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NUCLEI
Theory

Self-Consistent Approach to Off-Shell Transport*

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Abstract—The properties of two forms of the gradient expanded Kadanoff–Baym equations, i.e., the Kadanoff–Baym and Botermans–Malfliet forms, suitable for describing the transport dynamics of particles and resonances with broad spectral widths, are discussed in context of conservation laws, the definition of a kinetic entropy, and the possibility of numerical realization. Recent results on exact conservations of charge and energy–momentum within Kadanoff–Baym form of quantum kinetics based on local coupling schemes are extended to two cases relevant in many applications. These concern the interaction via a finite-range potential and, relevant in nuclear and hadron physics, e.g., for the pion–nucleon interaction, the case of derivative coupling. © *2003 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

Ever since L. Boltzmann suggested his famous kinetic equation, the field of non-equilibrium physics and stochastic processes has grown tremendously, expanding in various directions. The interactions among particles driven by mean fields were included, quasiparticles were introduced in order to include much of the medium effects, the kinematics was extended to the relativistic case, and ultimately theoretical foundations of the transport equation were given from an underlying quantum many-body or field theory. In this line of achievements also stands the work of Budker and Belyaev, who demonstrated the Lorentz invariance of the relativistic distribution function and derived relativistic Fokker–Planck kinetic equation [1]. The work entered into many textbooks and found numerous applications in atomic physics and electron–positron plasma. Presently, the relativistic transport concepts are a conventional tool to analyze the dynamics of dense and highly excited matter produced in relativistic heavy-ion collisions.

Along with semiphenomenological extensions, great progress was achieved in microscopic foundation of the kinetic theory, which is mainly associated with the names of Bogolyubov, Born, Green, Kirkwood, Yvon, and Zubarev. The appropriate framework for describing non-equilibrium processes within the real-time formalism of quantum-field theory was developed by Schwinger, Kadanoff, Baym, and Keldysh [2–4]. This formalism allows extensions of the kinetic picture beyond conventional approximations (like the quasiparticle one) and has found now numerous applications in many domains of physics.

The interest in transport descriptions of heavy-ion collisions beyond the quasiparticle approximation was initiated by Danielewicz [5], using the gradient expanded Kadanoff–Baym (KB) equations. These attempts have recently been revived [6–13] in order to properly describe the transport properties of broad resonances (like the ρ meson and Δ isobar). In a dense environment, a stable particle also acquires a considerable width because of collisional broadening. A proper dynamical treatment of their widths in a dense nuclear medium within transport theoretical concepts is still a challenging problem. Transport approaches for treating such off-shell dynamics were proposed in [7–13]. They all were based on the KB equations [3, 14], which describe the nonequilibrium quantum evolution at the truncation level of the Schwinger–Dyson equation. Expanded up to the first spacetime gradients, the KB equations provide transport equations for the one-body phase-space distribution functions with a collision term and Poisson bracket terms arising from the first-order gradient terms. Presently, two slightly different forms of the gradient-expanded KB equations are used: the original KB form [3], as follows right after the gradient expansion without any further approximations, and the Botermans–Malfliet (BM) one [15], which is derived from the KB form by omitting certain second-order spacetime gradient corrections.

In this paper, we would like to compare these two forms of "quantum" kinetic equations and discuss

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their advantages and disadvantages from the point of view of their conserving properties, the possibility of numerical realization, etc. (Section 2). Technical details on the conserving properties are deferred to Appendices B–D, since they illustrate some of the general consideration of [10] together with some extensions to cases particularly relevant in nuclear physics. Appendix C treats nonrelativistic nucleon–nucleon interactions via a potential of finite range. The derivative coupling is considered in Appendix D using the example of P-wave pion–nucleon interaction. In Section 3, we supplement some considerations about the construction of a kinetic entropy within these two transport schemes. To make the paper self-contained, we summarize the time-contour matrix notation in Appendix A and introduce the Φ-functional formalism for derivative coupling in Appendix B. A summary is given in Section 4.

2. OFF-SHELL KADANDFF–BAYM AND BOTERMANS–MALFLIET KINETICS

In this section, we summarize the formulation of the off-shell kinetic equations in two different forms: in the KB form, i.e., as follows right after the gradient expansion of the exact KB equations, and in the BM form [15], which differs from the KB form only in the second order of the gradient expansion. We assume that the reader is familiar with the real-time formulation of nonequilibrium many-body theory and use the contour matrix notation, detailed in Appendix A.

The starting point of all considerations is the set of KB equations which express the spacetime changes of the Wigner transformed correlation function $iG^{-+}(X,p)$ in terms of the real-time contour convolution of the self-energy Σ with the Green's function G. We give the kinetic equation in compact notation [cf. below Eq. (A.4)]:

$$v_\mu \partial_X^\mu iG^{-+}(X,p) = [\Sigma \otimes G - G \otimes \Sigma]_{X,p}^{-+} \quad (1)$$
$$\text{with} \quad v^\mu = \frac{\partial}{\partial p_\mu} G_0^{-1}(p),$$

where $G_0^{-1}(p)$ is the Fourier transform of the inverse free Green's function:

$$G_0^{-1}(p) = \begin{cases} p^2 - m^2 & \text{for relativistic bosons} \\ p_0 - \mathbf{p}^2/(2m) & \text{for nonrelativistic fermions or bosons.} \end{cases} \quad (2)$$

For a complete definition, Eq. (1) has to be supplemented with further equations, e.g., for the retarded Green's function together with the retarded relations (A.6). In addition to these equations, the exact set of KB equations also includes the prototype of the mass-shell equation, which we also discuss below. If a system under consideration is only slightly spatially inhomogeneous and slowly evolving in time, a good approximation is provided by an expansion up to first order in spacetime gradients. Then, the main problem to arrive at a proper kinetic equation consists in accurately disentangling the rather complicated right-hand side of Eq. (1). This problem in the context of conserving approximations will be addressed here.

A. Φ-Derivable Approximations

In actual calculations, one has to use certain approximations or truncation schemes to the exact nonequilibrium theory, which make conserving properties (such as charge and energy–momentum conservations) and thermodynamic consistency of the transport theory not evident. It was shown [8, 16, 17] that there exists a class of self-consistent approximations, called Φ-derivable approximations, which are conserving at the expectation value level and at the same time thermodynamically consistent, i.e., they provide true Noether currents and a conserved energy–momentum tensor. In these schemes, the self-consistent self-energies are generated from a functional $\Phi[G]$ through the following variational procedure [8]:

$$-i\Sigma_{ik}(X,p) = \mp \frac{\delta i\Phi[G]}{\delta iG^{ki}(X,p)} \quad (3)$$
$$\times \begin{cases} 2 & \text{for real fields,} \\ 1 & \text{for complex fields,} \end{cases} \qquad i,k \in \{-+\}.$$

The functional $\Phi[G]$ specifies the truncation scheme. It consists of a set of properly chosen closed two-particle irreducible diagrams, where lines denote the self-consistent propagator G, while vertices are bare. The functional variation with respect to G diagrammatically implies an opening of a propagator line of Φ.

Particular examples of Φ-derivable approximations can be found in Appendices C and D, which consider applications of the general formalism to cases important in nuclear physics. The treatment of the pion–nucleon derivative coupling in Appendix D requires the corresponding extension of the Φ-derivable formalism, which has not been done up to now. Therefore, in Appendix B, we perform such an extension and derive the relevant modifications of the variational rules and the ensuing additional terms in the current and energy–momentum tensor expressions.

The conserving properties of these approximations are exact at the level of KB Eqs. (1), while after the expansion up to the first spacetime gradients they are generally expected to be only approximately fulfilled.

B. Physical Notation

It is helpful to eliminate the imaginary factors inherent in the standard Green's function formulation and introduce quantities that are real and, in the quasi-homogeneous limit, positive, with clear physical meaning, thereby. Thus, instead of Green's functions $G^{ij}(X,p)$ and self-energies $\Sigma^{ij}(X,p)$ with $i,j \in \{-+\}$ (see Appendix A) in the Wigner representation, we use the kinetic notation of [8]. We define[3]

$$F(X,p) = A(X,p)f(X,p) = (\mp)iG^{-+}(X,p), \quad (4)$$
$$\tilde{F}(X,p) = A(X,p)[1 \mp f(X,p)] = iG^{+-}(X,p)$$

for the generalized Wigner functions F and $\tilde{F}$ and the corresponding *four*-phase-space distribution functions $f(X,p)$ and Fermi/Bose factors $[1 \mp f(X,p)]$. Here,

$$A(X,p) \equiv -2\mathrm{Im}G^R(X,p) = \tilde{F} \pm F \quad (5)$$

is the spectral function, and G^R is the retarded propagator. The spectral function satisfies the sum rule

$$\int\limits_0^\infty \frac{dp_0}{2\pi} A(X,p) = 1 \text{ for nonrelativistic particles,} \quad (6)$$

$$\int\limits_{-\infty}^\infty \frac{dp_0}{2\pi} p_0 A(X,p) = 1 \text{ for relativistic bosons,} \quad (7)$$

which follows from the canonical equal-time (anti)commutation relations for (fermionic) bosonic field operators. Likewise, the gain and loss rates of the collision integral are defined as

$$\Gamma^{\mathrm{in}}(X,p) = \mp i\Sigma^{-+}(X,p), \quad (8)$$
$$\Gamma^{\mathrm{out}}(X,p) = i\Sigma^{+-}(X,p)$$

with the damping width

$$\Gamma(X,p) \equiv -2\mathrm{Im}\Sigma^R(X,p) = \Gamma^{\mathrm{out}}(X,p) \pm \Gamma^{\mathrm{in}}(X,p), \quad (9)$$

where Σ^R is the retarded self-energy.

In terms of the above kinetic notation, the gradient-expanded KB equations are reduced to equations for four real quantities: two equations for the real and imaginary parts of the retarded Green's function, while there are two equations for the phase-space occupation F: the KB kinetic equation and the prototype "mass-shell equation." The latter doubling of equations reflects the well-known redundancy of the KB equations. Before the gradient expansion, both equations are completely identical. However, after the gradient expansion, their interrelation is no longer obvious and deserves special care (see below).

The equations for the retarded propagator in first-order gradient approximation can be immediately solved with the result [3, 15]

$$G^R = \frac{1}{M(X,p) + i\Gamma(X,p)/2} \quad (10)$$
$$\rightarrow \begin{cases} A(X,p) = \dfrac{\Gamma(X,p)}{M^2(X,p) + \Gamma^2(X,p)/4} \\ \mathrm{Re}G^R(X,p) = \dfrac{M(X,p)}{M^2(X,p) + \Gamma^2(X,p)/4}, \end{cases}$$

with the "mass" function

$$M(X,p) = G_0^{-1}(p) - \mathrm{Re}\Sigma^R(X,p). \quad (11)$$

Although solution (10) is simply algebraic, it is valid up to first-order gradients.

C. Kadanoff–Baym Form

In terms of the above notation, the KB kinetic equation for F in the first-order gradient approximation takes the form

$$\mathcal{D}F(X,p) - \left\{\Gamma^{\mathrm{in}}, \mathrm{Re}G^R\right\} = C(X,p). \quad (12)$$

We denote this as the *quantum transport equation in the KB choice.*[4] Here, the differential drift operator is defined as

$$\mathcal{D} = \left(v_\mu - \frac{\partial \mathrm{Re}\Sigma^R}{\partial p^\mu}\right)\partial_X^\mu + \frac{\partial \mathrm{Re}\Sigma^R}{\partial X^\mu}\frac{\partial}{\partial p_\mu}, \quad (13)$$

and $\{\ldots,\ldots\}$ denotes the four-dimensional Poisson bracket,

$$\{f(X,p), \varphi(X,p)\} = \frac{\partial f}{\partial p^\mu}\frac{\partial \varphi}{\partial X_\mu} - \frac{\partial f}{\partial X^\mu}\frac{\partial \varphi}{\partial p_\mu} \quad (14)$$

in covariant notation. Please note that, now, after the gradient approximation, all quantities on the left-hand side are to be taken in the local approximation, i.e., void of any further gradient terms. Thus, the occurring self-energies are obtained from evaluating

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[3] Here and below, the upper sign corresponds to fermions, while the lower sign corresponds to bosons.

[4] If the system consists of several different particle species, there is a set of coupled kinetic equations corresponding to each species (e.g., see Appendix D).

the diagrams as in the momentum representation with the coordinates X of all Green's functions kept identical. The right-hand side specifies the collision term[5)]

$$C(X,p) = \Gamma^{\rm in}(X,p)\tilde{F}(X,p) - \Gamma^{\rm out}(X,p)F(X,p). \tag{15}$$

If the diagrams for the self-energy contain internal vertices, which give rise to memory or nonlocal effects, the gain and loss rates contain additional gradient terms, which have to be constructed, e.g., according to the rules given in [10]. The resulting local part of the collision term is charge (e.g., the baryonic number) and energy–momentum conserving by itself:

$$\mathrm{tr}\int \frac{d^4p}{(2\pi)^4}\begin{pmatrix} e \\ p^\mu \end{pmatrix} C^{\rm loc} = 0. \tag{16}$$

Here and below, e denotes the elementary charge, while tr implies the sum over all possible internal degrees of freedom, like spin, and over possible particle species. We do not explicitly introduce the particle-species label to avoid overcomplication of equations. In terms of a local functional $\Phi^{\rm loc}$, the explicit form of the local collision term is

$$C^{\rm loc}(X,p) = \frac{\delta i\Phi^{\rm loc}}{\delta\tilde{F}(X,p)}\tilde{F}(X,p) - \frac{\delta i\Phi^{\rm loc}}{\delta F(X,p)}F(X,p) \tag{17}$$

[cf. Eq. (3)]. In this paper, we limit the considerations to cases void of memory effects in this collision term. The latter effects were studied in [8].

Relation (16) permits us to derive the current

$$j^\mu_{\text{KB-eff}}(X) = e\,\mathrm{tr}\int \frac{d^4p}{(2\pi)^4}\left[v^\mu F(X,p) + \mathrm{Re}\Sigma^R\frac{\partial F}{\partial p_\mu} - \mathrm{Re}G^R\frac{\partial \Gamma^{\rm in}}{\partial p_\mu}\right] \tag{18}$$

of a charge e (e.g., the baryonic one) from the KB kinetic equation (12), which is conserved:

$$\partial_\mu j^\mu_{\text{KB-eff}}(X) = 0. \tag{19}$$

Note that this current formally differs from the true Noether current

$$j^\mu(X) = e\,\mathrm{tr}\int \frac{d^4p}{(2\pi)^4}v^\mu F(X,p) + j^\mu_{(\rm der)}(X), \tag{20}$$

which follows right from the operator expression for this quantity (cf. [8] and Appendix B). The additional term $j^\mu_{(\rm der)}$ appears only in the case of derivative coupling [see Eq. (B.19)]. In view of the gradient approximation employed, one could generally expect both currents to differ beyond the validity range of the gradient approximation. However, as demonstrated in detail in [10], these two currents are exactly equal for Φ-derivable approximations if a consistent gradient expansion is performed also in the gain and loss rates (8) of the collision term (15). In this case, the exact conservation of the Noether current results from the corresponding invariance of the Φ functional {Eq. (6.9) in [17]}, which survives the gradient expansion:

$$e\,\mathrm{tr}\int \frac{d^4p}{(2\pi)^4}\left[\left\{\mathrm{Re}\Sigma^R, F\right\} - \left\{\mathrm{Re}G^R, \Gamma^{\rm in}\right\} + C\right] = \partial_\mu j^\mu_{(\rm der)}. \tag{21}$$

The latter relation is written for the general case of memory or nonlocal effects included in C. If such effects in C are absent, the collision term drops out of Eq. (21) according to Eq. (16).

Within Φ-derivable approximations, also the conservation of energy–momentum can be established for local (pointlike) couplings providing a local energy–momentum tensor. The p^ν-weighted 4-momentum integral of the KB kinetic equation leads to the following consistency relation [8][6)]:

$$\partial^\nu\left(\mathcal{E}^{\rm pot} - \mathcal{E}^{\rm int}\right) - \partial_\mu\mathcal{E}^{\mu\nu}_{(\rm der)} = \mathrm{tr}\left(\frac{1}{2}\right)_{\rm n.b} \times \int \frac{p^\nu d^4p}{(2\pi)^4}\left[\left\{\mathrm{Re}\Sigma^R, F\right\} - \left\{\mathrm{Re}G^R, \Gamma^{\rm in}\right\} + C\right], \tag{22}$$

which is again *exact* after the gradient expansion, as shown in [10] (see also Appendices C and D). It implies that the Noether energy–momentum tensor

$$\Theta^{\mu\nu}(X) = \mathrm{tr}\left(\frac{1}{2}\right)_{\rm n.b}\int \frac{d^4p}{(2\pi)^4}v^\mu p^\nu F(X,p) + g^{\mu\nu}\left(\mathcal{E}^{\rm int} - \mathcal{E}^{\rm pot}\right) + \mathcal{E}^{\mu\nu}_{(\rm der)} \tag{23}$$

is exactly conserved by the kinetic equation (12)

$$\partial_\mu\Theta^{\mu\nu}(X) = 0. \tag{24}$$

Here, potential energy density $\mathcal{E}^{\rm pot}(X)$, which a probe particle with Wigner density $F(X,p)$ would experience due to the interaction with all other particles in the system, is

$$\mathcal{E}^{\rm pot}(X) = \mathrm{tr}\left(\frac{1}{2}\right)_{\rm n.b}\int \frac{d^4p}{(2\pi)^4}\left[\mathrm{Re}\Sigma^R F + \mathrm{Re}G^R\Gamma^{\rm in}\right]. \tag{25}$$

[5)] See an example in Eq. (C.3).

[6)] Here, in compliance with Eq. (3), we define the factor $\left(\frac{1}{2}\right)_{\rm n.b} = \begin{cases} 1/2 & \text{for neutral bosons (real fields)} \\ 1 & \text{else.}\end{cases}$

The interaction energy density $\mathcal{E}^{\text{int}}(X)$ specifies that part of the total energy density which is due to interactions. In simple cases, it relates to $\mathcal{E}^{\text{pot}}$ by a simple counting factor, namely, if all the interaction vertices of a theory have the same number n_l of lines attached to them

$$\mathcal{E}^{\text{int}}(X) = \frac{2}{n_l}\mathcal{E}^{\text{pot}}(X). \tag{26}$$

In particular, for two-body interactions, one has $n_l = 4$ and thus $\mathcal{E}^{\text{int}} = \frac{1}{2}\mathcal{E}^{\text{pot}}$, while for the fermion–boson interaction $n_l = 3$, which results in $\mathcal{E}^{\text{int}} = \frac{2}{3}\mathcal{E}^{\text{pot}}$. In Appendices C and D, we discuss cases of this type. The additional term $\mathcal{E}^{\mu\nu}_{(\text{der})}$ appears in Eq. (23) only in the case of derivative coupling [cf. Eq. (B.18)].

The considerations given above summarize the results of [10], which are quite general. However, they are restricted to local (pointlike) interactions and are void of derivative couplings. This excludes two important cases relevant to many areas in physics, nuclear physics in particular. These are the cases of interaction mediated by finite-range nonrelativistic potentials and of derivative couplings like the P-wave pion–nucleon interaction. Since the considerations are rather technical, they are exemplified in Appendices C and D. There, the results of [10] are generalized, proving that, also in these cases, conserved currents and expressions for a conserved total energy and total momentum can be constructed. These two appendices also provide further illustrations of the discussion given in the present section.

The conserving feature is especially important for devising numerical simulation codes based on this kinetic equation. Indeed, if a test-particle method is used, one should be sure that the number of test particles is conserved exactly rather than approximately. For a direct application of this method, however, there is a particular problem with the KB kinetic equation. In the test-particle method, the distribution functions are represented by an ensemble of test particles as follows:

$$F(X,p) \sim \sum_i \delta^{(3)}\left(\mathbf{X} - \mathbf{X}_i(T)\right)\delta^{(4)}\left(p - p_i(T)\right), \tag{27}$$

where the i sum runs over test particles. Then, the $\mathcal{D}F$ term in Eq. (12) just corresponds to the classical motion of these test particles subjected to forces inferred from $\text{Re}\Sigma^R$, while the collision term C gives a stochastic change of test-particles' momenta, when their trajectories "cross." The additional term, i.e., the Poisson bracket term $\left\{\Gamma^{\text{in}}, \text{Re}G^R\right\}$, spoils this simplistic picture, since derivatives acting on the distribution function F appear here only indirectly. Namely, they are encoded through derivatives of Γ^{in}. This term is responsible for backflow effects, which restore the Noether current to be the conserved one. However, such backflow phenomena are difficult to absorb into test particles, since they describe the response of the medium to the motion of the charges. In order to conserve the number of test particles between subsequent collisions, one would have to unite the additional term $\left\{\Gamma^{\text{in}}, \text{Re}G^R\right\}$ with the drift term $\mathcal{D}F$ even in the simplest case, when the collision term is charge conserving by itself [see Eq. (16)] and derivative currents vanish, $j^{\mu}_{(\text{der})} = 0$. However, the interpretation of the additional term $\left\{\Gamma^{\text{in}}, \text{Re}G^R\right\}$ causes problems within this picture, since it is not just proportional to the same δ functions as in Eq. (27) and thus cannot be included in the collisionless propagation of test particles. This problem, of course, does not prevent a direct solution of the kinetic equation. For example, one can apply well-developed lattice methods, which are, however, much more complicated and time consuming as compared to the test-particle approach.

Within the same approximation level, the set of Dyson equations for Green's functions $G^{ij}(X,p)$ provides us with an alternative equation for F,

$$MF - \text{Re}G^R\Gamma^{\text{in}} = \frac{1}{4}\left(\{\Gamma, F\} - \{\Gamma^{\text{in}}, A\}\right), \tag{28}$$

which is called the mass-shell equation, since in the quasiparticle limit it provides the mass condition $M = 0$. This equation coincides with the kinetic one (12) only within the first-order gradient approximation [8, 11, 12, 15], while both equations are exactly identical before the gradient expansion. In view of this still remaining difference, the practical recipe is to forget about the mass-shell equation (28), since the retarded Eq. (10) determines the spectral distribution, and to treat Eq. (12) as a proper quantum kinetic equation. Still, this is an ambiguous recipe, which historically was one of the motivations to proceed to the Botermans–Malfliet form of the quantum kinetic equation.

D. Botermans–Malfliet Form

As can be seen from the mass-shell Eq. (28) and Eq. (12) [8, 11, 12, 15], the gain rate Γ^{in} departs from $F\Gamma/A$ only by corrections of first order in the gradients:

$$\Gamma^{\text{in}} = \Gamma F/A + O(\partial_X) \tag{29}$$

(in equilibrium, both equate to each another). This fact permits one to substitute the right-hand side estimate for Γ^{in} in any of the gradient terms, i.e., in the Poisson-bracketed terms of Eqs. (12) and (28), and neglect the correction $O(\partial_X)$, as it already leads to terms of second order in the gradients. Upon this substitution, first proposed by Botermans and Malfliet [15], one arrives at the following form of the kinetic and mass-shell equations:

$$\mathcal{D}F(X,p) - \left\{\Gamma\frac{F}{A}, \mathrm{Re}G^R\right\} = C(X,p), \tag{30}$$

$$MF - \mathrm{Re}G^R\Gamma^{\mathrm{in}} = \frac{1}{4}\left(\{\Gamma, F\} - \left\{\frac{\Gamma F}{A}, A\right\}\right), \tag{31}$$

which are already exactly identical, as they were before the gradient expansion, and still equivalent to those in the KB form within the first-order gradient approximation. The so-obtained Eq. (30) will be called the *quantum kinetic equation in BM choice*. This equation exactly conserves the following effective current:

$$j^\mu_{\text{BM-eff}}(X) = e\mathrm{tr}\int \frac{d^4p}{(2\pi)^4}\left[v^\mu F(X,p) \right. \tag{32}$$
$$\left. + \mathrm{Re}\Sigma^R\frac{\partial F}{\partial p_\mu} - \mathrm{Re}G^R\frac{\partial(\Gamma F/A)}{\partial p_\mu}\right],$$

which differs from the Noether current j^μ in terms of the order of $O(\partial_X)$, provided a Φ-derivable approximation is used for self-energies. All the properties of the KB-choice kinetic equation within a Φ-derivable approximation also transcribe to Eq. (30) in the BM choice through the substitution $\Gamma^{\mathrm{in}} = \Gamma F/A$ in Eqs. (21), (22), and (25). This substitution, however, touches the accuracy of those relations. For instance, the conservation laws of the Noether currents (20) and the energy–momentum tensor (23) are then only approximately fulfilled together with the corresponding consistency relations (21) and (22), which now appear as

$$e\mathrm{tr}\int \frac{d^4p}{(2\pi)^4}\left[\left\{\mathrm{Re}\Sigma^R, F\right\}\right. \tag{33}$$
$$\left. - \left\{\mathrm{Re}G^R, \Gamma F/A\right\} + C\right] \simeq \partial_\mu j^\mu_{(\mathrm{der})},$$

$$\partial^\nu\left(\mathcal{E}^{\mathrm{pot}} - \mathcal{E}^{\mathrm{int}}\right) - \partial_\mu\mathcal{E}^{\mu\nu}_{(\mathrm{der})} \simeq \mathrm{tr}\left(\frac{1}{2}\right)_{\mathrm{n.b}} \tag{34}$$
$$\times\int\frac{p^\nu d^4p}{(2\pi)^4}\left[\left\{\mathrm{Re}\Sigma^R, F\right\} - \left\{\mathrm{Re}G^R, \frac{\Gamma F}{A}\right\} + C\right],$$

respectively, and hold only up to first-order gradients.

The effective BM current (32) was used by Leupold [12] as a basis for the construction of a test-particle ansatz for the nonrelativistic case. In this case, the additional term $\left\{\Gamma F/A, \mathrm{Re}G^R\right\}$ in the BM kinetic equation (30) is expressed in terms of the same distribution function as the drift term $\mathcal{D}F$. Therefore, one can unify these terms to construct equations of motion for test particles, which provide exact conservation of $j^\mu_{\text{BM-eff}}$. To automatically fulfill this effective-current conservation, the test-particle ansatz is introduced for the combination

$$\frac{1}{2}\Gamma A\left(1 - \frac{\partial\mathrm{Re}\Sigma^R}{\partial p_0} - \frac{M}{\Gamma}\frac{\partial\Gamma}{\partial p_0}\right)F(X,p) \tag{35}$$
$$\sim \sum_i \delta^{(3)}\left(\mathbf{X} - \mathbf{X}_i(T)\right)\delta^{(4)}\left(p - p_i(T)\right),$$

rather than for the distribution function itself. Note that the energy $p_i^0(T)$ of the test particle is a free coordinate, not restricted by a mass-shell condition. Cassing and Juchem [11] extended this test-particle ansatz to the relativistic case. The equations of motion for the test particle, which follow from this ansatz, in particular, give the time evolution of the off-shellness of a test particle [11, 12]

$$\frac{dM}{dT} = \frac{M}{\Gamma}\frac{d\Gamma}{dT}, \tag{36}$$

the origin of which can be traced back to the additional term $\left\{\Gamma F/A, \mathrm{Re}G^R\right\}$ in the BM kinetic Eq. (30). Here, M is the mass of the test particle relative to its on-shell value [see Eq. (11)], and this equation of motion implies that, once the width drops in time, the particles are driven towards the on-shell mass $M = 0$. This clarifies the meaning of the additional term $\left\{\Gamma F/A, \mathrm{Re}G^R\right\}$ in the off-shell BM transport: it provides the time evolution of the off-shellness.

3. ENTROPY

Another important feature of the kinetic description is the approach to thermal equilibrium during evolution of a closed system. In terms of transport theory, the sufficient (while not necessary!) condition of it is the existence of an H theorem. Leaving aside all complications associated with nonlocal effects in the collision term and possible lack of positive definiteness of the transition rates, discussed in [8], we confine our consideration to simple approximations [cf. (C.1)]. As demonstrated in [8], in the BM approximation to the quantum kinetic Eq. (30), the H theorem can indeed be formulated:

$$\partial_\mu s^\mu_{\mathrm{BM}}(X) = \mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\ln\frac{\tilde{F}}{F}C^{\mathrm{loc}}(X,p) \geq 0, \tag{37}$$

where the quantity

$$s^\mu_{\mathrm{BM}} = \mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left[\left(v^\mu - \frac{\partial\mathrm{Re}\Sigma^R}{\partial p_\mu}\right)\right. \tag{38}$$
$$\times\left(\mp\tilde{F}\ln\frac{\tilde{F}}{A} - F\ln\frac{F}{A}\right) - \mathrm{Re}G^R$$

$$\times \left(\mp \ln \frac{\tilde{F}}{A} \frac{\partial}{\partial p_\mu} \left(\Gamma \frac{\tilde{F}}{A} \right) - \ln \frac{F}{A} \frac{\partial}{\partial p_\mu} \left(\Gamma \frac{F}{A} \right) \right) \Bigg]$$

obtained from the left-hand side of the BM kinetic Eq. (30) is interpreted as an entropy flow for the BM choice. For the Φ-derivable approximation (C.1), the right-hand side of relation (37) takes the following form:

$$\operatorname{tr} \int \frac{d^4p}{(2\pi)^4} \ln \frac{\tilde{F}}{F} C^{\mathrm{loc}}(X,p) \tag{39}$$

$$= \operatorname{tr} \frac{1}{4} \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4} \frac{d^4p_4}{(2\pi)^4}$$

$$\times R^{\mathrm{loc}} (2\pi)^4 \delta^4 \left(p_1 + p_2 - p_3 - p_4 \right)$$

$$\times \left(F_1 F_2 \tilde{F}_3 \tilde{F}_4 - \tilde{F}_1 \tilde{F}_2 F_3 F_4 \right) \ln \frac{F_1 F_2 \tilde{F}_3 \tilde{F}_4}{\tilde{F}_1 \tilde{F}_2 F_3 F_4},$$

where R^{loc} is the transition rate determined by Eq. (C.4). This expression is indeed non-negative, since $(x-y)\ln(x/y) \geq 0$ for any positive x and y, and is of the second order in deviation from equilibrium $(F - F_{\mathrm{eq}})$, as both $(F_1 F_2 \tilde{F}_3 \tilde{F}_4 - \tilde{F}_1 \tilde{F}_2 F_3 F_4)$ and $\ln(F_1 F_2 \tilde{F}_3 \tilde{F}_4 / \tilde{F}_1 \tilde{F}_2 F_3 F_4)$ approach zero at equilibrium. From the kinetic equation, it follows that the deviation from equilibrium is of the first order in time gradients: $(F - F_{\mathrm{eq}}) \propto O(\partial_T F)$. This implies that the right-hand side of relation (37) is of the second order in time gradients, which is, strictly speaking, beyond our first-order gradient approximation. However, from the point of view of practical use, this feature is highly welcome as it guarantees equilibration. A further advantage of the kinetic entropy flux (38) is that, in equilibrium, its zero component merges the thermodynamic expression for the entropy deduced from the thermodynamic potential in the Φ-derivable scheme [8, 18, 19].

In the case of the KB choice (12), the situation is more controversial. Performing all the same manipulations with the KB kinetic Eq. (12) as those in [8], we arrive at the following relation:

$$\partial_\mu s^\mu_{\mathrm{KB}}(X) = \operatorname{tr} \int \frac{d^4p}{(2\pi)^4} \ln \frac{\tilde{F}}{F} C^{\mathrm{loc}} - \delta H_{\mathrm{KB}}, \tag{40}$$

where

$$s^\mu_{\mathrm{KB}} = \operatorname{tr} \int \frac{d^4p}{(2\pi)^4} \left[\left(v^\mu - \frac{\partial \mathrm{Re}\Sigma^R}{\partial p_\mu} \right) \right. \tag{41}$$

$$\times \left(\mp \tilde{F} \ln \frac{\tilde{F}}{A} - F \ln \frac{F}{A} \right)$$

$$\left. - \mathrm{Re}G^R \left(\mp \ln \frac{\tilde{F}}{A} \frac{\partial \Gamma^{\mathrm{out}}}{\partial p_\mu} - \ln \frac{F}{A} \frac{\partial \Gamma^{\mathrm{in}}}{\partial p_\mu} \right) \right],$$

$$\delta H_{\mathrm{KB}} = - \int \frac{d^4p}{(2\pi)^4} \mathrm{Re}G^R \left\{ \ln \frac{\tilde{F}}{F}, \frac{C^{\mathrm{loc}}}{A} \right\}. \tag{42}$$

The KB entropy flow s^μ_{KB} is identical to the BM one s^μ_{BM} up to zeroth-order gradients, while they differ in the first-order gradient corrections. One can easily obtain s^μ_{BM} from the KB entropy flow by doing replacement (29) in Γ^{in} and a similar replacement in Γ^{out}.

The additional term δH_{KB} on the right-hand side of relation (40) is of the second order in gradients, due to the Poisson bracket and $C^{\mathrm{loc}} \propto O(\partial_T F)$. Therefore, the right-hand side of (40) consists of two terms, which are of the same order of magnitude, and one of them (δH_{KB}) is sign indefinite. This prevents us from concluding the positive definiteness of the right-hand side of Eq. (40). Alternatively, we were not able to cast this term into a full divergence as to be included into the definition of the KB entropy flow. This fact by itself does not imply that the system does not approach equilibrium or even the absence of an H theorem for the KB kinetic equation, but suggests that equilibration should be tested in actual calculations. The local H theorem we are looking for is a very stringent condition, providing a monotonic approach to equilibrium. In fact, equilibration may well be nonmonotonic in time.

Still, for the KB kinetic equation, we are able to prove the H theorem in a limiting case, i.e., close to local thermal equilibrium or for a quasi-stationary state, which slowly evolves in space and time. To be definite, let us talk about the local thermal equilibrium. In terms of the distribution function

$$F(X,p) = F_{\mathrm{loc.eq}}(X,p) + \delta F(X,p), \tag{43}$$

the above assumption implies that $|\delta F| \ll F_{\mathrm{loc.eq}}$ and $|\partial_X F_{\mathrm{loc.eq}}| \lesssim |\partial_X \delta F|$. Then, we can write

$$\delta H_{\mathrm{KB}} = \partial_\mu \delta s^\mu_{\mathrm{KB}}(X) \tag{44}$$

$$+ \int \frac{d^4p}{(2\pi)^4} \frac{C^{\mathrm{loc}}}{A} \left\{ \ln \frac{\tilde{F}}{F}, \mathrm{Re}G^R \right\},$$

where

$$\delta s^\mu_{\mathrm{KB}}(X) = -\operatorname{tr} \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Re}G^R}{A} \tag{45}$$

$$\times \frac{\partial \ln(\tilde{F}/F)}{\partial p_\mu} C^{\mathrm{loc}}(X,p).$$

Here, the remaining term

$$\int \frac{d^4p}{(2\pi)^4} \frac{C^{\mathrm{loc}}}{A} \left\{ \ln \frac{\tilde{F}}{F}, \mathrm{Re}G^R \right\} \propto O(\delta F \partial_X \delta F) \tag{46}$$

$$+ O(\delta F \partial_X F_{\mathrm{loc.eq}})$$

can be neglected, as it has additional gradient smallness as compared to the first term on the right-hand side of Eq. (40). Here, we have taken into account that $C^{\rm loc} \propto \delta F$ and $\left\{\ln(\tilde{F}/F), {\rm Re}G^R\right\} \propto \partial_X(F_{\rm loc.eq} + \delta F)$. Thus, from Eq. (40), we conclude that

$$\partial_\mu \left(s^\mu_{\rm KB} + \delta s^\mu_{\rm KB}\right) \geq 0 \quad \text{near local equilibrium,} \quad (47)$$

which is the H theorem for this case with the total entropy flow $s^\mu_{\rm KB} + \delta s^\mu_{\rm KB}$. Note that $\delta s^\mu_{\rm KB}$ is proportional to the collision term and hence equals zero in equilibrium. The applicability range of this result is the same as that for the memory entropy derived in [8] for the BM choice.

4. SUMMARY AND PERSPECTIVES

In conclusion, we would like to summarize the present status of the two considered approaches to off-shell transport.

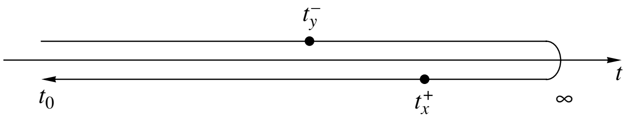
From a consistency point of view, the BM choice looks more appealing, since it preserves the exact identity between the kinetic and mass-shell equation, a property inherent in the original KB equations [8]. For the KB choice, this identity between the kinetic and mass-shell equations is only *approximately* preserved, namely, within the validity range of the first-order gradient approximation. However, this disadvantage is not of great practical use, since, in any case, only one of these two equations, namely, the kinetic one, should be used in actual calculations.

For the construction of conservation laws related to global symmetries or energy and momentum, the *local* collision term entirely drops out of the balance. Thus, the conservation laws solely depend on the properties of the first-order gradient terms in the kinetic equation. In this respect, the KB kinetic equation has a conceptual advantage as it leads to *exact* [10] rather than approximate conservation laws, provided the scheme is based on Φ-derivable approximations. Thereby, the expectation values of the original operator expressions of conserved quantities (e.g., Noether currents) are exactly conserved. The reason is that the KB kinetic equation preserves certain contour symmetries among the various gradient terms, while they are violated for the BM choice. Of course, within their range of applicability, these two approaches are equivalent, because the BM kinetic equation conserves the charge and energy–momentum within the theoretical accuracy of the gradient approximation. Still, the fact that the KB choice possesses exact conservation laws puts this version at the level of a generic equation, much like the Boltzmann or hydrodynamic equations, to be used as phenomenological dynamical equations for practical applications. Such conserving dynamical schemes may be useful even though the applicability condition of the approximation might be violated at some stages of evolution. For instance, such a situation happens at the initial stage of heavy-ion collisions. As the conservations are exact, we can still use the gradient approximation, relying on a minor role of this rather short initial stage in the total evolution of a system. Moreover, exact conservation laws allow us to keep control of numerical codes.

Although the KB kinetic equations possess exactly conserved Noether currents, a practical numerical approach (e.g., by a test-particle method) to its solution has not yet been established. The obstacle is the special Poisson bracket term in the KB kinetic Eq. (12) which lacks proper interpretation since the phase-space occupation function $F(X, p)$ enters only indirectly through the gain-rate gradient terms. What is known is that this term encodes the backflow component, which ensures that the Noether currents are conserved. However, such backflow features are difficult to implement in a test-particle scheme. This problem, of course, does not exclude solution of the KB kinetic equation, e.g., within well-adapted lattice methods, which are, however, much more complicated and time consuming as compared to the test-particle approach. For the BM kinetic equation, on the other hand, an efficient test-particle method is already available [11, 12], for the price that it deals with an alternative current rather than the Noether current.

As a novel part, we showed (cf. Appendices) that the exact conservation laws in the KB kinetic equations, originally derived for local interaction terms, which lead to a local energy–momentum tensor, also do hold for derivative couplings and for interactions of finite range, like a nonrelativistic potential. For the latter case, of course, only global conservation of energy and momentum can be achieved. In order to deal with the derivative coupling, we extended the Φ-derivable approach to this case.

An important feature of kinetic descriptions is the approach to thermal equilibrium during evolution of a closed system. A sufficient (while not necessary!) condition is provided by an H theorem. As was demonstrated in [8], at least within simplest Φ-derivable approximations for the kinetic equation in BM choice, an H theorem indeed can be proven. In equilibrium, the so-derived kinetic entropy merges the corresponding equilibrium expression, which in the context of Φ-derivable approximations results from the thermodynamic potential (cf. [8, 18, 19]). For the KB kinetic equation, the result is by far weaker. Here, we were able to prove the H theorem only within simplest Φ-derivable approximations and for a system very close to an almost spatially homogeneous



Closed real-time contour with two external points x, y on the contour.

thermal local equilibrium or stationary state. These results, in general, do not imply that the system does not approach equilibrium but suggest that equilibration should be tested in actual calculations. Furthermore, the local H theorem with a local entropy current that we considered for the BM case may be far too restrictive, providing a monotonic approach to equilibrium. In fact, for kinetic equations with memory or nonlocal effects, equilibration may well be nonmonotonic in time.

Though the discussion in this paper is confined to problems of Φ-derivable off-shell transport based on the first-order gradient expansion, significant progress has recently been achieved also in solving KB equations directly without any gradient expansion for selected examples. These concern nonequilibrium processes in scalar and spinor–scalar models on one- and three-space dimensions (see, e.g., [20–22] and references therein). It was found [20, 21] that, after a comparably short but violent nonequilibrium evolution, the time dependence of the Wigner-transformed spectral function becomes rather weak even for moderate coupling constants. During this slow evolution, the system is still far away from equilibrium. This fact provides a necessary condition for a successful gradient expansion and hence indicates a wide range of applicability of the approaches discussed in this paper. Even though the rapid far-from-equilibrium dynamics is formally beyond the scope of applicability of the gradient-expanded quantum kinetics, nevertheless, the KB choice includes all the ingredients required for such a treatment, i.e., the proper mean-field dynamics, together with the off-shell transport of particles, thereby satisfying exact rather than approximate conservation laws even far away from equilibrium.

Further progress in understanding the properties of Φ-derivable approximations to finite-temperature quantum field theory was reported concerning the question of renormalizability. The new results are equally applicable to quantum kinetic equations, both in KB and in BM choices. In [23], it was shown that truncated nonperturbative self-consistent Dyson resummation schemes can be renormalized with local counterterms defined at the vacuum level. The requirements are that the underlying theory is renormalizable and that the self-consistent scheme follows Baym's Φ-derivable concept. This result proves that there is no arbitrariness in studying the in-medium modifications of model parameters, like the mass and the coupling constants, within this class of approximation schemes. It is sufficient to adjust them in the vacuum, for instance, by fitting them to scattering data, in order to predict their changes in the medium without ambiguity. This result also guarantees the standard Φ-derivable properties, like thermodynamic consistency and exact conservation laws, for the renormalized approximation schemes to hold. In [24], the theoretical concepts for the renormalization devised in [23] were applied to the ϕ^4 model, demonstrating the practicability of the method.

In general, the symmetries of the classical action that lead to Ward–Takahashi identities for the proper vertex functions are violated for Φ-derivable approximations for functions beyond the one-point level, i.e., on the correlator level. This causes problems concerning the Nambu–Goldstone modes [25] in the broken symmetry case or concerning local symmetries (gauge symmetries) [26] on a level where the gauge fields are treated beyond the classical field approximation, i.e., on the propagator level. In [26], it was shown that, on top of any solution of a Φ-derivable approximation, which is constructed from a symmetric Lagrangian, there exists a nonperturbative effective action which generates proper vertex functions in the same sense as the 1PI effective action. These external vertex functions fulfill the Ward–Takahashi identities of the underlying symmetry. However, in general, they coincide with the self-consistent ones only up to one-point order. Thus, usually, the so-generated external self-energy and higher vertex functions are different from the Φ-derivable expressions. Therefore, the pleasant property of the Φ-derivable approximations, namely, the conserving one, proves to be lost. The derivation of approximation schemes that satisfy all symmetry properties of the underlying classical action and at the same time are fully self-consistent and conserving still remains as an open task.

As has been already mentioned, the gauge invariance may be lost in Φ-derivable approximations too. In particular, this problem prevents applications of Φ-derivable approximations (including kinetic ones) to description of quark–gluon plasma based on QCD. This occurs because, in general, solutions for dressed propagators and vertices do not satisfy Ward–Takahashi identities. This pathology shows up as an explicit dependence of results on the choice of the gauge condition. In [27], it was demonstrated, in fact, that Φ-derivable approximations have a controlled gauge dependence, i.e., the gauge-dependent terms appear at orders higher than the truncation order. Furthermore, using the stationary point obtained for the approximation to evaluate the complete

2PI effective action boosts the order at which the gauge-dependent terms appear, to twice the order of truncation. This is still not a solution of the gauge problem in the rigorous sense but certain progress to its better control and understanding.

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APPENDIX A

Matrix Notation

In calculations that apply to the Wigner transformations, it is necessary to decompose the full contour into its two branches—the *time-ordered* and *antitime-ordered* branches. One then has to distinguish between the physical spacetime coordinates x and the corresponding contour coordinates $x^{\mathcal{C}}$, which for a given $x = (t, \mathbf{x})$ take two values $x^- = (t^-, \mathbf{x})$ and $x^+ = (t^+, \mathbf{x})$ on the two branches of the contour (see figure).

Closed real-time contour integrations can then be decomposed as

$$\int_{\mathcal{C}} dx \cdots = \int_{t_0}^{\infty} dx \cdots + \int_{\infty}^{t_0} dx \ldots \tag{A.1}$$
$$= \int_{t_0}^{\infty} dx \cdots - \int_{t_0}^{\infty} dx \ldots,$$

where only the time limits are explicitly given. The extra minus sign of the antitime-ordered branch can conveniently be formulated by a $\{-+\}$ "metric" with the metric tensor in $\{-+\}$ indices

$$\left(\sigma^{ij}\right) = \left(\sigma_{ij}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.2}$$

which provides a proper matrix algebra for multipoint functions on the contour with "co"- and "contra"-contour values. Thus, for any two-point function F, the contour values are defined as

$$F^{ij}(x,y) := F(x^i, y^j), \quad i,j \in \{-,+\}, \tag{A.3}$$

with

$$F_i^{\ j}(x,y) := \sigma_{ik} F^{kj}(x,y),$$
$$F^i_{\ j}(x,y) := F^{ik}(x,y)\sigma_{kj},$$
$$F_{ij}(x,y) := \sigma_{ik}\sigma_{jl}F^{kl}(x,y)$$

on the different branches of the contour. Here summation over repeated indices is implied. Then, contour folding of contour two-point functions, e.g., in Dyson equations, simply becomes

$$H(x^i, y^k) = H^{ik}(x,y) = [F \otimes G]^{ik} \tag{A.4}$$
$$\equiv \int_{\mathcal{C}} dz F(x^i, z) G(z, y^k) = \int dz F^i_{\ j}(x,z) G^{jk}(z,y)$$

in the matrix notation.

For any multipoint function, the external point $x_{\max}$, which has the largest physical time, can be placed on either branch of the contour without changing the value, since the contour-time evolution from $x^-_{\max}$ to $x^+_{\max}$ provides unity. Therefore, one-point functions have the same value on both sides on the contour.

Due to the change of operator ordering, genuine multipoint functions are, in general, discontinuous whenever two contour coordinates become identical. In particular, two-point functions like $iF(x,y) = \langle \mathcal{T}_{\mathcal{C}} \hat{A}(x) \hat{B}(y) \rangle$ become[7]

$$iF(x,y) = \begin{pmatrix} iF^{--}(x,y) & iF^{-+}(x,y) \\ iF^{+-}(x,y) & iF^{++}(x,y) \end{pmatrix} \tag{A.5}$$
$$= \begin{pmatrix} \langle \mathcal{T} \hat{A}(x) \hat{B}(y) \rangle & \mp \langle \hat{B}(y) \hat{A}(x) \rangle \\ \langle \hat{A}(x) \hat{B}(y) \rangle & \langle \mathcal{T}^{-1} \hat{A}(x) \hat{B}(y) \rangle \end{pmatrix},$$

where $\mathcal{T}$ and $\mathcal{T}^{-1}$ are the usual time- and antitime-ordering operators. Since there are altogether only two possible orderings of the two operators, in fact given by the Wightman functions F^{-+} and F^{+-}, which are both continuous, not all four components of F are independent. Equation (A.5) implies the following relations between nonequilibrium and usual retarded and advanced functions:

$$F^R(x,y) = F^{--}(x,y) - F^{-+}(x,y) \tag{A.6}$$
$$= F^{+-}(x,y) - F^{++}(x,y)$$
$$:= \Theta(x_0 - y_0)$$
$$\times \left(F^{+-}(x,y) - F^{-+}(x,y)\right),$$
$$F^A(x,y) = F^{--}(x,y) - F^{+-}(x,y)$$
$$= F^{-+}(x,y) - F^{++}(x,y)$$

---

[7] Frequently used alternative notation is $F^< = F^{-+}$ and $F^> = F^{+-}$.

$$:= -\Theta(y_0 - x_0) \times \left(F^{+-}(x,y) - F^{-+}(x,y)\right),$$

where $\Theta(x_0 - y_0)$ is the step function of the time difference. The rules for the co-contour functions F^{--}, etc., follow from Eq. (A.3).

For such two-point functions, complex conjugation implies

$$\left(iF^{-+}(x,y)\right)^* = iF^{-+}(y,x) \to iF^{-+}(X,p) = \text{real}, \tag{A.7}$$

$$\left(iF^{+-}(x,y)\right)^* = iF^{+-}(y,x) \to iF^{+-}(X,p) = \text{real},$$

$$\left(iF^{--}(x,y)\right)^* = iF^{++}(y,x) \to \left(iF^{--}(X,p)\right)^* = iF^{++}(X,p),$$

$$\left(F^{R}(x,y)\right)^* = F^{A}(y,x) \to \left(F^{R}(X,p)\right)^* = F^{A}(X,p),$$

where the right parts specify the corresponding properties in the Wigner representation. Diagrammatically, these rules imply the simultaneous swapping of all "+" vertices into "−" vertices and vice versa together with reversing of the line arrow sense of all propagator lines in the diagram.

Contrary to the common case (A.5), the symmetrized contour convolution

$$E(x) = \int_{\mathcal{C}} dz\left[F(x,z)G(z,x) + G(x,z)F(z,x)\right] \tag{A.8}$$

is continuous when two contour coordinates become identical. This can be easily checked, proceeding from relations (A.6) for F and G functions. Moreover, for this symmetrized convolution with two coincident points, we obtain a very simple expression in the Wigner representation if all gradient corrections to the convolution are neglected (so-called local approximation),

$$E^{\text{loc}}(X) = \int \frac{d^4p}{(2\pi)^4}\left[F^{--}(X,p)G^{--}(X,p) - F^{++}(X,p)G^{++}(X,p)\right]. \tag{A.9}$$

In particular, this form is applicable to the potential (25) and derivative (D.4) energy densities.

APPENDIX B

Derivative Coupling

To be specific, we consider a multicomponent system with different constituents a described by nonrelativistic fermionic and relativistic scalar bosonic field operators, summarized as $\hat\phi = \{\hat\phi_a(x)\}$. The free Lagrangians of these fields are

$$\hat{\mathcal{L}}L_a^0 = \begin{cases} \dfrac{1}{2}\left(i\hat\phi_a^\dagger\partial_t\hat\phi_a - i\partial_t\hat\phi_a^\dagger\hat\phi_a - \dfrac{1}{m_a}\nabla\hat\phi_a^\dagger\nabla\hat\phi_a\right) & \text{for nonrelativistic fermions} \\ \dfrac{1}{2}\dfrac{1}{m_a}\left(\partial_\mu\hat\phi_a\partial^\mu\hat\phi_a - m_a^2\hat\phi_a^2\right) & \text{for neutral relativistic bosons} \\ \dfrac{1}{m_a}\partial_\mu\hat\phi_a^\dagger\partial^\mu\hat\phi_a - m_a^2\hat\phi_a^\dagger\hat\phi_a & \text{for charged relativistic bosons.} \end{cases} \tag{B.1}$$

We assume that these fields interact via linear derivative coupling, such that the interaction Lagrangian depends not only on these fields but also on their derivatives: $\hat{\mathcal{L}}^{\text{int}} = \hat{\mathcal{L}}^{\text{int}}\{\hat\phi_a, \hat\phi_a^\dagger, \partial^\mu\hat\phi_a, \partial^\mu\hat\phi_a^\dagger\}$. The variational principle of stationary action determines Euler–Lagrange equations of motion for the field operators

$$\partial_\mu\frac{\partial\hat{\mathcal{L}}L^0}{\partial(\partial_\mu\hat\phi_a^\dagger)} - \frac{\partial\hat{\mathcal{L}}L^0}{\partial\left(\hat\phi_a^\dagger\right)} = \frac{\partial\hat{\mathcal{L}}L^{\text{int}}}{\partial\left(\hat\phi_a^\dagger\right)} - \partial_\mu\frac{\partial\hat{\mathcal{L}}L^{\text{int}}}{\partial(\partial_\mu\hat\phi_a^\dagger)} =: \frac{\delta\hat{\mathcal{L}}L^{\text{int}}}{\delta\hat\phi_a^\dagger(x)} \tag{B.2}$$

and the corresponding adjoint equation, where the "variational" δ derivative is defined as

$$\frac{\delta}{\delta f(x)}\cdots := \frac{\partial}{\partial f(x)}\cdots - \partial_\mu\left(\frac{\partial}{\partial(\partial_\mu f(x))}\cdots\right). \tag{B.3}$$

This is the key definition, which allows us to recast the local-coupling formulas to the derivative coupling case. In fact, the "variational" δ derivative specifies the *full* derivative over $f(x)$, implying that all derivatives acting on $f(x)$ in the action should be redirected to other terms by means of partial integration before taking variational derivatives of $f(x)$.

The equations of motion can also be written in the differential form,

$$G_0^{-1}(x)\hat{\phi}_a(x) = -\hat{J}_a(x) \equiv -\frac{\delta \hat{\mathcal{L}}L^{\text{int}}}{\delta \hat{\phi}_a^\dagger(x)} \qquad \text{(B.4)}$$

and similarly for the corresponding adjoint equation. The $\hat{J}_a(x)$ operator is the local source current of field a with mass m_a, and $G_0^{-1}(x)$ is the free evolution operator[8]

$$G_0^{-1}(x) \qquad \text{(B.5)}$$
$$= \begin{cases} -\partial_\mu \partial^\mu - m_a^2 & \text{for relativistic bosons} \\ i\partial_t + \dfrac{1}{2m_a}\partial_{\mathbf{x}}^2 & \text{for nonrelativstic particles} \end{cases}$$

with free propagator $G_0(y, x)$ as resolvent [cf. Eq. (2)].

Invariances of the Lagrangian provide a set of conservation laws, the most prominent of which are those for the energy—momentum and certain currents. In addition to the standard canonical energy—momentum tensor [28], different representations of this tensor have been considered [29, 30]. Using the Euler—Lagrange equations of motion and the definition of the source current (B.4), one can show that the following form also defines a conserving energy—momentum tensor:

$$\hat{\Theta}^{\mu\nu}(x) = -\frac{1}{2}\left[\sum_a \left(\frac{1}{2}\right)_{\text{n.b}} \left(\partial_x^\nu - \partial_y^\nu\right) \right. \qquad \text{(B.6)}$$
$$\left. \times \left(\frac{\partial \hat{\mathcal{L}}L^0(x)}{\partial(\partial_\mu \hat{\phi}_a)}\hat{\phi}_a(y) - \hat{\phi}_a^\dagger(x)\frac{\partial \hat{\mathcal{L}}L^0(y)}{\partial(\partial_\mu \hat{\phi}_a^\dagger)}\right)\right]_{x=y}$$
$$+ g^{\mu\nu}\left(\hat{\mathcal{E}}^{\text{int}}(x) - \hat{\mathcal{E}}^{\text{pot}}(x)\right) + \hat{\mathcal{E}}^{\mu\nu}_{(\text{der})}(x).$$

Here we have introduced the operators of the interaction-energy density $\hat{\mathcal{E}}^{\text{int}}$ and the potential-energy density $\hat{\mathcal{E}}^{\text{pot}}$:

$$\hat{\mathcal{E}}^{\text{int}}(x) = -\hat{\mathcal{L}}L^{\text{int}}(x), \qquad \text{(B.7)}$$

$$\hat{\mathcal{E}}^{\text{pot}}(x) = -\frac{1}{2}\sum_a \left(\frac{1}{2}\right)_{\text{n.b}} \qquad \text{(B.8)}$$
$$\times \left(\hat{J}_a^\dagger(x)\hat{\phi}_a(x) + \hat{\phi}_a^\dagger(x)\hat{J}_a(x)\right).$$

Furthermore, we have singled out the contribution

$$\hat{\mathcal{E}}^{\mu\nu}_{(\text{der})}(x) = \sum_a \left(\frac{1}{2}\right)_{\text{n.b}} \qquad \text{(B.9)}$$
$$\times \left(\frac{\partial \hat{\mathcal{L}}L^{\text{int}}}{\partial(\partial_\mu \hat{\phi}_a)}\partial^\nu \hat{\phi}_a + \partial^\nu \hat{\phi}_a^\dagger \frac{\partial \hat{\mathcal{L}}L^{\text{int}}}{\partial(\partial_\mu \hat{\phi}_a^\dagger)}\right)$$

arising in the case of derivative coupling. Here and below, the case of neutral bosons results from equating $\hat{\phi}_a = \hat{\phi}_a^\dagger$ in all the formulas. Proper counting is assured by the extra $\left(\frac{1}{2}\right)_{\text{n.b}}$ factor which takes the value 1/2 for neutral boson (real fields) and 1 for complex fields.

If the Lagrangian is invariant under some global transformation of charged fields (with the charges e_a), e.g.,

$$\hat{\phi}_a(x) \to e^{-ie_a\Lambda}\hat{\phi}_a(x); \qquad \hat{\phi}_a^\dagger(x) \to e^{ie_a\Lambda}\hat{\phi}_a^\dagger(x), \qquad \text{(B.10)}$$

there exists a Noether current defined as [28]

$$\hat{j}^\mu = -i\sum_a e_a \left(\frac{\partial \hat{\mathcal{L}}L}{\partial(\partial_\mu \hat{\phi}_a)}\hat{\phi}_a - \hat{\phi}_a^\dagger \frac{\partial \hat{\mathcal{L}}L}{\partial(\partial_\mu \hat{\phi}_a^\dagger)}\right) \qquad \text{(B.11)}$$
$$= \hat{j}^\mu_{(\text{conv})} + \hat{j}^\mu_{(\text{der})},$$

which is conserved, i.e., $\partial_\mu \hat{j}^\mu = 0$. Here we have decomposed it into two terms: the conventional one

$$\hat{j}^\mu_{(\text{conv})} = -i\sum_a e_a \left(\frac{\partial \hat{\mathcal{L}}L^0}{\partial(\partial_\mu \hat{\phi}_a)}\hat{\phi}_a - \hat{\phi}_a^\dagger \frac{\partial \hat{\mathcal{L}}L^0}{\partial(\partial_\mu \hat{\phi}_a^\dagger)}\right), \qquad \text{(B.12)}$$

which is associated with the free Lagrangian, and the derivative term

$$\hat{j}^\mu_{(\text{der})} = -i\sum_a e_a \left(\frac{\partial \hat{\mathcal{L}}L^{\text{int}}}{\partial(\partial_\mu \hat{\phi}_a)}\hat{\phi}_a - \hat{\phi}_a^\dagger \frac{\partial \hat{\mathcal{L}}L^{\text{int}}}{\partial(\partial_\mu \hat{\phi}_a^\dagger)}\right), \qquad \text{(B.13)}$$

which is nonzero only for derivative coupling.

To define the Φ functional for the case under consideration, all the steps described in [17] should be repeated. Then, we arrive at the Φ functional that depends also on the gradients of mean fields ($\partial_\mu \phi_a$ and $\partial_\mu {\phi_a}^*$) and Green's functions ($\partial_x^\mu G(x, y)$ and $\partial_y^\mu G(x, y)$), rather than on their values only. The variational rules of this functional formally look similar to those in [17],

$$iJ_a(x) = \frac{\delta i\Phi}{\delta {\phi_a}^*(x)}, \qquad \text{(B.14)}$$

$$-i\Sigma_a(x, y) = (\mp)\frac{\delta i\Phi}{\delta iG_a(y, x)} \qquad \text{(B.15)}$$
$$\times \begin{cases} 2 & \text{for real fields} \\ 1 & \text{for complex fields,} \end{cases}$$

---

[8] Note that the first line in (B.5) is not the nonrelativistic limit of the second one. We have already divided the second line by $2m_a$, to take into account different normalizations of relativistic and nonrelativistic wave functions.

$$-\mathcal{E}^{\text{int}}(x) = \frac{\delta i\Phi}{\delta i\lambda(x)}, \tag{B.16}$$

but should be understood in terms of the variational δ derivative of Eq. (B.3) for one-point functions (like $\phi_a(x)$ and $\lambda(x)$) and its generalization

$$\begin{aligned}\frac{\delta i\Phi}{\delta iG(y,x)} &:= \frac{\delta_0 i\Phi}{\delta_0 iG(y,x)} \\ &- \partial_x^\mu\left(\frac{\delta_0 i\Phi}{\delta_0(\partial_x^\mu iG(y,x))}\right) - \partial_y^\mu\left(\frac{\delta_0 i\Phi}{\delta_0(\partial_y^\mu iG(y,x))}\right) \\ &+ \partial_x^\mu\partial_y^\nu\left(\frac{\delta_0 i\Phi}{\delta_0(\partial_x^\mu\partial_y^\nu iG(y,x))}\right)\end{aligned} \tag{B.17}$$

to two-point functions. Here, $\delta_0/\delta_0 iG(y,x)$ means the conventional variation over $G(y,x)$, which does not touch $\partial_x^\mu iG$, $\partial_y^\mu iG$, and $\partial_x^\mu\partial_y^\nu iG$ terms in $i\Phi$. Similar to the variational δ derivative of Eq. (B.3), the δ variation of Eq. (B.17) means the *full* variation over $G(y,x)$, implying that all derivatives acting on $G(y,x)$ in the Φ functional should be redirected to other terms by means of partial integration before taking variation over $G(y,x)$. The factor $\lambda(x)$ appearing in Eq. (B.16) is an auxiliary scaling parameter of the coupling constant. In terms of the Φ functional, the additional derivative contributions to mean values of the energy–momentum tensor (B.9) and current (B.13) take the form

$$\begin{aligned}\mathcal{E}^{\mu\nu}_{(\text{der})} &=: \langle\hat{\mathcal{E}}^{\mu\nu}_{(\text{der})}\rangle \\ &= \sum_a\Bigg(\left(\frac{1}{2}\right)_{\text{n.b}}\left[\frac{\delta\Phi}{\delta(\partial_\mu\phi_a(x))}\partial^\nu\phi_a(x) + \frac{\delta\Phi}{\delta(\partial_\mu{\phi_a}^*(x))}\partial^\nu{\phi_a}^*(x)\right] \\ &+ \int\limits_{\mathcal{C}} dz\left[\frac{\delta\Phi}{\delta(\partial^x_\mu iG_a(z,x))}\partial^\nu_x iG_a(z,x) + \partial^\nu_x iG_a(x,z)\frac{\delta\Phi}{\delta(\partial^x_\mu iG_a(x,z))}\right]\Bigg),\end{aligned} \tag{B.18}$$

$$\begin{aligned}j^\mu_{(\text{der})} &=: \langle\hat{j}^\mu_{(\text{der})}\rangle = -i\sum_a e_a \\ &\times\Bigg(\left[\frac{\delta\Phi}{\delta(\partial_\mu\phi_a(x))}\phi_a(x) - \frac{\delta\Phi}{\delta(\partial_\mu{\phi_a}^*(x))}{\phi_a}^*(x)\right] \\ &+ \int\limits_{\mathcal{C}} dz\left[iG_a(x,z)\frac{\delta\Phi}{\delta(\partial^x_\mu iG_a(x,z))} - \frac{\delta\Phi}{\delta(\partial^x_\mu iG_a(z,x))}iG_a(z,x)\right]\Bigg),\end{aligned} \tag{B.19}$$

while the remaining terms of $\Theta^{\mu\nu}$ and j^μ retain the same form as that for local coupling (cf. [17]). Here, the variation is also understood in terms of Eq. (B.17) to take account of the $\partial_x^\mu\partial_y^\nu iG_a(y,x)$ dependence of the Φ functional.

The next step to the kinetic description consists in gradient expansion of KB equations and all the related quantities. Expansion of the equations of motion up to the first order in gradients implies that the conserving quantities and self-energies, except for possible memory terms in the collision integral, are required only up to zero order in gradients. These zero-order quantities are determined by the local Φ functional, where all gradient corrections are neglected. Since in the local approximation $\partial_x^\mu iG$, $\partial_y^\nu iG$, and $\partial_x^\mu\partial_y^\nu iG$ transform into $-iq^\mu iG(X,q)$, $iq^\nu iG(X,q)$, and $-iq^\mu iq^\nu iG(X,q)$, respectively, no partial integrations are needed for the variations of Eqs. (B.15), (B.16). This means that conventional variation rules of Eq. (3) still hold in this case. At the same time, derivative contributions to the conserving quantities, Eqs. (B.18) and (B.19), involve only variations over derivatives of the Green's functions and, hence, should be carefully defined within the local approximation for the particular application considered.

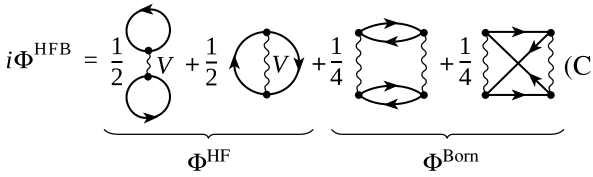
APPENDIX C

Nonrelativistic Nuclear Matter

Currently, calculations of ground-state and low-temperature properties of nuclear matter are performed within the G- or T-matrix approximations to the self-energy [31–33]. Based on realistic nonrelativistic nucleon–nucleon potentials, they quantitatively reproduce phenomenological properties of nuclear matter. However, already for the ground state, the resulting chemical potential, i.e., the single-particle separation energy, deviates from the binding energy per particle, violating the Hugenholtz–van Hove theorem. This is a manifestation of problems with the thermodynamic consistency in these approximations, which gets even worse at nonzero temperatures. This problem was discussed in [31, 34, 35]. A consistent way to overcome this problem consists in using a self-consistent T-matrix approximation [31] based on the Φ-derivable approximation.

Dynamic simulations of nuclear matter are even more demanding to the choice of approximation to the self-energy, because the requirement of charge and energy–momentum conservations should be met except for that of the thermodynamic consistency. Again, all these requirements are met provided the approximation is Φ-derivable. Since dynamic simulations are much more complicated as compared to static ones, up to now they have been performed in a simpler approximation to the self-energy, i.e., the direct Born approximation [5, 7], which provides a qualitative description of the dynamics. These simulations were based on the KB equations without any gradient expansion. Here, we would like to call attention to the fact that the use of the gradient expansion in the KB form (see Section 2C) would simplify these dynamic simulations and, at the same time, preserve the pleasant features of exact conservations and thermodynamic consistency.

In view of a reasonable level of complexity feasible for current computing, we confine our consideration to the full Born approximation to the Φ functional



(C.1)

which includes the Hartree–Fock contribution Φ^{HF} [the first two diagrams in Eq. (C.1)] and the true Born contribution Φ^{Born} (the last two diagrams). Here, the wavy line symbolizes a nonlocal nucleon–nucleon potential $V(|\mathbf{x}_1-\mathbf{x}_2|)$, or $V(|\mathbf{q}|)$ in the momentum representation. For simplicity, below we denote the latter as $V(q)$, keeping in mind that, in fact, it does not depend on either q_0 or direction of $\mathbf{q}$.

Note that the Φ^{Born} part gives rise to the self-energy containing internal vertices. This implies that the corresponding collision term involves nonlocal effects (see discussion in [8]). However, only "spatial nonlocality" appears in the collision term, while the memory in time is absent since $V(|\mathbf{x}_1-\mathbf{x}_2|)\delta(t_1-t_2)$ is time-local. According to the general consideration of [10], exact conservations in the gradient approximation take place if all the nonlocal terms are consistently taken into account up to first-order gradients. Below, we show that, in the particular case of the Φ^{HFB} functional, the exact conservations hold true even if we neglect the spatial nonlocality generated by Φ^{HFB}. These exact conservations imply global rather than local conservation of the energy–momentum, which is in fact natural for the case of instant interaction of finite range considered here.

Neglecting the gradient terms induced by the finite range of V, we consider the Φ^{Born} functional in the local approximation, where all Green's functions in the Wigner representation are taken at the same centroid coordinate X. Alongside some variational expressions, we use an X-dependent local Φ functional, $\Phi(X)$, where the last spatial integration is omitted, i.e.,

$$\Phi=\int dX\Phi(X). \tag{C.2}$$

The Φ^{HFB} of Eq. (C.1) gives rise to the following local collision term:

$$C^{\text{HFB-loc}}=\int\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\frac{d^4p_4}{(2\pi)^4}R^{\mathrm{HFB}} \tag{C.3}$$
$$\times\left(\tilde{F}_1\tilde{F}_2F_3F_4-F_1F_2\tilde{F}_3\tilde{F}_4\right)\delta^4(p_1+p_2-p_3-p_4),$$

$$R^{\mathrm{HFB}}=\frac{(2\pi)^4}{2}\left[V(p_1-p_3)+V(p_1-p_4)\right]^2, \tag{C.4}$$

where $F_1=F(X,p_1)$, etc. [cf. Eq. (17)].

1. Charge Conservation

In Φ^{HFB}, the G^{--} and G^{++} Green's functions are encountered only in different $+-$ Φ diagrams, and hence we can vary G^{--} and G^{++} independently. Therefore, Φ^{HFB} is invariant under the following transformation:

$$G^{--}(X,p)\to G^{--}(X,p+\xi(X)), \tag{C.5}$$
$$G^{++}(X,p)\to G^{++}(X,p-\xi(X)),$$

with F, $\tilde{F}$, and V kept unchanged. Here, $\xi(X)$ is an arbitrary function. If $|\xi(X)|\ll1$, transformation (C.5) reads

$$\delta G^{--}=\xi_\mu(X)\frac{\partial G^{--}}{\partial p_\mu}, \tag{C.6}$$
$$\delta G^{++}=-\xi_\mu(X)\frac{\partial G^{++}}{\partial p_\mu}.$$

Performing variation of Φ^{Born} under the transformation (C.6) within the canonical variation rules (3), we arrive at

$$i\delta\Phi^{\mathrm{loc}}=\int dX\xi_\mu(X)\mathrm{Tr}\int\frac{d^4p}{(2\pi)^4} \tag{C.7}$$
$$\times\left(i\Sigma_{--}\frac{\partial iG^{--}}{\partial p_\mu}-i\Sigma_{++}\frac{\partial iG^{++}}{\partial p_\mu}\right)$$
$$=2i\int dX\xi_\mu(X)\mathrm{Tr}\int\frac{d^4p}{(2\pi)^4}$$
$$\times\left(\Gamma^{\mathrm{in}}\frac{\partial\mathrm{Re}G^R}{\partial p_\mu}+\mathrm{Re}\Sigma^R\frac{\partial F}{\partial p_\mu}\right)=0.$$

Here, we have used the fact that the integral

$$\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(\Gamma\frac{\partial\mathrm{Re}G^R}{\partial p_\mu}+\mathrm{Re}\Sigma^R\frac{\partial A}{\partial p_\mu}\right) \tag{C.8}$$

$$= -\frac{1}{2}\mathrm{Im\,tr}\int\frac{d^4p}{(2\pi)^4}\Sigma^R\frac{\partial G^R}{\partial p_\mu} = 0$$

equals zero due to analyticity of G^R and Σ^R. Thus, we obtain the relation

$$\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(\Gamma^{\mathrm{in}}\frac{\partial \mathrm{Re}G^R}{\partial p_\mu} + \mathrm{Re}\Sigma^R\frac{\partial F}{\partial p_\mu}\right) = 0, \quad \text{(C.9)}$$

which guarantees the Noether current conservation [cf. Eq. (21)] with $j^\mu_{(\mathrm{der})} = 0$.

2. Energy–Momentum Conservation

In order to construct the conservation laws related to spacetime homogeneity, we perform the following transformation:

$$G^{--}(X,p) \to G^{--}\left(X + \xi(X), p\right), \qquad \text{(C.10)}$$
$$G^{++}(X,p) \to G^{++}\left(X - \xi(X), p\right),$$

with F, $\tilde{F}$, and V kept unchanged. This transformation only acts on $\Phi^{\mathrm{HFB}-}$ and $\Phi^{\mathrm{HFB}+}$, i.e., those involving only "$-$" or "$+$" vertices, respectively,

$$\delta\Phi^{\mathrm{HFB}} \qquad \text{(C.11)}$$
$$= \int dX \xi_\mu(X)\partial^\mu\left(\Phi^{\mathrm{HFB}}(X^-) - \Phi^{\mathrm{HFB}}(X^+)\right),$$

where $\Phi^{\mathrm{HFB}}(X^i)$ are understood in the sense of (C.2). Note that

$$i\left(\Phi^{\mathrm{HFB}}(X^-) - \Phi^{\mathrm{HFB}}(X^+)\right) \qquad \text{(C.12)}$$
$$= i\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\Sigma^{\mathrm{HF}}F + \frac{1}{2}i\mathrm{Tr}\int\frac{d^4p}{(2\pi)^4}$$
$$\times\left(\Gamma^{\mathrm{in}}\mathrm{Re}G^R + (\mathrm{Re}\Sigma^R - \Sigma^{\mathrm{HF}})F\right)$$
$$\underbrace{-\,\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(\Gamma\mathrm{Re}G^R + (\mathrm{Re}\Sigma^R - \Sigma^{\mathrm{HF}})A\right)}_{=0},$$

where the last integral is again zero due to analyticity, similar to (C.8). Here, the first term on the right-hand side results from the first two (Hartree–Fock) diagrams in Eq. (C.1), while the last two integrals follow from the last two (Born) diagrams. Alternatively, we can perform variation of Φ^{Born} applying the canonical variation rules (3):

$$i\delta\Phi^{\mathrm{HFB}} = \int dX\xi_\mu(X)\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(i\Sigma_{--}\frac{\partial iG^{--}}{\partial X_\mu} - i\Sigma_{++}\frac{\partial iG^{++}}{\partial X_\mu}\right) = 2i\int dX\xi_\mu(X)\mathrm{tr}\int\frac{d^4p}{(2\pi)^4} \qquad \text{(C.13)}$$
$$\times\left(\Gamma^{\mathrm{in}}\frac{\partial\mathrm{Re}G^R}{\partial X_\mu} + \mathrm{Re}\Sigma^R\frac{\partial F}{\partial X_\mu}\right) - i\underbrace{\int dX\xi_\mu(X)\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(\Gamma\frac{\partial\mathrm{Re}G^R}{\partial X_\mu} + \mathrm{Re}\Sigma^R\frac{\partial A}{\partial X_\mu}\right)}_{=0},$$

where the last integral is again zero due to analyticity. Therefore, we arrive at the important identity

$$\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(\Gamma^{\mathrm{in}}\frac{\partial\mathrm{Re}G^R}{\partial X_\mu} + \mathrm{Re}\Sigma^R\frac{\partial F}{\partial X_\mu}\right) \qquad \text{(C.14)}$$
$$= \partial^\mu\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left[\frac{1}{2}\Sigma^{\mathrm{HF}}F\right.$$
$$\left.+\frac{1}{4}\left(\Gamma^{\mathrm{in}}\mathrm{Re}G^R + (\mathrm{Re}\Sigma^R - \Sigma^{\mathrm{HF}})F\right)\right].$$

Next, we investigate the transformation

$$G^{--}(X,p) \to G^{--}\left(X, \Lambda_{\mu\nu}(X)p^\nu\right), \qquad \text{(C.15)}$$
$$G^{++}(X,p) \to G^{++}\left(X, \Lambda^{-1}_{\mu\nu}(X)p^\nu\right),$$

$$V^-(q) \to V^-(\Lambda_{\mu\nu}(X)q^\nu), \qquad \text{(C.16)}$$
$$V^+(q) \to V^+(\Lambda^{-1}_{\mu\nu}(X)q^\nu)$$

for the entire Φ^{HFB}, while F and $\tilde{F}$ are kept unchanged. For the Hartree–Fock part Φ^{HF}, one finds

$$\tilde{p}_\mu = \Lambda_{\mu\nu}p^\nu, \qquad \text{(C.17)}$$
$$\tilde{p}'_\mu = \Lambda_{\mu\nu}p'^\nu \to d^4pd^4p' = (\det\Lambda)^{-2}d^4\tilde{p}d^4\tilde{p}',$$

$$\delta\Phi^{\mathrm{HF}-} = \int dX\left[(\det\Lambda)^{-2} - 1\right]\Phi^{\mathrm{HF}}(X^-), \quad \text{(C.18)}$$

$$\delta\Phi^{\mathrm{HF}+} = \int dX\left[(\det\Lambda)^{2} - 1\right]\Phi^{\mathrm{HF}}(X^+), \quad \text{(C.19)}$$

where again $\Phi^{\mathrm{HF}-}$ and $\Phi^{\mathrm{HF}+}$ are the Φ^{HF} diagrams involving only "$-$" or "$+$" vertices, respectively. In general, an arbitrary diagram $\Phi^{n_G,n_\lambda(-)}$, consisting of n_G Green's functions, n_λ of "$-$" interactions, and no "$+$" interactions, transforms as

$$\delta\Phi^{n_G,n_\lambda(-)} \qquad \text{(C.20)}$$
$$= \int dX\left[(\det\Lambda)^{-(n_G - n_\lambda + 1)} - 1\right]\Phi^{n_G,n_\lambda(-)}(X),$$

since the change of each momentum integration gives $(\det\Lambda)^{-1}$ [cf. Eq. (C.17)], while the transformation of the $\delta(p)$ function at each vertex (apart from one vertex due to global momentum conservation of the diagram) produces $\det\Lambda$:

$$\delta^4(p+p'-q)=\det\Lambda\,\delta^4(\tilde{p}+\tilde{p}'-\tilde{q}). \tag{C.21}$$

Similarly,

$$\begin{aligned}&\delta\Phi^{n_G,n_\lambda(+)} \\ &=\int dX\left[(\det\Lambda)^{n_G-n_\lambda+1}-1\right]\Phi^{n_G,n_\lambda(+)}(X).\end{aligned} \tag{C.22}$$

According to these rules the "−" and "+" diagrams of the second order in the interaction, $\Phi^{\text{Born}-}$ and $\Phi^{\text{Born}+}$, are transformed as follows:

$$\delta\Phi^{\text{Born}-}=\int dX\left[(\det\Lambda)^{-3}-1\right]\Phi^{\text{Born}-}(X), \tag{C.23}$$

$$\delta\Phi^{\text{Born}+}=\int dX\left[(\det\Lambda)^{3}-1\right]\Phi^{\text{Born}+}(X). \tag{C.24}$$

If the Λ transformation is infinitesimal, $\Lambda_{\mu\nu}(X)=1+\omega_{\mu\nu}(X)$ with $|\omega_{\mu\nu}|\ll 1$ and $\det\Lambda=1+\text{Tr}\omega$, $\det(\Lambda^{-1})=1-\text{tr}\omega$, we obtain

$$\begin{aligned}&\delta\left(\Phi^{\text{HFB}-}+\Phi^{\text{HFB}+}\right) \\ &=-\int dX 2\text{tr}\omega\left(\Phi^{\text{HF}-}(X)-\Phi^{\text{HF}+}(X)\right) \\ &-\int dX 3\text{tr}\omega\left(\Phi^{\text{Born}-}(X)-\Phi^{\text{Born}+}(X)\right)\end{aligned} \tag{C.25}$$

[cf. Eq. (C.12)]. The $\Phi^{\text{Born}-+}$ and $\Phi^{\text{Born}+-}$ components, i.e., those containing both "−" and "+" vertices, are modified by only the V transformation. Moreover, this transformation leaves them invariant,

$$\begin{aligned}&i\delta\Phi^{\text{Born}-+} \\ &=\int dX\omega_{\mu\nu}\text{tr}\int\frac{d^4p}{(2\pi)^4}p^\nu\left(\frac{\delta i\Phi^{\text{Born}-+}(X)}{\delta iV^-}\frac{\partial iV^-}{\partial p_\mu}\right. \\ &\left.-\frac{\delta i\Phi^{\text{Born}-+}(X)}{\delta iV^+}\frac{\partial iV^+}{\partial p_\mu}\right) \\ &=\int dX\omega_{\mu\nu}\text{tr}\int\frac{d^4p}{(2\pi)^4}p^\nu\left(\frac{\delta i\Phi^{\text{Born}-+}(X)}{\delta V^-}V^-\right. \\ &\left.-\frac{\delta i\Phi^{\text{Born}-+}(X)}{\delta V^+}V^+\right)\frac{1}{V}\frac{\partial V}{\partial p_\mu}=0,\end{aligned} \tag{C.26}$$

since $\Phi^{\text{Born}-+}$ is symmetric with respect to V^- and V^+. Thus,

$$\begin{aligned}&i\delta\Phi^{\text{HFB}}=-\int dX(\text{tr}\omega)i\text{tr}\int\frac{d^4p}{(2\pi)^4} \\ &\times\left[2\Sigma^{\text{HF}}F-\frac{3}{2}\left(\Gamma^{\text{in}}\text{Re}G^R+(\text{Re}\Sigma^R-\Sigma^{\text{HF}})F\right)\right].\end{aligned} \tag{C.27}$$

Alternatively, we can perform variation of Φ^{HFB} applying the canonical variation rules (3):

$$\begin{aligned}i\delta\Phi^{\text{HFB}}&=\int dX\omega_{\mu\nu}\left[\text{tr}\int\frac{d^4p}{(2\pi)^4}p^\nu\left(i\Sigma_{--}\frac{\partial iG^{--}}{\partial p_\mu}-i\Sigma_{++}\frac{\partial iG^{++}}{\partial p_\mu}\right)+2iQ^{\mu\nu}(X)\right] \\ &=2i\int dX\omega_{\mu\nu}\left[\text{tr}\int\frac{d^4p}{(2\pi)^4}p^\nu\left(\Gamma^{\text{in}}\frac{\partial\text{Re}G^R}{\partial p_\mu}+\text{Re}\Sigma^R\frac{\partial F}{\partial p_\mu}\right)+Q^{\mu\nu}(X)\right] \\ &\quad\underbrace{-i\int dX\omega_{\mu\nu}\text{tr}\int\frac{d^4p}{(2\pi)^4}p^\nu\left(\Gamma\frac{\partial\text{Re}G^R}{\partial p_\mu}+\text{Re}\Sigma^R\frac{\partial A}{\partial p_\mu}\right)}_{=0},\end{aligned} \tag{C.28}$$

where the last integral again equals zero due to analyticity. Here, we have introduced the quantity

$$\begin{aligned}&2iQ^{\mu\nu}(X)=\text{tr}\int\frac{d^4p}{(2\pi)^4}p^\nu \\ &\times\left(\frac{\delta i\Phi^{\text{HFB}-}(X)}{\delta iV^-}\frac{\partial iV^-}{\partial p_\mu}-\frac{\delta i\Phi^{\text{HFB}+}(X)}{\delta iV^+}\frac{\partial iV^+}{\partial p_\mu}\right)\end{aligned} \tag{C.29}$$

arising from variation over V. All we have to know about this quantity is that $Q^{\mu\nu}=0$ when $\mu=0$ and/or $\nu=0$. This property results from p_0 independence of $V(|\mathbf{p}|)$. In particular, this property yields

$$\int d^3X\partial_\mu Q^{\mu\nu}(X)=0, \tag{C.30}$$

where the X integration runs only over space.

Hence, comparing Eq. (C.28) to Eq. (C.27), we arrive at another important identity:

$$\begin{aligned}&\text{tr}\int\frac{d^4p}{(2\pi)^4}p^\nu\left(\Gamma^{\text{in}}\frac{\partial\text{Re}G^R}{\partial p_\mu}+\text{Re}\Sigma^R\frac{\partial F}{\partial p_\mu}\right) \\ &+Q^{\mu\nu}(X)=-g^{\mu\nu}\text{tr}\int\frac{d^4p}{(2\pi)^4}\Sigma^{\text{HF}}F\end{aligned} \tag{C.31}$$

$$-g^{\mu\nu}\frac{3}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(\Gamma^{\mathrm{in}}\mathrm{Re}G^R+(\mathrm{Re}\Sigma^R-\Sigma^{\mathrm{HF}})F\right).$$

We turn now to the right-hand side of the consistency relation for energy–momentum conservation (22)

$$K^\nu=\mathrm{tr}\int d^3X\frac{p^\nu d^4p}{(2\pi)^4}\left[\{\mathrm{Re}\Sigma^R,F\}-\{\mathrm{Re}G^R,\Gamma^{\mathrm{in}}\}\right] \tag{C.32}$$

integrated over space, which is suitable for the global conservation. In this expression, the local collision term (C.3) drops out according to Eq. (16). It can be transformed by means of the identity

$$\int\frac{d^4p}{(2\pi)^4}p^\nu\{\varphi,f\} \tag{C.33}$$

$$=\int\frac{d^4p}{(2\pi)^4}\left[\partial^\mu\left(p^\nu f\frac{\partial\varphi}{\partial p^\mu}\right)+f\partial^\nu\varphi\right],$$

where φ and f are arbitrary functions, with the result

$$K^\nu=-\mathrm{tr}\int d^3X\partial^\mu\int\frac{d^4p}{(2\pi)^4}p^\nu \tag{C.34}$$

$$\times\left(\mathrm{Re}\Sigma^R\frac{\partial F}{\partial p^\mu}+\Gamma^{\mathrm{in}}\frac{\partial\mathrm{Re}G^R}{\partial p^\mu}\right)$$

$$-\,\mathrm{tr}\int d^3X\frac{d^4p}{(2\pi)^4}\left(\mathrm{Re}\Sigma^R\partial^\nu F+\Gamma^{\mathrm{in}}\partial^\nu\mathrm{Re}G^R\right).$$

Now, applying identities (C.14), (C.30), and (C.31) to the right-hand side of Eq. (C.34), we obtain

$$K^\nu=\int d^3X\partial_\mu g^{\mu\nu}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{1}{2} \tag{C.35}$$

$$\times\left(\Gamma^{\mathrm{in}}\mathrm{Re}G^R+\mathrm{Re}\Sigma^R F\right),$$

which is precisely needed for the global conservation of the Noether energy–momentum

$$\frac{\partial}{\partial T}\int d^3X\Theta^{0\nu}(X)=0, \tag{C.36}$$

since, for the case under consideration, $\mathcal{E}^{\mathrm{int}}=\frac{1}{2}\mathcal{E}^{\mathrm{pot}}$ [cf. Eq. (26)].

APPENDIX D

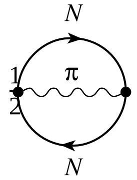
Nucleon–Pion System

For the discussion of the physical aspects of the nucleon–pion problem, we refer to [9]. Here, we would like to clarify some technical details. We choose the nonrelativistic form of pion–nucleon interaction [36]

$$\hat{\mathcal{L}}L^{\mathrm{int}}=g\hat{\psi}^\dagger\left[(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})(\boldsymbol{\tau}\cdot\hat{\boldsymbol{\phi}})\right]\hat{\psi}, \tag{D.1}$$

where $\hat{\psi}$ and $\hat{\boldsymbol{\phi}}$ are nonrelativistic nucleon and Klein–Gordon pion field operators, respectively. Below, subscripts N or π correspondingly attribute a quantity to either nucleon or pion subsystems, respectively. We accept a simple approximation defined by the following Φ functional:

$$i\Phi^{\pi N}=\frac{1}{2}$$



$$=\frac{1}{2}\mathrm{tr}\int dX\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}$$

$$\times\delta^4(p_1-p_2+q)iG^{ij}(X,p_1)(-ig)(-i\mathbf{q}\cdot\sigma)\tau \tag{D.2}$$

$$\times i\Delta_{ij}(X,q)(-ig)(i\mathbf{q}\cdot\sigma)\tau iG^{ji}(X,p_2),$$

where G and Δ are the nucleon and pion Green's functions, respectively, and tr runs over spin and isospin indices. Here, we have assumed an isotopically symmetric system, where the pion Green's functions of all isotopic charges coincide. Though this approximation is evidently oversimplified to produce quantitative results (cf. [9]), it is able to reproduce qualitative features of the dynamics. Moreover, this approximation is at the edge of present computing abilities. The formal basics of the Φ-functional formalism are given in Appendix B.

The charge current, defined by Eqs. (20) and (B.11), relates to the baryon number conservation and hence is trivial from the point of view of the pion–nucleon interaction. Indeed, to prove the baryon number conservation, we should perform transformation (C.5) with F_N, $\tilde{F}_N$, and Δ^{ij} kept unchanged. The pion Green's functions Δ_{ij} are not subjected to this transformation, since pions are neutral from the point of view of baryonic charge. All the subsequent considerations are completely identical to that of the Fock diagram [the second term in Eq. (C.1)] and lead to the same final result (C.9), i.e., to the exact Noether current conservation.

The energy–momentum conservation is more instructive in this respect. Before proceeding to the conservation laws themselves, we should define the derivative contribution to the energy–momentum tensor (B.18). In our case of vanishing mean fields, the pion Green's function enters the Φ functional only doubly differentiated. Therefore, expression (B.18) takes the form

$$\mathcal{E}^{\mu\nu}_{(\mathrm{der})}=\frac{1}{2}\int_{\mathcal{C}}dz\left(\frac{\delta_0\Phi}{\delta_0(\partial^z_\lambda\partial^x_\mu i\Delta(z,x))}\partial^z_\lambda\partial^\nu_x i\Delta(z,x)\right. \tag{D.3}$$

$$\left.+\,\partial^x_\nu\partial^\lambda_z i\Delta(x,z)\frac{\delta_0\Phi}{\delta_0(\partial^x_\mu\partial^z_\lambda i\Delta(x,z))}\right),$$

where δ_0 is already the conventional variation. In the Wigner representation with due regard for Eq. (A.9), it transforms into

$$\begin{aligned}&\mathcal{E}^{\mu\nu}_{(\mathrm{der})}(X) \qquad \text{(D.4)}\\ &= -\mathrm{tr}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\delta^4(p_1-p_2+q)g^2\\ &\times\left[iG^{--}(X,p_1)(-q^\nu\sigma^\mu)\boldsymbol{\tau} i\Delta^{--}(X,q)\right.\\ &\times(i\mathbf{q}\cdot\boldsymbol{\sigma})\boldsymbol{\tau} iG^{--}(X,p_2)-iG^{++}(X,p_1)\\ &\left.\times(-q^\nu\sigma^\mu)\boldsymbol{\tau} i\Delta^{++}(X,q)(i\mathbf{q}\cdot\boldsymbol{\sigma})\boldsymbol{\tau} iG^{++}(X,p_2)\right].\end{aligned}$$

Contrary to usual convention, here we use Greek indices μ and ν for the components of 3-vectors in order to distinguish them from the "$+\,-$" summation indices. The potential energy density is still determined by the standard expression (25) but in terms of the sum over nucleons and pions. Within the approximation of Eq. (D.2), $\mathcal{E}^{\mathrm{pot}}$ can be alternatively expressed as

$$\mathcal{E}^{\mathrm{pot}}(X) = \mathcal{E}^{\mathrm{pot}}_N + \mathcal{E}^{\mathrm{pot}}_\pi = \frac{3}{2}[\Phi^{\pi N-}(X) - \Phi^{\pi N+}(X)], \tag{D.5}$$

where $\Phi^{\pi N-}$ ($\Phi^{\pi N+}$) refers to the $\Phi^{\pi N}$ functional with removed integration over dX and all vertices being of "$-$" ("$+$") type. In view of relation (26),

$$\mathcal{E}^{\mathrm{int}}(X) - \mathcal{E}^{\mathrm{pot}}(X) = -\frac{1}{2}[\Phi^{\pi N-}(X) - \Phi^{\pi N+}(X)]. \tag{D.6}$$

Energy–Momentum Conservation

We briefly repeat the steps proving the exact energy–momentum conservation for nonrelativistic nuclear matter (Appendix C) with the emphasis on the specifics of the derivative coupling.

First, the transformation of Eq. (C.10) for the nucleon Green's functions together with the corresponding transformation of the pion Green's functions

$$\begin{aligned}\Delta^{--}(X,p) &\to \Delta^{--}\left(X+\xi(X),p\right), \qquad \text{(D.7)}\\ \Delta^{++}(X,p) &\to \Delta^{++}\left(X-\xi(X),p\right),\end{aligned}$$

with F_π and $\tilde{F}_\pi$ being kept unchanged, has to be performed. This transformation is unaffected by the derivative coupling, and in a similar way as before, we arrive at the identity

$$\begin{aligned}&\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left[\left(\Gamma^{\mathrm{in}}_N\frac{\partial\mathrm{Re}G^R}{\partial X_\mu}+\mathrm{Re}\Sigma^R\frac{\partial F_N}{\partial X_\mu}\right)\right. \qquad \text{(D.8)}\\ &\left.+\frac{1}{2}\left(\Gamma^{\mathrm{in}}_\pi\frac{\partial\mathrm{Re}\Delta^R}{\partial X_\mu}+\mathrm{Re}\Pi^R\frac{\partial F_\pi}{\partial X_\mu}\right)\right]\\ &= -\partial^\mu(\mathcal{E}^{\mathrm{int}}-\mathcal{E}^{\mathrm{pot}}).\end{aligned}$$

Here, Σ and Π refer to nucleon and pion self-energies, respectively, and subscripts N or π correspondingly attribute a quantity to either nucleon or pion subsystems. The right-hand side of this identity is written with due regards to Eq. (D.6).

Let us now perform the transformation (C.15) for both nucleon and pion Green's functions, as well as q factors encountered in vertices of $\Phi^{\pi N}$. Then, the variation of $\Phi^{\pi N}$ gives

$$\begin{aligned}\delta\Phi^{\pi N} &= \int dX\omega_{\mu\nu} \qquad \text{(D.9)}\\ &\times\left[-2g^{\mu\nu}\left(\Phi^{\pi N-}(X)-\Phi^{\pi N+}(X)\right)+2\mathcal{E}^{\mu\nu}_{(\mathrm{der})}\right]\\ &= \int dX\omega_{\mu\nu}\left[4g^{\mu\nu}\left(\mathcal{E}^{\mathrm{int}}-\mathcal{E}^{\mathrm{pot}}\right)+2\mathcal{E}^{\mu\nu}_{(\mathrm{der})}\right],\end{aligned}$$

where the $\mathcal{E}^{\mu\nu}_{(\mathrm{der})}$ results from the variation of q factors in vertices of $\Phi^{\pi N}$. Alternatively, performing variation of $\Phi^{\pi N}$ according to the canonical variation rules (3) and equating the result to expression (D.9), we arrive at another identity,

$$\begin{aligned}&\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left[\left(\Gamma^{\mathrm{in}}_N\frac{\partial\mathrm{Re}G^R}{\partial p_\mu}+\mathrm{Re}\Sigma^R\frac{\partial F_N}{\partial p_\mu}\right)\right. \qquad \text{(D.10)}\\ &\left.+\frac{1}{2}\left(\Gamma^{\mathrm{in}}_\pi\frac{\partial\mathrm{Re}\Delta^R}{\partial p_\mu}+\mathrm{Re}\Pi^R\frac{\partial F_\pi}{\partial p_\mu}\right)\right]\\ &= 2g^{\mu\nu}(\mathcal{E}^{\mathrm{int}}-\mathcal{E}^{\mathrm{pot}})+\mathcal{E}^{\mu\nu}_{(\mathrm{der})}.\end{aligned}$$

The right-hand side of the consistency relation for energy–momentum conservation (22) now reads

$$\begin{aligned}K^\nu &= \mathrm{tr}\int\frac{p^\nu d^4p}{(2\pi)^4}\Big[(\{\mathrm{Re}\Sigma^R,F_N\} \qquad \text{(D.11)}\\ &-\{\mathrm{Re}G^R,\Gamma^{\mathrm{in}}_N\})+\frac{1}{2}(\{\mathrm{Re}\Pi^R,F_\pi\}-\{\mathrm{Re}\Delta^R,\Gamma^{\mathrm{in}}_\pi\})\Big].\end{aligned}$$

By means of identity (C.33), it is transformed to the form

$$\begin{aligned}K^\nu &= -\partial^\mu\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}p^\nu\left[\left(\mathrm{Re}\Sigma^R\frac{\partial F_N}{\partial p^\mu}\right.\right. \qquad \text{(D.12)}\\ &\left.\left.+\Gamma^{\mathrm{in}}_N\frac{\partial\mathrm{Re}G^R}{\partial p^\mu}\right)+\frac{1}{2}\left(\mathrm{Re}\Pi^R\frac{\partial F_\pi}{\partial p^\mu}+\Gamma^{\mathrm{in}}_\pi\frac{\partial\mathrm{Re}\Delta^R}{\partial p^\mu}\right)\right]\\ &-\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\Big[\left(\mathrm{Re}\Sigma^R\partial^\nu F_N+\Gamma^{\mathrm{in}}_N\partial^\nu\mathrm{Re}G^R\right)\\ &+\frac{1}{2}\left(\mathrm{Re}\Pi^R\partial^\nu F_\pi+\Gamma^{\mathrm{in}}_\pi\partial^\nu\mathrm{Re}\Delta^R\right)\Big].\end{aligned}$$

Now, applying identities (D.8) and (D.10) to the right-hand side of Eq. (D.12), we obtain

$$K^\nu = \partial^\nu\left(\mathcal{E}^{\mathrm{pot}}-\mathcal{E}^{\mathrm{int}}\right)-\partial_\mu\mathcal{E}^{\mu\nu}_{(\mathrm{der})}, \tag{D.13}$$

which is precisely needed for the local conservation of the Noether energy–momentum.

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