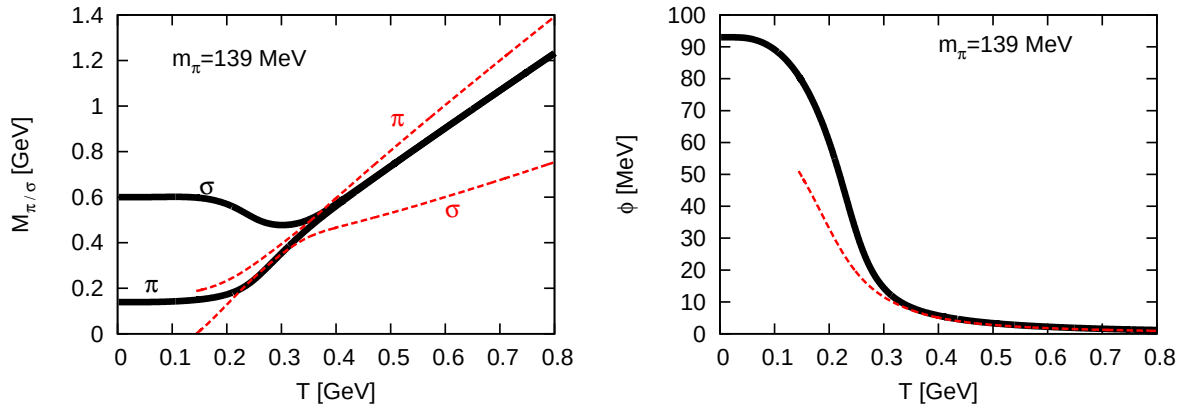


FIG. 4: Meson masses (left panel) and ϕ (right panel) as functions of temperature for $m_\pi = 139$ MeV case (stable and unstable branches presented by solid and dashed lines, respectively).

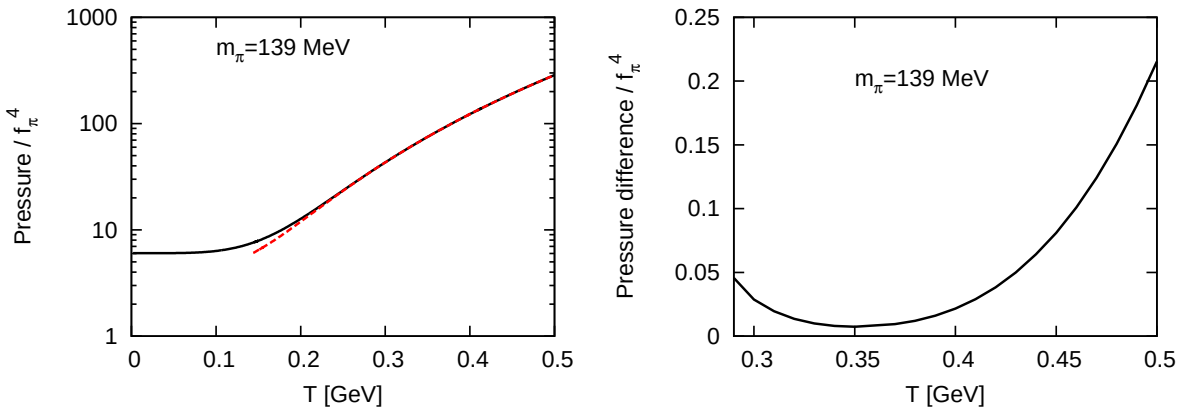


FIG. 5: Pressure (left panel) and pressure difference between stable and less stable branches (right panel) as functions of temperature for $m_\pi = 139$ MeV case (stable and unstable branches presented by solid and dashed lines, respectively).

Hartree–Fock equation for classical field, since the Φ-derivable and gapless schemes provide the same classical-field equation already without any modifications, (iii) it does not change results in the phase of restored $O(N)$ symmetry because there is no need for it. If two last requirements are released, the $\Delta\Phi$ correction become ambiguous: any

$$\Delta\Phi_{\rm alternative} = -\frac{(N-1)\lambda}{2N}\left[(Q_\pi - Q_\sigma)^2 + f\left(\phi^2/(N-1) + Q_\pi - Q_\sigma, \phi^2/(N-1) + Q_\pi + Q_\sigma/(N-1) + const\right)\right], \quad (5.1)$$

where f is an arbitrary function satisfying $\partial f(x,y)/\partial x = 0$ at $x = 0$, is as good as (3.1). The arguments of the f function are taken to be $O(N)$ invariants. For the first argument it was demonstrated in Eqs. (3.1) and (3.2), whereas for the second argument it follows from the identity $Q_{aa} = Q_\sigma + (N-1)Q_\pi$. Note that our actual choice (3.1) is as close to the RPA result [19, 31] as possible.

The treatment of this paper was based on a naive renormalization scheme, when all divergent parts of diagrams were simply omitted. Presently, consistent renormalization schemes of Φ-derivable approximations are available [19, 26–29]. Our first experience of dealing with them indicates that they give rise to new (yet not always desirable) features of Φ-derivable approximations to theories with spontaneously broken symmetry. These new features deserve special studies. To avoid mixing of effects of the NG-theorem restoring term $\Delta\Phi$ with those of the renormalization, we defer a discussion of the latter to a subsequent paper.

The introduced gapless Hartree–Fock approximation differs from approximations available at present. It differs from the $1/N$ expansion, since it keeps all the terms in N, from the Baym–Grinstein approximation [14], as it still is of the Φ-derivable nature, from gapless approximations [19, 31], etc. Of course, at the Hartree–Fock level there is no "collisional" dissipation in the scheme (the Landau damping may exist [32]). Nevertheless, the "collisional" dissipation can be introduced phenomenologically at the quasiparticle level, i.e. by adding a collision term to the Vlasov kinetic equation resulted from this modified Hartree–Fock approximation. Such an approach looks preferable, e.g., for the DCC simulations [25] as compared to that based on the Baym–Grinstein approximation. The "collisional" dissipation naturally appears in higher-order Φ-derivable approximations, e.g. through the sunset self-energy diagrams. The corresponding modifications of these higher-order approximations in the way described here are much more involved and will be reported elsewhere.

Acknowledgements

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Appendices

APPENDIX A: TADPOLE

In this appendix we present the limiting cases of the tad-pole integral Q^T introduced in Eq. (3.20):

$$Q^T(M,T) = \frac{T^2}{2\pi^2}\int_{M/T}^{\infty}\frac{[\epsilon^2-(M/T)^2]^{1/2}}{e^\epsilon-1}d\epsilon = \frac{1}{2\pi^2}M^2\sum_{n=1}^{\infty}\frac{T}{nM}K_1\left(\frac{nM}{T}\right), \tag{A.1}$$

where $\epsilon = (k^2+M^2)^{1/2}/T$. For $T \ll M$

$$Q^T(M, T\to 0) = (2\pi)^{-3/2}T^{3/2}M^{1/2}\exp(-M/T), \tag{A.2}$$

while for $M \ll T$

$$Q^T(M\to 0, T) = T^2\left[\frac{1}{12} - \frac{1}{4\pi}\frac{M}{T} - \frac{1}{8\pi^2}\left(\frac{M}{T}\right)^2\ln\frac{M}{T}\right]. \tag{A.3}$$

APPENDIX B: SECOND DERIVATIVE OF CJT EFFECTIVE POTENTIAL

The equations of motion (3.16)–(3.18) in fact are conditions which determine the extremum of the CJT effective potential. However, stable solutions correspond only to minima of this potential. To distinguish between minimum and maximum of the potential, the sign of the second derivative of the potential over ϕ, under the condition that the Green functions solve the gap equations with this ϕ, should be determined. The first derivative leads to

$$\frac{dV_{\rm gHF}(\phi, G(\phi))}{d\phi} = \frac{\partial V_{\rm gHF}(\phi,G)}{\partial\phi} + \frac{\delta V_{\rm gHF}(\phi,G)}{\delta G}\frac{dG}{d\phi} = \phi M_\pi^2 - H, \tag{B.1}$$

since

$$\frac{\delta V_{\rm gHF}(\phi, G)}{\delta G} = 0 \tag{B.2}$$

due to equations of motion for Green functions. To obtain Eq. (B.1) we have used the equations of motion (3.17) and (3.18). Note that $dV_{\rm gHF}(\phi, G(\phi))/d\phi = 0$, if ϕ is a solution to Eq. (3.18). Taking the next derivative one arrives at

$$\frac{d^2 V_{\rm gHF}(\phi, G(\phi))}{d\phi^2} = M_\pi^2 + \phi \frac{dM_\pi^2}{d\phi}. \tag{B.3}$$

In order to find $dM_\pi^2/d\phi$ we can use the equations of motion (3.16) and (3.17). Taking derivatives over ϕ in Eqs (3.16) and (3.17) and then resolving them with respect to $dM_\pi^2/d\phi$, we arrive at

$$\frac{dM_\pi^2}{d\phi} = \frac{\frac{\lambda}{N} 2\phi M_\pi \left[M_\sigma + \frac{\lambda}{N} 2(N+2) R_\sigma\right]}{M_\sigma \left[M_\pi - \frac{\lambda}{N}(N-1) R_\pi\right] + \frac{\lambda}{N} R_\sigma \left[(2N-5) M_\pi - \frac{\lambda}{N} 2(N-1)(N+2) R_\pi\right]} \tag{B.4}$$

with

$$R_a = \frac{1}{2} \frac{\partial Q^T(M_a, T)}{\partial M_a} \tag{B.5}$$

To avoid uncertainties in the calculation of $\partial Q^T(M, T)/\partial M$ at $M \ll T$, it is reasonable to directly use Eq. (A.3).

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[33] This notion reflects the fact that the massless Goldstone boson has a gapless spectrum.
[34] graphs that fall apart, if two line meeting at the same point, are cut; in the Hatree–Fock approximation it is identical to the standard 2PI condition.
[35] This approximation is also frequently referred as the Hartree approximation, e.g. see [30]. In fact, exchange terms, i.e. the Fock terms, are included into this approximation, therefore the term "Hartree–Fock" is more precise. For contact interaction the distinction between direct and exchange terms is not so obvious. It becomes visible, if one replaces the contact interaction by a heavy-boson exchange interaction. In the $1/N$ counting scheme the Fock terms prove to be of subleading order [31].
[36] All the considerations below are performed in terms of the thermodynamic Φ functional which differs from the real-time Φ in the factor of $\mathrm{i}\beta$, where $\beta = 1/T$ is the inverse temperature. In the case of a spatially homogeneous thermodynamic system, an additional factor appears: the volume V of the system. Thus, our $\Phi = (-\mathrm{i}T/V)\Phi_{\text{real-time}}$. This Φ is similar to the V_2 part of the effective potential in the CJT formalism [8] with the exception that, contrary to V_2, by definition Φ includes all 2PI interaction terms, i.e. also those of zero and first loop order which result from interactions with the classical field (first two graphs in (2.2)).
[37] of course, provided we introduce a chemical potential
[38] These values are relevant to the case $N = 4$.