Relativistic hydrodynamics with spin

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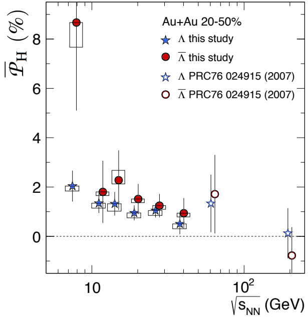
recent works with **F. Becattini, B. Friman, A. Jaiswal, A. Kumar, R. Ryblewski, and E. Speranza**
PRC97 (2018) 041901, PRD97 (2018) 116017, PRC98 (2018) 044906
PRC99 (2019) 011901, PLB789 (2019) 419

Hirschegg 2019, From QCD matter to hadrons
Hirschegg, Kleinwalsertal, Austria, January 13-19, 2019

- **Non-central heavy-ion collisions create fireballs with large global angular momenta** which may generate a spin polarization of the hot and dense matter in a way similar to the Barnett effects
- **Much effort has recently been invested in studies of polarization and spin dynamics of particles produced in high-energy nuclear collisions**, both from the experimental and theoretical point of view
  L. Adamczyk et al. (**STAR**), (2017), **Nature 548 (2017) 62-65**
  Global Λ hyperon polarization in nuclear collisions: evidence for the most vortical fluid
  www.sciencenews.org/article/smashing-gold-ions-creates-most-swirly-fluid-ever



PART 1:

energy-momentum and spin tensors

global and local equilibria

spin polarization and vorticity

HOW CAN WE INCLUDE SPIN POLARIZATION IN A HYDRODYNAMIC FRAMEWORK?
THE LATTER FORMS THE BASIC INGREDIENT OF THE STANDARD MODEL OF HIC

PERFECT-FLUID HYDRODYNAMICS = LOCAL EQUILIBRIUM + CONSERVATION LAWS

one usually includes energy, linear momentum, baryon number, ...

T (temperature), u^μ (three independent components of flow), $\mu = T\xi$ (chemical potential)

FOR PARTICLES WITH SPIN, THE CONSERVATION OF ANGULAR MOMENTUM IS NOT TRIVIAL

new hydrodynamic variables should be introduced

$\Omega_{\mu\nu} = T\omega_{\mu\nu}$ (spin chemical potential = temperature $\times$ spin polarization tensor)

HYDRO WITH SPIN SHOULD EXPLAIN THE SPACE TIME CHANGES OF POLARIZATION
MAY BE NOT POSSIBLE TO EXPLAIN THE ORIGINS OF SPIN POLARIZATION

Canonical energy-momentum $\widehat{T}^{\mu\nu}_{\rm can}$ and angular-momentum $\widehat{J}^{\mu,\lambda\nu}_{\rm can}$ tensors from the Noether Theorem:

$$\partial_\mu \widehat{T}^{\mu\nu}_{\rm can} = 0, \quad \partial_\mu \widehat{J}^{\mu,\lambda\nu}_{\rm can} = 0. \tag{1}$$

In general, the energy-momentum tensor is not symmetric

$$T^{\mu\nu}_{\rm can} \neq T^{\nu\mu}_{\rm can} \tag{2}$$

although classical $T^{\mu\nu}$ is always symmetric

$$T^{\mu\nu}_{\rm class} = \frac{\Delta p^\mu}{\Delta\Sigma_\nu} = \frac{\Delta p^\nu}{\Delta\Sigma_\mu} = T^{\nu\mu}_{\rm class} \quad \text{if} \quad \boldsymbol{v} = \frac{\boldsymbol{p}}{E_p} \quad \text{(1906 Planck)} \tag{3}$$

here $p^\mu = (E_p, \boldsymbol{p})$ is the four-momentum, while $\Delta\Sigma_\nu$ is a space-time volume element

Total angular momentum $\widehat{J}^{\mu,\lambda\nu}_{\rm can}$ has orbital $\widehat{L}^{\mu,\lambda\nu}_{\rm can}$ and spin $\widehat{S}^{\mu,\lambda\nu}_{\rm can}$ parts:

$$\widehat{J}^{\mu,\lambda\nu}_{\rm can} = x^\lambda \widehat{T}^{\mu\nu}_{\rm can} - x^\nu \widehat{T}^{\mu\lambda}_{\rm can} + \widehat{S}^{\mu,\lambda\nu}_{\rm can} \equiv \widehat{L}^{\mu,\lambda\nu}_{\rm can} + \widehat{S}^{\mu,\lambda\nu}_{\rm can}, \tag{4}$$

$$\partial_\mu \widehat{J}^{\mu,\lambda\nu}_{\rm can} = \widehat{T}^{\lambda\nu}_{\rm can} - \widehat{T}^{\nu\lambda}_{\rm can} + \partial_\mu \widehat{S}^{\mu,\lambda\nu}_{\rm can} = 0, \qquad \partial_\mu \widehat{S}^{\mu,\lambda\nu}_{\rm can} = \widehat{T}^{\nu\lambda}_{\rm can} - \widehat{T}^{\lambda\nu}_{\rm can}. \tag{5}$$

Antisymmetry in the last two indices:

$$\widehat{J}^{\mu,\lambda\nu}_{\rm can} = -\widehat{J}^{\mu,\nu\lambda}_{\rm can}, \quad \widehat{L}^{\mu,\lambda\nu}_{\rm can} = -\widehat{L}^{\mu,\nu\lambda}_{\rm can}, \quad \widehat{S}^{\mu,\lambda\nu}_{\rm can} = -\widehat{S}^{\mu,\nu\lambda}_{\rm can} \tag{6}$$

Pseudo-gauge transformation
(different localization of energy density and angular momentum, global charges not changed)

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\lambda\left(\widehat{\Phi}^{\lambda,\mu\nu} - \widehat{\Phi}^{\mu,\lambda\nu} - \widehat{\Phi}^{\nu,\lambda\mu}\right), \tag{7}$$

$$\widehat{S}'^{\lambda,\mu\nu} = \widehat{S}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu}. \tag{8}$$

Belinfante's construction: superpotential defined as $\widehat{\Phi} = \widehat{S}^{\lambda,\mu\nu}_{\rm can}$

$$\widehat{T}^{\mu\nu}_{\rm Bel} = \widehat{T}^{\mu\nu}_{\rm can} + \frac{1}{2}\partial_\lambda\left(\widehat{S}^{\lambda,\mu\nu}_{\rm can} - \widehat{S}^{\mu,\lambda\nu}_{\rm can} - \widehat{S}^{\nu,\lambda\mu}_{\rm can}\right), \quad \widehat{S}^{\lambda,\mu\nu}_{\rm Bel} = 0. \tag{9}$$

in this talk the canonical tensors are considered.
physical system under consideration: hadronic gas (Λ hyperons + ...).
Dirac's equation treated as an effective description of baryons with spin 1/2
no EM fields included

Local-equilibrium density operator (Zubarev, Becattini)

Canonical operators

$$e^{-(E-\mu)/T} \longrightarrow e^{-p\cdot\beta(x)+\xi(x)} \longrightarrow \widehat{\rho}_{\rm LEQ} = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}^{\mu\nu}_{\rm can}\beta_\nu - \frac{1}{2}\omega_{\lambda\nu}\widehat{J}^{\mu,\lambda\nu}_{\rm can} - \xi\widehat{j}^\mu\right)\right] \quad (10)$$

the "orbital part" of $\widehat{J}^{\mu,\lambda\nu}_{\rm can}$ can be moved to the "energy-momentum part", hence we may equivalently use

$$\widehat{\rho}_{\rm LEQ} = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}^{\mu\nu}_{\rm can}\beta_\nu - \frac{1}{2}\omega_{\lambda\nu}\widehat{S}^{\mu,\lambda\nu}_{\rm can} - \xi\widehat{j}^\mu\right)\right] \quad (11)$$

Σ is a space-like hypersurface, for example, corresponding to a constant LAB time t, in this case $\widehat{\rho}_{\rm LEQ} = \widehat{\rho}_{\rm LEQ}(t)$

β_ν is the ratio between the local four-velocity u^μ and temperature T (a four-temperature vector)

an antisymmetric tensor field $\omega_{\lambda\nu}$ is dubbed the **spin polarization tensor**

$$\widehat{\rho}_{\rm LEQ} = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}^{\mu\nu}_{\rm can}\beta_\nu - \frac{1}{2}\omega_{\lambda\nu}\widehat{S}^{\mu,\lambda\nu}_{\rm can} - \xi\widehat{j}^\mu\right)\right] \tag{12}$$

Global equilibrium

in global equilibrium we require that $\widehat{\rho}_{\rm LEQ}$ = const. this implies

$$\partial_\mu \left(\widehat{T}^{\mu\nu}_{\rm can}\beta_\nu - \frac{1}{2}\omega_{\lambda\nu}\widehat{S}^{\mu,\lambda\nu}_{\rm can} - \xi\widehat{j}^\mu\right) = 0 \tag{13}$$

$\widehat{T}^{\mu\nu}_{\rm can} \neq \widehat{T}^{\nu\mu}_{\rm can}$ $\longrightarrow \beta_\mu$ **satisfies the Killing equation**
at the same time, spin polarization is given by thermal vorticity $\varpi_{\lambda\nu}$

$$\partial_\lambda\beta_\nu + \partial_\nu\beta_\lambda = 0, \qquad \partial_\lambda\beta_\nu - \partial_\nu\beta_\lambda \equiv -2\varpi_{\lambda\nu} = -2\omega_{\lambda\nu} = \text{const.}, \quad \xi = \text{const.} \tag{14}$$

solution of the Killing equation:

$$\beta_\mu = \beta^0_\mu + \varpi_{\mu\nu}x^\nu, \quad \beta^0_\mu = \text{const}, \quad \varpi_{\mu\nu} = -\varpi_{\nu\mu} = \text{const} \tag{15}$$

uniform motion, rigid rotation (special boundary conditions)
constant acceleration along the stream lines

Local equilibrium — first option:
based on the works by F. Becattini and collaborators
also in local equilibrium we have $\omega_{\lambda\nu}(x) = \varpi_{\lambda\nu}(x)$

1) Run any type of hydro, perfect or viscous, or transport, or whatsoever, without spin

2) Find $\beta_\mu(x) = u_\mu(x)/T(x)$ on the freeze-out hypersurface
(defined often by the condition T=const)

3) Calculate thermal vorticity $\varpi_{\alpha\beta}(x) \neq$ const

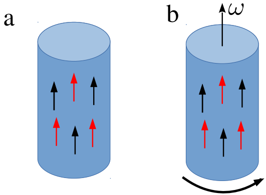
4) Identify thermal vorticity with the spin polarization tensor $\omega_{\mu\nu}$

5) Make predictions about spin polarization

SUCH A METHOD WORKS WELL FOR GLOBAL POLARIZATION (out-of-plane direction),

DOES NOT DESCRIBE THE LONGITUDINAL POLARIZATION

F. Becattini, WF, E. Speranza, Phys. Lett. B789 (2019) 419



1) POLARIZATION CANNOT BE INDUCED INSTANTANEOUSLY

2) POLARIZATION APPROACHES THERMAL VORTICITY THROUGH DISSIPATIVE PROCESSES — DECREASE OF THE ENTROPY OF SPINS MUST BE COMPENSATED BY PRODUCTION OF ENTROPY ELSEWHERE

Local equilibrium — second option (extended local equilibrium):
$\omega_{\lambda\nu}(x)$ and $\varpi_{\lambda\nu}(x)$ are independent, since β and $\omega_{\lambda\nu}(x)$ "control" different densities,

on a timescale smaller than that required to have $\omega_{\lambda\nu}(x) = \varpi_{\lambda\nu}(x)$
one can construct perfect-fluid hydrodynamics with spin
similar ideas by Giorgio Torrieri and Leonardo Tinti

in local equilibrium $\widehat{\rho}_{\rm LEQ}$ is approximately constant (with dissipation effects neglected)

$$T^{\mu\nu} = \mathrm{tr}\left(\widehat{\rho}_{\rm LEQ}\,\widehat{T}^{\mu\nu}\right), \quad S^{\mu,\lambda\nu} = \mathrm{tr}\left(\widehat{\rho}_{\rm LEQ}\,\widehat{S}^{\mu,\lambda\nu}\right), \quad j^{\mu} = \mathrm{tr}\left(\widehat{\rho}_{\rm LEQ}\,\widehat{j}^{\mu}\right). \tag{16}$$

these tensors are all functions of the hydrodynamic variables β_μ, $\omega_{\mu\nu}$, and ξ

$$T^{\mu\nu} = T^{\mu\nu}[\beta,\omega,\xi], \quad S^{\mu,\lambda\nu} = S^{\mu,\lambda\nu}[\beta,\omega,\xi], \quad j^{\mu} = j^{\mu}[\beta,\omega,\xi], \tag{17}$$

and satisfy the conservation laws

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}, \quad \partial_\mu j^{\mu} = 0 \tag{18}$$

PART 2:

equilibrium spin-density distributions and Wigner functions

hydrodynamic equations with spin

Pauli-Lubański vector

Spin dependent phase-space distribution functions 1

standard scalar functions $f(x,p)$ are generalized to 2×2 Hermitean matrices in spin space for each value of the space-time position x and four-momentum p

F. Becattini, V. Chandra, L. Del Zanna, E. Grossi, Annals Phys. 338 (2013) 32

$$\left[f^+(x,p)\right]_{rs} \equiv f^+_{rs}(x,p) = \bar{u}_r(p) X^+ u_s(p), \quad (19)$$

$$\left[f^-(x,p)\right]_{rs} \equiv f^-_{rs}(x,p) = -\bar{v}_s(p) X^- v_r(p). \quad (20)$$

$$X^\pm = \exp\left[\pm\xi(x) - \beta_\mu(x) p^\mu\right] M^\pm, \quad M^\pm = \exp\left[\pm\frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right]. \quad (21)$$

here $\Sigma^{\mu\nu}$ is the Dirac spin operator, electric- and magnetic-like three-vectors

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}. \quad (22)$$

special case in this talk

$$M^\pm = 1 \pm \frac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}. \quad (23)$$

The spin observables are represented by the Pauli matrices σ and the expectation values of σ provide information on the polarization of spin-1/2 particles in their rest frame

$$f^{\pm}(x,p) = e^{\pm\xi - p\cdot\beta}\left[1 - \frac{1}{2}\,\boldsymbol{P}\cdot\boldsymbol{\sigma}\right], \tag{24}$$

average polarization per particle

$$\boldsymbol{P} \quad = \quad \frac{1}{m}\left[E_p\,\boldsymbol{b} - \boldsymbol{p}\times\boldsymbol{e} - \frac{\boldsymbol{p}\cdot\boldsymbol{b}}{E_p+m}\boldsymbol{p}\right] = \boldsymbol{b}_* \quad (\boldsymbol{b}\text{ field in the particle rest frame}) \tag{25}$$

$$\langle \boldsymbol{P}(x,p)\rangle = \frac{1}{2}\,\frac{\mathrm{tr}_2\left[(f^+ + f^-)\boldsymbol{\sigma}\right]}{\mathrm{tr}_2\left[f^+ + f^-\right]} = -\frac{1}{4}\boldsymbol{P}. \tag{26}$$

polarization is measured → we have an access to the components of $\omega_{\mu\nu}$ (possibly integrated over space-time regions)

De Groot, van Leeuwen, van Weert: *Relativistic Kinetic Theory. Principles and Applications*

GLW framework

$$\mathcal{W}^{+}_{\rm eq}(x,k) = \frac{1}{2}\sum_{r,s=1}^{2}\int dP\,\delta^{(4)}(k-p)u^r(p)\bar{u}^s(p)f^{+}_{rs}(x,p), \qquad (27)$$

$$\mathcal{W}^{-}_{\rm eq}(x,k) = -\frac{1}{2}\sum_{r,s=1}^{2}\int dP\,\delta^{(4)}(k+p)v^s(p)\bar{v}^r(p)f^{-}_{rs}(x,p). \qquad (28)$$

Clifford-algebra expansion

(used in many early works on QED and QGP plasma, e.g., H.T. Elze, M. Gyulassy, D. Vasak, Phys.Lett. B177 (1986) 402)

for the equilibrium

$$\mathcal{W}^{\pm}_{\rm eq}(x,k) = \frac{1}{4}\left[\mathcal{F}^{\pm}_{\rm eq}(x,k) + i\gamma_5\mathcal{P}^{\pm}_{\rm eq}(x,k) + \gamma^\mu\mathcal{V}^{\pm}_{{\rm eq},\mu}(x,k) + \gamma_5\gamma^\mu\mathcal{A}^{\pm}_{{\rm eq},\mu}(x,k) + \Sigma^{\mu\nu}\mathcal{S}^{\pm}_{{\rm eq},\mu\nu}(x,k)\right].$$

and any other Wigner function

$$\mathcal{W}^{\pm}(x,k) = \frac{1}{4}\left[\mathcal{F}^{\pm}(x,k) + i\gamma_5\mathcal{P}^{\pm}(x,k) + \gamma^\mu\mathcal{V}^{\pm}_{\mu}(x,k) + \gamma_5\gamma^\mu\mathcal{A}^{\pm}_{\mu}(x,k) + \Sigma^{\mu\nu}\mathcal{S}^{\pm}_{\mu\nu}(x,k)\right].$$

Global-equilibrium Wigner function

WF, A. Kumar, R. Ryblewski, Phys. Rev. C98 (2018) 044906

$$\left(\gamma_\mu K^\mu - m\right)\mathcal{W}(x,k) = C[\mathcal{W}(x,k)]. \tag{29}$$

Here K^μ is the operator defined by the expression

$$K^\mu = k^\mu + \frac{i\hbar}{2}\,\partial^\mu, \tag{30}$$

In the case of global equilibrium, with the vanishing collision term, the Wigner function $\mathcal{W}(x,k)$ exactly satisfies the equation

$$\left(\gamma_\mu K^\mu - m\right)\mathcal{W}(x,k) = 0. \tag{31}$$

the leading order terms in $\hbar$ can be taken from $\mathcal{W}^{\pm}_{\rm eq}(x,k)$

$$\mathcal{F}^{(0)} = \mathcal{F}_{\rm eq}, \tag{32}$$

$$\mathcal{P}^{(0)} = 0, \tag{33}$$

$$\mathcal{V}^{(0)}_\mu = \mathcal{V}_{{\rm eq},\mu}, \tag{34}$$

$$\mathcal{A}^{(0)}_\mu = \mathcal{A}_{{\rm eq},\mu}, \tag{35}$$

$$\mathcal{S}^{(0)}_{\mu\nu} = \mathcal{S}_{{\rm eq},\mu\nu}. \tag{36}$$

in the NLO in $\hbar$ we get the kinetic equation (well known in the literature)

$$k^\mu \partial_\mu \mathcal{F}_{\rm eq}(x,k) = 0, \tag{37}$$

$$k^\mu \partial_\mu \mathcal{A}^\nu_{\rm eq}(x,k) = 0, \quad k_\nu \mathcal{A}^\nu_{\rm eq}(x,k) = 0, \tag{38}$$

Global equilibrium — Eq. (37) and Eq. (38) are exactly fulfilled, therefore

$$\partial_\lambda \beta_\nu + \partial_\nu \beta_\lambda = 0, \qquad \omega_{\lambda\nu} = \text{const.}, \quad \xi = \text{const.} \tag{39}$$

$$\beta_\mu = \beta^0_\mu + \Omega^0_{\mu\nu} x^\nu, \quad \beta^0_\mu = \text{const.}, \quad \Omega^0_{\mu\nu} = -\Omega^0_{\nu\mu} = \text{const.}, \quad \Omega^0_{\mu\nu} \neq \omega_{\mu\nu} \tag{40}$$

POLARIZATION NOT RELATED TO THERMAL VORTICITY!
we call this state an EXTENDED GLOBAL EQUILIBRIUM

Local equilibrium — only moments of Eq. (37) and Eq. (38) are satisfied

$$\int d^4k\, k^\alpha \partial_\alpha \mathcal{F}_{\rm eq}(x,k) = 0, \quad \int d^4k\, k^\beta\, k^\alpha \partial_\alpha \mathcal{F}_{\rm eq}(x,k) = 0 \tag{41}$$

$$\longrightarrow \quad \partial_\alpha N^\alpha_{\rm GLW}(x) = 0, \quad \partial_\alpha T^{\alpha\beta}_{\rm GLW}(x) = 0 \tag{42}$$

$$\int d^4k\, \epsilon^{\mu\beta\gamma\delta} k_\beta\, k^\lambda \partial_\lambda\, \mathcal{A}^\nu_{\rm eq}(x,k) = 0 \tag{43}$$

$$\longrightarrow \quad \partial_\lambda S^{\lambda,\mu\nu}_{\rm GLW}(x) = 0. \tag{44}$$

GLW — forms proposed in the textbook on the kinetic theory by
de Groot - van Leeuwen - van Weert

charge current

$$N^{\alpha}_{\rm GLW} = n u^{\alpha}, \qquad n = 4\sinh(\xi)\, n_{(0)}(T) = 2\left(e^{\xi} + e^{-\xi}\right) n_{(0)}(T) \tag{45}$$

energy-momentum tensor (with a perfect-fluid form)

$$T^{\mu\nu}_{\rm GLW}(x) = (\varepsilon + P) u^{\mu} u^{\nu} - P g^{\mu\nu}, \tag{46}$$

$$\varepsilon = 4\cosh(\xi)\, \varepsilon_{(0)}(T), \quad P = 4\cosh(\xi)\, P_{(0)}(T), \tag{47}$$

$n_{(0)}(T)$, $\varepsilon_{(0)}(T)$, $P_{(0)}(T)$ — particle density, energy density, and pressure of classical particles at the temperature T

spin tensor

$$S^{\lambda,\mu\nu}_{\rm GLW} = \frac{\hbar\cosh(\xi)}{m^2} \int dP\, e^{-\beta\cdot p} p^{\lambda} \left(m^2 \omega^{\mu\nu} + 2 p^{\alpha} p^{[\mu} \omega^{\nu]}{}_{\alpha}\right) = S^{\lambda,\mu\nu}_{\rm ph} + S^{\lambda,\mu\nu}_{\Delta}. \tag{48}$$

only $S^{\lambda,\mu\nu}_{\rm ph}$ was used in WF, B. Friman, A. Jaiswal, E.Speranza, Phys.Rev. C97 (2018) 041901

LO generates corrections in the NLO!

$$\mathcal{P}^{(1)} = -\frac{1}{2m}\,\partial^\mu \mathcal{A}_{\mathrm{eq},\mu}, \tag{49}$$

$$\mathcal{V}^{(1)}_\mu = -\frac{1}{2m}\partial^\nu \mathcal{S}_{\mathrm{eq},\nu\mu}, \tag{50}$$

$$\mathcal{S}^{(1)}_{\mu\nu} = \frac{1}{2m}\left(\partial_\mu \mathcal{V}_{\mathrm{eq},\nu} - \partial_\nu \mathcal{V}_{\mathrm{eq},\mu}\right). \tag{51}$$

they are IMPORTANT IF the canonical formalism is used

$$T^{\mu\nu}_{\mathrm{GLW}}(x) = \frac{1}{m}\mathrm{tr}_4 \int d^4k\, k^\mu\, k^\nu \mathcal{W}(x,k) = \frac{1}{m}\int d^4k\, k^\mu\, k^\nu \mathcal{F}(x,k). \tag{52}$$

$$T^{\mu\nu}_{\mathrm{can}}(x) = \int d^4k\, k^\nu \mathcal{V}^\mu(x,k) \tag{53}$$

quantum corrections induce asymmetry $T^{\mu\nu}_{\mathrm{can}}(x) \neq T^{\nu\mu}_{\mathrm{can}}(x)$

Including the components of $\mathcal{V}^\mu(x,k)$ up to the first order in the equilibrium case we obtain

$$T^{\mu\nu}_{\rm can}(x) = T^{\mu\nu}_{\rm GLW}(x) + \delta T^{\mu\nu}_{\rm can}(x) \tag{54}$$

where

$$\delta T^{\mu\nu}_{\rm can}(x) = -\frac{\hbar}{2m}\int d^4k\, k^\nu \partial_\lambda \mathcal{S}^{\lambda\mu}_{\rm eq}(x,k) = -\partial_\lambda S^{\nu,\lambda\mu}_{\rm GLW}(x). \tag{55}$$

The canonical energy-momentum tensor is conserved

$$\partial_\alpha T^{\alpha\beta}_{\rm can}(x) = 0. \tag{56}$$

$$\begin{aligned} S^{\lambda,\mu\nu}_{\rm can} &= \hbar\cosh(\xi)\int dP\, e^{-\beta\cdot p}\left(\omega^{\mu\nu}p^\lambda + \omega^{\nu\lambda}p^\mu + \omega^{\lambda\mu}p^\nu\right) \\ &\equiv \hbar\cosh(\xi)n_{(0)}(T)\left(u^\lambda\omega^{\mu\nu} + u^\mu\omega^{\nu\lambda} + u^\nu\omega^{\lambda\mu}\right) = S^{\lambda,\mu\nu}_{\rm GLW} + S^{\mu,\nu\lambda}_{\rm GLW} + S^{\nu,\lambda\mu}_{\rm GLW}. \end{aligned} \tag{57}$$

The canonical spin tensor is NOT conserved

$$\partial_\lambda S^{\lambda,\mu\nu}_{\rm can}(x) = T^{\nu\mu}_{\rm can} - T^{\mu\nu}_{\rm can} = -\partial_\lambda S^{\mu,\lambda\nu}_{\rm GLW}(x) + \partial_\lambda S^{\nu,\lambda\mu}_{\rm GLW}(x). \tag{58}$$

if we introduce the tensor $\Phi_{\rm can}^{\lambda,\mu\nu}$ defined by the relation

$$\Phi_{\rm can}^{\lambda,\mu\nu} \equiv S_{\rm GLW}^{\mu,\lambda\nu} - S_{\rm GLW}^{\nu,\lambda\mu}, \tag{59}$$

we can write

$$S_{\rm can}^{\lambda,\mu\nu} = S_{\rm GLW}^{\lambda,\mu\nu} - \Phi_{\rm can}^{\lambda,\mu\nu} \tag{60}$$

and

$$T_{\rm can}^{\mu\nu} = T_{\rm GLW}^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\Phi_{\rm can}^{\lambda,\mu\nu} + \Phi_{\rm can}^{\mu,\nu\lambda} + \Phi_{\rm can}^{\nu,\mu\lambda}\right). \tag{61}$$

The canonical and GLW frameworks are connected by a pseudo-gauge transformation. Similarly to Belinfante, it leads to a symmetric energy-momentum tensor, however, does not eliminate the spin degrees of freedom.

we introduce the phase-space density $\Delta\Pi_\mu$ of the **Pauli-Lubański vector**

$$E_p \frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\Delta\Sigma_\lambda(x)\,E_p\frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p}\,\frac{p^\beta}{m}. \tag{62}$$

only the spin-part contributes here, the results are the same for the canonical and GLW versions

$$\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}E_p\frac{dS^{\lambda,\nu\alpha}(x,p)}{d^3p} = \frac{\hbar\cosh(\xi)}{(2\pi)^3}\,e^{-p\cdot\beta}p^\lambda\tilde{\omega}_{\mu\beta}. \tag{63}$$

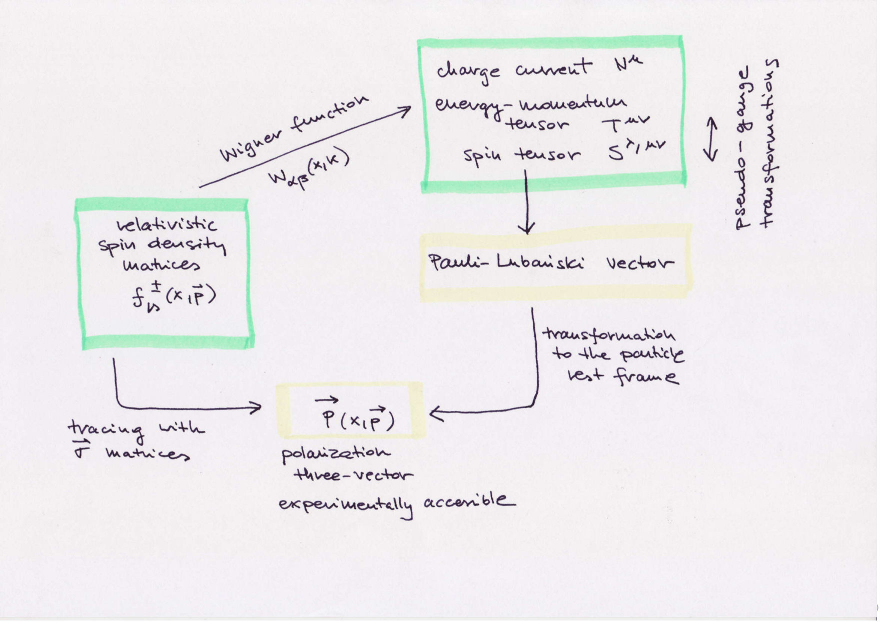
dividing by the total density of particles and antiparticles, we find

$$\pi_\mu(x,p) \equiv \frac{\Delta\Pi_\mu(x,p)}{\Delta\mathcal{N}(x,p)} = -\frac{\hbar}{4m}\,\tilde{\omega}_{\mu\beta}\,p^\beta. \tag{64}$$

in PRF

$$\pi_*^0 = 0, \qquad \boldsymbol{\pi}_* = -\frac{\hbar}{4}\boldsymbol{P}\,. \tag{65}$$

This is an important result showing that the space part of the PL vector in PRF agrees with the mean polarization three-vector


charge current N^μ
energy-momentum tensor $T^{\mu\nu}$
spin tensor $S^{\lambda,\mu\nu}$
Wigner function
$W_{\alpha\beta}(x,k)$
pseudo-gauge transformations
relativistic spin density matrices
$f_{rs}^{\pm}(x,\vec{p})$
Pauli-Lubanski vector
transformation to the particle rest frame
tracing with $\vec{\sigma}$ matrices
$\vec{P}(x,\vec{p})$
polarization three-vector
experimentally accessible

Summary 2

1. Still much work to do, e.g., one has to include mean EM and other fields, spin-orbit interaction, dissipation, ...
2. The arguments collected in our works suggest using **the de Groot - van Leeuwen - van Weert (GLW) forms of the energy-momentum and spin tensors**, together with their conservation laws, as the building blocks for construction of hydrodynamics with spin → **Avdhesh Kumar's talk**
3. **Using the classical concept of spin one can formulate a consistent framework of hydrodynamics with spin**, which for small values of the polarization agrees with the approach using relativistic spin-density matrices → **Radek Ryblewski's talk**

more in a mini-review: arXiv:1811.04409

Quark Matter 2021 in Kraków, Oct. 4-9, 2021, ICE Congress Center