GOETHE UNIVERSITÄT FRANKFURT AM MAIN

CRC–TR 211

Solution of the Wigner function up to order $\hbar$ and its application to hydrodynamics

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Outline

Massless fermions

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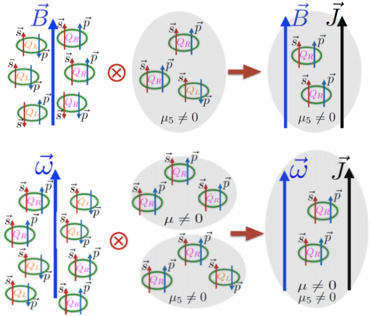
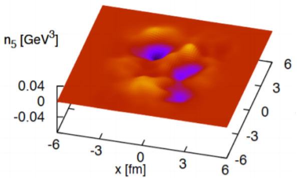
- Spin parallel/anti-parallel to momentum.
- **Chiral Magnetic Effect (CME)**
  - Electric current along **B**.
  - Non-vanishing chiral chemical potential.
  - Observed in Dirac semimetal: Q. Li, D. E. Kharzeev, et.al.,Nature Phys. 12 (2016) 550-554
- **Chiral Vortical Effect (CVE)**
  Y. Hirono, T. Hirano, and D. E. Kharzeev, arXiv:1412.0311



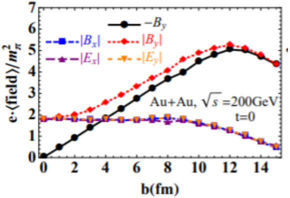
D.E. Kharzeev, J. Liao, S.A. Voloshin, and G. Wang, Prog.Part.Nucl.Phys. 88 (2016) 1-28

Chiral effects in heavy-ion collisions

- Mass of u, d quarks $\ll T$ of quark-gluon plasma
- Axial charge generated from
  - Topological fluctuations of gluonic sector
    D. Kharzeev, A. Krasnitz, and R. Venugopalan, Phys.Lett. B545 (2002) 298-306
  - Nonzero $\mathbf{E} \cdot \mathbf{B}$ in initial plasma
    Y. Hirono, T. Hirano, and D. E. Kharzeev, arXiv:1412.0311
- Magnetic field $B \sim m_\pi^2$ for 200GeV/A Au-Au collisions.
- Angular momentum $L \sim 10^5\hbar$ for 200GeV/A Au-Au collisions at $b = 5$fm.
  F. Becattini, F. Piccinini, and J. Rizzo, Phys.Rev. C77 (2008) 024906

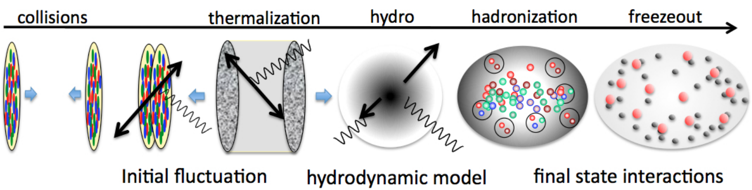


Y. Hirono, T. Hirano, and D. E. Kharzeev, arXiv:1412.0311



W-T. Deng and X-G. Huang, Phys.Rev. C85 (2012) 044907

Quantum transport



Relativistic Hydrodynamics

massive, w/o spin-orbit and spin-magnetic-field interactions

Chiral Kinetic Theory

massless

- Kinetic description for massive spin-1/2 particles?
- Recent works based on local spin-dependent distribution functions:
  F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, Annals Phys. 338 (2013) 32-49,
  W. Florkowski, B. Friman, A. Jaiswal, and E. Speranza, Phys.Rev. C97 (2018) no.4, 041901,
  W. Florkowski, A. Kumar, and R. Ryblewski, Phys.Rev. C98 (2018) no.4, 044906.

Wigner function: definition

- Wigner function

$$W_{\alpha\beta}(x,p)=\int\frac{d^4y}{(2\pi)^4}e^{-\frac{i}{\hbar}p\cdot y}\left\langle :\bar{\psi}_\beta\left(x+\frac{y}{2}\right)U\left(x+\frac{y}{2},x-\frac{y}{2}\right)\psi_\alpha\left(x-\frac{y}{2}\right):\right\rangle . \tag{1}$$

- Gauge link

$$U\left(x+\frac{y}{2},x-\frac{y}{2}\right)=\exp\left[-\frac{i}{\hbar}y^\mu\int_{-1/2}^{1/2}ds\ \mathbb{A}_\mu(x+sy)\right] . \tag{2}$$

- Decomposed in terms of gamma matrices

$$W=\frac{1}{4}\left\{\mathcal{F}+i\gamma^5\mathcal{P}+\gamma^\mu\mathcal{V}_\mu+\gamma^5\gamma^\mu\mathcal{A}_\mu+\frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right\} , \tag{3}$$

where the 16 coefficients are real functions of $\{x,p\}$.

- Transport equation

$$\left[\gamma_\mu\left(\Pi^\mu+i\frac{\hbar}{2}\nabla^\mu\right)-m\right]W(x,p)=0 , \tag{4}$$

where $\nabla_\mu\equiv\partial_\mu-j_0(\frac{\hbar}{2}\triangle)F_{\mu\nu}\partial_p^\nu$, $\Pi_\mu=p_\mu-\frac{\hbar}{2}j_1(\frac{\hbar}{2}\triangle)F_{\mu\nu}\partial_p^\nu$ and $\Delta\equiv\partial_x\cdot\partial_p$.
Spherical Bessel functions j_0, j_1.
H. T. Elze, M. Gyulassy, and D. Vasak, Nucl. Phys. B276, 706 (1986)

Semi-classical expansion and solution

- Semi-classical expansion in $\hbar$,
- valid if (i) $\hbar\gamma^\mu\nabla_\mu W \ll mW$ and (ii) $\hbar \ll \Delta R \Delta P$.

Up to order $\hbar$:

$$\begin{aligned}
\mathcal{F} &= m\left[V\delta(p^2-m^2) - \frac{\hbar}{2}F^{\mu\nu}\Sigma_{\mu\nu}A\delta'(p^2-m^2)\right] + \mathcal{O}(\hbar^2),\\
\mathcal{P} &= \frac{\hbar}{4m}\epsilon^{\mu\nu\alpha\beta}\nabla_\mu\left[p_\nu\Sigma_{\alpha\beta}A\delta(p^2-m^2)\right] + \mathcal{O}(\hbar^2),\\
\mathcal{V}^\mu &= \delta(p^2-m^2)\left[p^\mu V + \frac{\hbar}{2}\nabla_\nu(\Sigma^{\mu\nu}A)\right],\\
&\quad -\hbar\left(\frac{1}{2}p^\mu F^{\alpha\beta}\Sigma_{\alpha\beta} + \Sigma^{\mu\nu}F_{\nu\alpha}p^\alpha\right)A\delta'(p^2-m^2) + \mathcal{O}(\hbar^2),\\
\mathcal{A}^\mu &= -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}p_\nu\Sigma_{\alpha\beta}A\delta(p^2-m^2) + \hbar\tilde{F}^{\mu\nu}p_\nu V\delta'(p^2-m^2) + \mathcal{O}(\hbar^2),\\
\mathcal{S}^{\mu\nu} &= m\left[\Sigma^{\mu\nu}A\delta(p^2-m^2) - \hbar F^{\mu\nu}V\delta'(p^2-m^2)\right] + \mathcal{O}(\hbar^2),
\end{aligned} \tag{5}$$

where V, A, and $\Sigma_{\mu\nu}$ are functions of phase-space coordinates $\{x^\mu, p^\mu\}$.
see Nora Weickgenannt's presentation

Semi-classical expansion and solution

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- Meaning of functions V and A can be obtained by comparing our solution with the one from first-principles calculation,

$$V = \frac{2}{(2\pi\hbar)^3} \sum_s \left[\theta(p^0) f_s^+(x, \mathbf{p}) + \theta(-p^0) f_s^-(x, -\mathbf{p})\right],$$

$$A = \frac{2}{(2\pi\hbar)^3} \sum_s s \left[\theta(p^0) f_s^+(x, \mathbf{p}) - \theta(-p^0) f_s^-(x, -\mathbf{p})\right], \quad (6)$$

  where s denotes helicity (massless) or spin-up/spin-down along given quantization direction (massive).
- Distributions $f_s^\pm$, as well as functions V and A can contain contributions of arbitrary order in $\hbar$.

Fluid-dynamical quantities

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- Inserting solution for Wigner function into J^μ and $T^{\mu\nu}_{mat}$, we obtain
- Net particle-number current:

$$J^\mu = \int dP\ p^\mu V + \frac{\hbar}{2}\partial_{x\nu}\int dP\ \Sigma^{\mu\nu}A + \frac{\hbar}{4}F^{\alpha\beta}\int dP\ \partial_p^\mu(\Sigma_{\alpha\beta}A) + \mathcal{O}(\hbar^2). \tag{7}$$

- Canonical energy-momentum tensor:

$$T^{\mu\nu}_{mat} = \int dP\ p^\mu p^\nu V + \frac{\hbar}{2}\partial_{x\alpha}\int dP\ p^\nu \Sigma^{\mu\alpha}A + \frac{\hbar}{4}g^{\mu\nu}F^{\alpha\beta}\int dP\ \Sigma_{\alpha\beta}A - \frac{\hbar}{2}F^\nu{}_\alpha\int dP\ \Sigma^{\mu\alpha}A + \frac{\hbar}{4}F^{\alpha\beta}\int dP\ p^\nu\partial_p^\mu(\Sigma_{\alpha\beta}A) + \mathcal{O}(\hbar^2). \tag{8}$$

Canonical energy-momentum tensor is in general not symmetric.

- $dP \equiv d^4p\,\delta(p^2 - m^2)$.

Truncation

- Consider only first two leading orders in $\hbar$.
  $\rightarrow$ Functions contributing to J^{μ} and $T^{\mu\nu}_{mat}$:

$$V^{(0)} + \hbar V^{(1)}, \quad A^{(0)}, \quad \Sigma^{(0)\mu\nu},$$

  where superscripts (0) and (1) label different orders in $\hbar$.
- $\Sigma^{(0)\mu\nu}$: anti-symmetric, normalized $\Sigma^{(0)}_{\mu\nu}\Sigma^{(0)\mu\nu} = 2$, $\Sigma^{(0)\mu\nu} p_{\nu} = 0$.
  $\rightarrow$ two independent degrees of freedom.

$$\Sigma^{(0)\mu\nu} \equiv -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta} p_{\alpha} n^{(0)}_{\beta} \,. \tag{9}$$

- $n^{(0)\mu}$: normalized $n^{(0)} \cdot n^{(0)} = -1$, $n^{(0)} \cdot p = 0$.
  $\rightarrow$ Space-like unit vector along the direction of polarization density, since $\mathcal{A}^{(0)} = mn^{(0)\mu} A^{(0)} \delta(p^2 - m^2)$.

Kinetic equations for $V^{(0)} + \hbar V^{(1)}$, $A^{(0)}$, and $n^{(0)\mu}$

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Kinetic equations

$$
\begin{aligned}
0 &= p_\mu(\partial_x^\mu - F^{\mu\nu}\partial_{p\nu})n^{(0)\alpha} - F^{\alpha\beta} n_\beta^{(0)}, \\
0 &= p_\mu(\partial_x^\mu - F^{\mu\nu}\partial_{p\nu})A^{(0)}, \\
0 &= p_\mu(\partial_x^\mu - F^{\mu\nu}\partial_{p\nu})\left[V^{(0)} + \hbar V^{(1)}\right] + \frac{\hbar}{4}(\partial_{x\alpha} F^{\mu\nu})\partial_p^\alpha(\Sigma_{\mu\nu}^{(0)} A^{(0)}). \quad (10)
\end{aligned}
$$

- Coincide with Bargmann-Michel-Telegdi (BMT) equation and collisionless Boltzmann-Vlasov equation.
- On-mass shell $p^0 = \pm\sqrt{\mathbf{p}^2 + m^2}$.
- Contains contributions from particles/anti-particles.
- $V^{(0)}$ and $V^{(1)}$ do not influence the evolution of $A^{(0)}$ and $n^{(0)\mu}$.
- Solve these equations: $n^{(0)\mu}$ and $A^{(0)} \to \Sigma_{\mu\nu}^{(0)} \to V^{(0)} + \hbar V^{(1)}$.
- Collision terms.

Longitudinally polarized

- Purely longitudinally polarized:

$$n^{(0)\mu} = \frac{1}{m}\sqrt{\frac{(u_{lab}\cdot p)^2}{(u_{lab}\cdot p)^2 - m^2}}\left(p^\mu - \frac{m^2}{u_{lab}\cdot p}u^\mu_{lab}\right) ,$$

$$\Sigma^{(0)\mu\nu} = \frac{\mathrm{sgn}(u_{lab}\cdot p)}{\sqrt{(u_{lab}\cdot p)^2 - m^2}}\epsilon^{\mu\nu\alpha\beta}p_\alpha u_{lab,\beta} , \quad (11)$$

spin $\parallel$ to momentum $\mathbf{p}$ in lab frame u^μ_{lab}.

- Substituting into BMT equation, we obtain

Case 1:

$m = 0$.
Massless case. $m\,n^{(0)\mu} = p^\mu$.

Longitudinally polarized

- Purely longitudinally polarized:

$$n^{(0)\mu} = \frac{1}{m}\sqrt{\frac{(u_{lab}\cdot p)^2}{(u_{lab}\cdot p)^2 - m^2}}\left(p^\mu - \frac{m^2}{u_{lab}\cdot p}u^\mu_{lab}\right) ,$$
$$\Sigma^{(0)\mu\nu} = \frac{\mathrm{sgn}(u_{lab}\cdot p)}{\sqrt{(u_{lab}\cdot p)^2 - m^2}}\epsilon^{\mu\nu\alpha\beta}p_\alpha u_{lab,\beta} , \tag{11}$$

spin $\parallel$ to momentum **p** in lab frame u^μ_{lab}.

- Substituting into BMT equation, we obtain

Case 1:

$m = 0$.
Massless case. $m\, n^{(0)\mu} = p^\mu$.

Case 2:

$\partial^\mu u^\nu_{lab} = 0$ and $F^{\mu\nu}u_{lab,\nu} = 0$.
Constant u^μ_{lab} and vanishing electric field in lab frame.

Equilibrium distributions

- Assume distribution functions V and A take equilibrium form

$$\begin{aligned} V &= \frac{2}{(2\pi\hbar)^3}\sum_s \left[\theta(u\cdot p)f_s^+(x,p)+\theta(-u\cdot p)f_s^-(x,-p)\right], \\ A &= \frac{2}{(2\pi\hbar)^3}\sum_s s\left[\theta(u\cdot p)f_s^+(x,p)+\theta(-u\cdot p)f_s^-(x,p)\right], \end{aligned} \quad (12)$$

where

$$f_s^r(x,p)=\left\{\exp\left[r\frac{1}{T}(u\cdot p-\mu-s\mu_5)+rs\frac{\hbar}{4}\omega^{\mu\nu}\Sigma_{\mu\nu}^{(0)}\right]+1\right\}^{-1}. \quad (13)$$

- Fluid four-velocity u^μ, temperature T, chemical potential μ, and chiral chemical potential μ_5 are assumed to be x-dependent.
- Term proportional to $\hbar$ represents spin-vorticity coupling. Thermal vorticity tensor:

$$\omega^{\mu\nu}\equiv\frac{1}{2}\left(\partial^\mu\frac{u^\nu}{T}-\partial^\nu\frac{u^\mu}{T}\right). \quad (14)$$

Constraints for the distributions

- Substituting distributions into kinetic equations we obtain constraints,

$$\begin{aligned} \partial_\mu \omega^{\mu\nu} &= 0, \\ \partial^\mu \frac{u^\nu}{T} + \partial^\nu \frac{u^\mu}{T} &= 0, \\ F^{\mu\nu} u_\nu - T \partial^\mu \frac{\mu}{T} &= 0, \\ \partial^\mu \frac{\mu_5}{T} &= 0. \end{aligned} \tag{15}$$

Particle-number current

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- Inserting dipole-moment tensor for longitudinally polarized case and equilibrium distributions into particle-number current results in

$$J^\mu = n\, u^\mu + \xi \omega^\mu + \xi_B B^\mu \,, \tag{16}$$

where thermal vorticity vector $\omega^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu \omega_{\alpha\beta}$.

- Assuming $u^\mu \simeq u^\mu_{lab} + T\omega^{\mu\nu} x_\nu$, expand to linear order in ω^μ and $F^{\mu\nu}$:

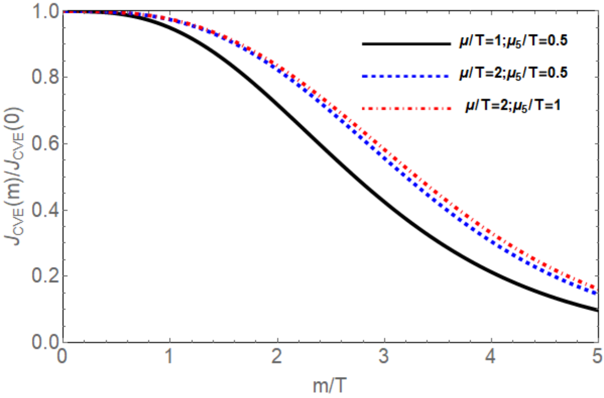
$$\begin{aligned} n &= \frac{1}{2\pi^2\hbar^3} \sum_{rs} r \int_0^\infty dp\, p^2\, f^{(0)r}_{eq,s}\,, \\ \xi &= \frac{1}{2\pi^2\hbar^2} T \sum_{rs} s \int_0^\infty dp\, p\, f^{(0)r}_{eq,s}\,, \\ \xi_B &= \frac{1}{\pi^2\hbar^2}\mu_5 + \frac{1}{4\pi^2\hbar^2} T \sum_{rs} rs \ln\left[1 + \exp\left(\frac{m - r\mu - rs\mu_5}{T}\right)\right], \end{aligned} \tag{17}$$

where

$$f^{(0)r}_{eq,s} \equiv \left\{\exp\left[\left(\sqrt{p^2+m^2} - r\mu - rs\mu_5\right)/T\right] + 1\right\}^{-1} . \tag{18}$$

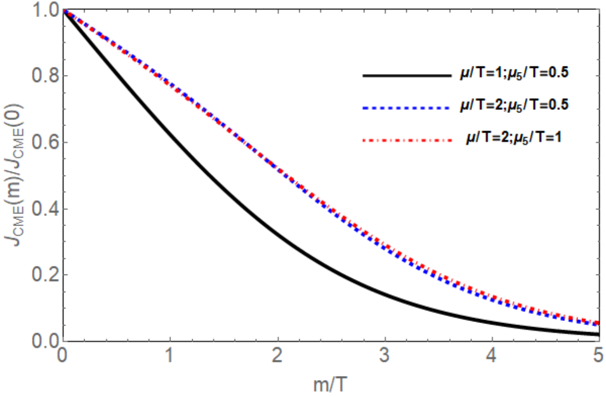
Chiral Vortical Effect

- CVE coefficient ξ



Chiral Magnetic Effect

- CME coefficient ξ_B



Energy-momentum tensor

- Up to linear order in $\hbar$, ω^μ and B^μ

$$T^{\mu\nu}_{mat} = (\varepsilon + P)u^\mu u^\nu - P g^{\mu\nu} + (u^\nu \omega^\mu + u^\mu \omega^\nu)\zeta_\omega + (u^\nu B^\mu + u^\mu B^\nu)\zeta_B + \frac{1}{2}(u^\nu \omega^\mu - u^\mu \omega^\nu)\zeta_\omega \,, \quad (19)$$

where

$$\varepsilon = \frac{1}{2\pi^2\hbar^3}\sum_{rs}\int_0^\infty dp\, p^2 \sqrt{p^2+m^2}\, f^{(0)r}_{eq,s} \,,$$
$$P = \frac{1}{6\pi^2\hbar^3}\sum_{rs}\int_0^\infty dp \frac{p^4}{\sqrt{p^2+m^2}} f^{(0)r}_{eq,s} \,,$$
$$\zeta_\omega = \frac{1}{6\pi^2\hbar^2}\sum_{rs} rs \int_0^\infty dp\, p^3\, f^{(0)r}_{eq,s}\left[1 - f^{(0)r}_{eq,s}\right] \,,$$
$$\zeta_B = \frac{1}{4\pi^2\hbar^2}\sum_{rs} s \int_0^\infty dp\, p\, f^{(0)r}_{eq,s} \,. \quad (20)$$

- $m = 0 \rightarrow$ coincide with D-L. Yang, Phys.Rev. D98 (2018) no.7, 076019 .
- $\hbar = 0$ or $\mu_5 = 0 \rightarrow \zeta_\omega = \zeta_B = 0 \rightarrow$ spinless ideal fluid.

Summary & Outlook

- Summary
  - Presented formal solution of Wigner function up to order $\hbar$
  - Computed order–$\hbar$ corrections to fluid-dynamical quantities.
  - Derived kinetic equations for undetermined functions.
  - Presented specific solution which smoothly recover massless limit.
- Outlook
  - Dynamical calculations.
  - Contribution from collisions.

Kinetic equations

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- Kinetic equations

$$\begin{aligned} p\cdot\nabla V+\frac{\hbar}{4}(\partial_{x\alpha}F^{\mu\nu})\partial_p^\alpha(\Sigma_{\mu\nu}A)+\mathcal{O}(\hbar^2) &= 0, \\ p\cdot\nabla(\Sigma^{\mu\nu}A)-F^{\alpha[\mu}\Sigma^{\nu]}{}_\alpha A+\frac{\hbar}{2}(\partial_{x\alpha}F^{\mu\nu})\partial_p^\alpha V+\mathcal{O}(\hbar^2) &= 0, \end{aligned} \tag{21}$$

On-shell momentum $p^2-m^2=0$.

- Constraint for dipole-moment tensor

$$p^\nu\Sigma_{\mu\nu}A=\frac{\hbar}{2}\nabla_\mu V, \tag{22}$$

where $\nabla_\mu\equiv\partial_{x\mu}-F_{\mu\nu}\partial_p^\nu+\mathcal{O}(\hbar^2)$.

Number current and energy-momentum tensor

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- Lagrangian for a Dirac spinor in an electromagnetic field:

$$\mathcal{L} = \frac{\hbar}{2} i\bar{\psi}\left[\gamma \cdot \left(\overrightarrow{D} - \overleftarrow{D}^\dagger\right) - m\right]\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \ , \quad (23)$$

where $D^\mu = \partial_x^\mu + i\mathbb{A}^\mu$ is covariant derivative.

- Net particle-number current

$$J^\mu(x) = \langle : \bar{\psi}\gamma^\mu\psi : \rangle = \int d^4p \ \mathcal{V}^\mu(x,p) \ . \quad (24)$$

- Canonical energy-momentum tensor:

$$\begin{aligned} T^{\mu\nu}(x) &= \frac{\hbar}{2}\left\langle : i\bar{\psi}\gamma^\mu(\overrightarrow{D}^\nu - \overleftarrow{D}^\nu)\psi : \right\rangle + \mathbb{A}^\nu \langle : \bar{\psi}\gamma^\mu\psi : \rangle + T^{\mu\nu}_{em}(x) \\ &= \int d^4p \ p^\nu \mathcal{V}^\mu(x,p) + \mathbb{A}^\nu \int d^4p \ \mathcal{V}^\mu(x,p) + T^{\mu\nu}_{em}(x) \\ &\equiv T^{\mu\nu}_{mat}(x) + T^{\mu\nu}_{int}(x) + T^{\mu\nu}_{em}(x) \ , \end{aligned} \quad (25)$$

where

$$T^{\mu\nu}_{em}(x) \equiv \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\mu\alpha}\partial_x^\nu\mathbb{A}_\alpha \quad (26)$$

Angular-momentum tensor

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- Total canonical angular-momentum tensor:

$$M^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} - (F^{\lambda\mu}\mathbb{A}^\nu - F^{\lambda\nu}\mathbb{A}^\mu) + \frac{1}{4}\left\langle :\bar{\psi}\left\{\gamma^\lambda,\ \sigma^{\mu\nu}\right\}\psi:\right\rangle . \tag{27}$$

- Spin tensor of matter

$$S^{\lambda,\mu\nu}_{mat}(x) \equiv \frac{1}{4}\left\langle :\bar{\psi}\left\{\gamma^\lambda,\ \sigma^{\mu\nu}\right\}\psi:\right\rangle = -\frac{1}{2}\epsilon^{\lambda\mu\nu\rho}\int d^4p\,\mathcal{A}_\rho(x,p) . \tag{28}$$

Conservation laws

$$\begin{aligned} &\partial_{x\mu} J^\mu(x) = 0 , \\ &\partial_{x\nu} T^{\mu\nu}_{mat}(x) = F^{\nu\alpha}(x) J_\alpha(x) , \\ &\hbar\partial_{x\lambda} S^{\lambda,\mu\nu}_{mat}(x) = T^{\nu\mu}_{mat}(x) - T^{\mu\nu}_{mat}(x) . \end{aligned} \tag{29}$$

Longitudinally- or transversely- polarized

- The polarization direction $n^{(0)\mu}(x,p)$ can be decomposed as

$$n^{(0)\mu} = n_{\parallel}\frac{1}{m}\sqrt{\frac{(u_{lab}\cdot p)^2}{(u_{lab}\cdot p)^2 - m^2}}\left(p^\mu - \frac{m^2}{u_{lab}\cdot p}u^\mu_{lab}\right) + n^\mu_\perp \tag{30}$$

where $n_\perp \cdot u_{lab} = n_\perp \cdot p = 0$ and $n_\parallel^2 - n_\perp \cdot n_\perp = 1$.

Example

Longitudinally polarized
$n_\parallel = 1, \quad n^\mu_\perp = 0$:
polarization parallel to momentum in lab frame

Longitudinally- or transversely- polarized

- The polarization direction $n^{(0)\mu}(x,p)$ can be decomposed as

$$n^{(0)\mu} = n_{\parallel} \frac{1}{m} \sqrt{\frac{(u_{lab} \cdot p)^2}{(u_{lab} \cdot p)^2 - m^2}} \left(p^\mu - \frac{m^2}{u_{lab} \cdot p} u^\mu_{lab} \right) + n^\mu_\perp \tag{30}$$

where $n_\perp \cdot u_{lab} = n_\perp \cdot p = 0$ and $n_\parallel^2 - n_\perp \cdot n_\perp = 1$.

Example

Longitudinally polarized
$n_\parallel = 1, \quad n^\mu_\perp = 0$:
polarization parallel to momentum in lab frame

Example

Transversely polarized
$n_\parallel = 0, \quad n^\mu_\perp \neq 0$:
polarization is perpendicular to momentum in lab frame

Angular-momentum tensor

$$S^{(0)\lambda,\mu\nu} = \epsilon^{\lambda\mu\nu\rho} u_\rho \xi_s \tag{31}$$

with

$$\xi_s = -\frac{1}{2\pi^2\hbar^3} \sum_{rs} rs \int_0^\infty dp \tag{32}$$