Dynamics of a boson-fermion mixture at very low temperatures

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About Ultra-Cold Trapped Atoms

Interesting features

- High interest on behaviour of matter at ultra-low temperatures.
- Bosons and fermions.
- Mesoscopic quantum systems.
- Laser cooling and magnetic traps.
- All relevant parameters can be modified experimentally (particle numbers, densities...).
- Atom laser.

Involves aspects of

- Atomic physics.
- Quantum Mechanics proceedings.
- Many-body physics.
- Ultra-low temperature physics.
- Solid State physics.
- ...

Our Goal

Describe a mixture of bosons and fermions in a magnetic trap theoretically:

- Density profiles.
- Dependence on the strength of the interaction.
- Dependence on the number of particles.
- Stability of the system against collapse.

1. Motivation

Experimental Work

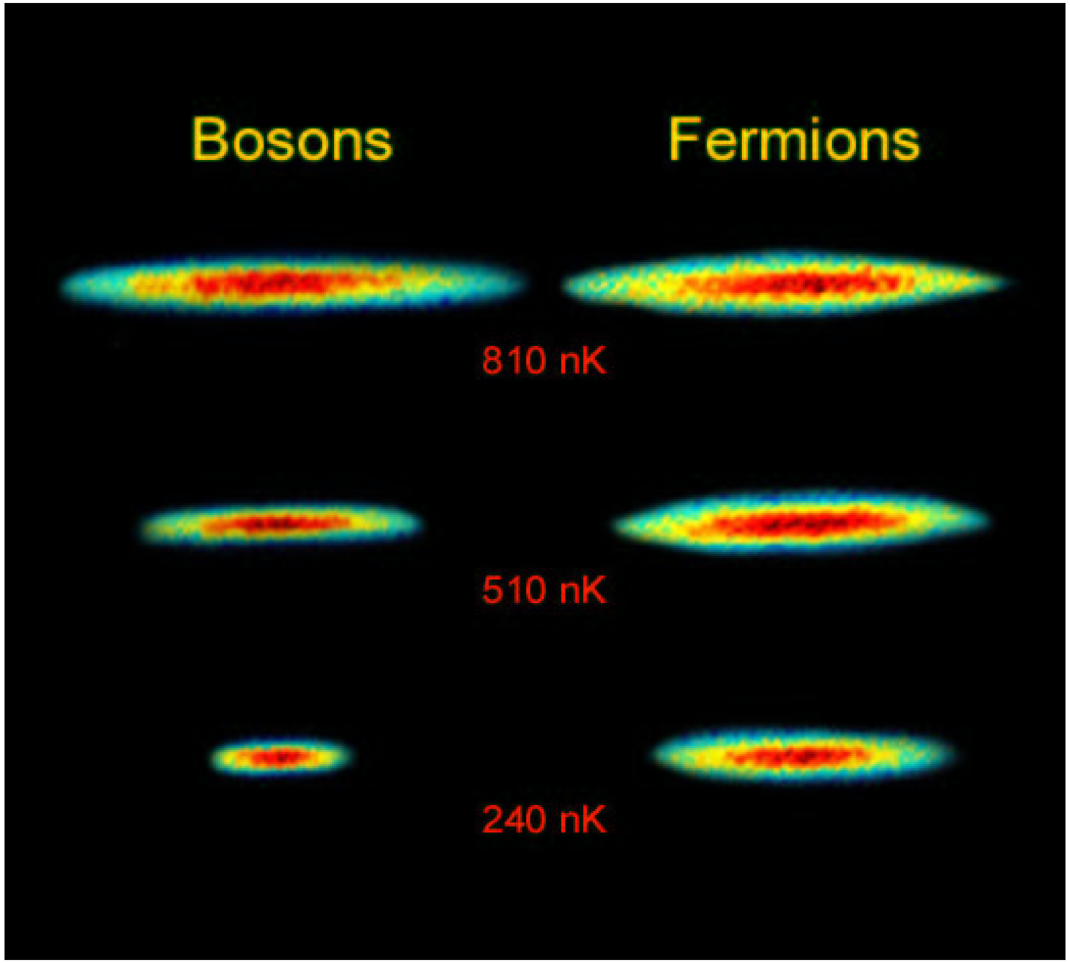
- Binary mixture of ^{7}Li and ^{6}Li in a magnetic trap $(N \sim 10^5)$
- Dependence on temperature
- BEC transition
- Fermi pressure

^{7}Li ^{6}Li

$T/T_C = 1.5,$
$T/T_F = 1.0$

$T/T_F = 0.56$

$T/T_F = 0.25$



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The Challenge

Task

- Solve a many-body problem.
- Determine ground state properties (density distributions)

Method

- Construct energy functional $E[n_b, n_f]$
- Functional minimization gives ground state

Dificulties

- Many-body problem: $N \sim 10^4 - 10^7$.
- Non-homogeneous system.
- Real atom-atom interaction not easy to include.

Approximations

- Mean-Field Approximation: $|\Psi\rangle$ product of single particle states.
- Effective Contact Interaction (dilute system): $\mathbf{U}_{atom-atom} \sim \delta^{(3)}(\vec{r})$.
- Thomas-Fermi Approximation: energy density of the inhomogeneous system locally given by the homogeneous one.

Homogeneous

- $U_{trap} = 0$
- n_b, n_f
- $\mathcal{E} = \mathcal{E}(n_b, n_f)$

Inhomogeneous

- $U_{trap} = U(\vec{x})$
- $n_b(\vec{x}), n_f(\vec{x})$
- $\mathcal{E} = \mathcal{E}[n_b(\vec{x}), n_f(\vec{x})]$

Hamiltonian and Many-Body State

Building up the hamiltonian

- $\mathbf{H} = \mathbf{H}_{int} + \mathbf{U}_{trap}$

- $$\mathbf{H}_{\mathbf{int}} = \frac{1}{2m}\sum_i \vec{\mathbf{p}}_i^2 + \frac{4\pi}{m}a_b \sum_{i,j>i} \delta^{(3)}\left(\vec{\mathbf{r}}_{ij}\right)\ \Pi_i^b\Pi_j^b + \frac{4\pi}{m}a_f \sum_{i,j>i} \delta^{(3)}\left(\vec{\mathbf{r}}_{ij}\right)\ \Pi_i^f\Pi_j^f + \frac{4\pi}{m}a_{bf} \sum_{i,j>i} \delta^{(3)}(\vec{\mathbf{r}}_{ij})(\Pi_i^b\Pi_j^f + \Pi_i^f\Pi_j^b)$$

- $\mathbf{U}_{\mathbf{trap}} = \sum_i \left(U_b\left(\vec{x}_i\right)\Pi_i^b + U_f\left(\vec{x}_i\right)\Pi_i^f\right)$,

  ➨ $U(\vec{x}) = \frac{m\omega^2}{2}\vec{x}^2 = \frac{1}{2m\ell^4}\vec{x}^2$

Quantum state (homogeneous system)

- Many-body state

$$|\Psi\rangle = |\Psi_b\rangle \otimes |\Psi_f\rangle$$

- Boson state (BEC)

$$|\Psi_b\rangle = |\phi\rangle \otimes \cdots \otimes |\phi\rangle$$

$$\langle \vec{x}|\phi\rangle = \frac{1}{\sqrt{V}}$$

- Fermion state

$$|\Psi_f\rangle = \mathcal{A}(|\vec{k}_1\rangle \otimes \cdots \otimes |\vec{k}_n\rangle)$$

$$\langle \vec{x}|\vec{k}_i\rangle = \frac{1}{\sqrt{V}}e^{i\vec{k}_i\cdot\vec{x}}$$

GSI theory

Energy Functional and Minimization

✔ Energy functional

- Energy density ($\mathcal{E}[n_b, n_f] = \frac{\langle\Psi|H|\Psi\rangle}{V}$)

$$\mathcal{E}[n_b, n_f] = \frac{3^{5/3}\pi^{4/3}}{5\,2^{1/3}m} n_f^{5/3}(\vec{x}) + \frac{2\pi}{m} a_b n_b^2(\vec{x}) + \frac{4\pi}{m} a_{bf} n_b(\vec{x}) n_f(\vec{x}) + \frac{m\omega^2}{2} x^2 n_b(\vec{x}) + \frac{m\omega^2}{2} x^2 n_f(\vec{x}).$$

- Energy functional

$$E[n_b, n_f] = \int_V d^3x \mathcal{E}[n_b, n_f]$$

Functional Variation

- Ground State = minimum for energy.
- Fixed number of particles $\rightarrow$ Lagrange multipliers

$$\mathcal{F}[n_b, n_f] = \mathcal{E}[n_b, n_f] - \mu_b n_b(\vec{x}) - \mu_f n_f(\vec{x})$$

$$F[n_b, n_f] = \int_V d^3x \;\; \mathcal{F}[n_b, n_f]$$

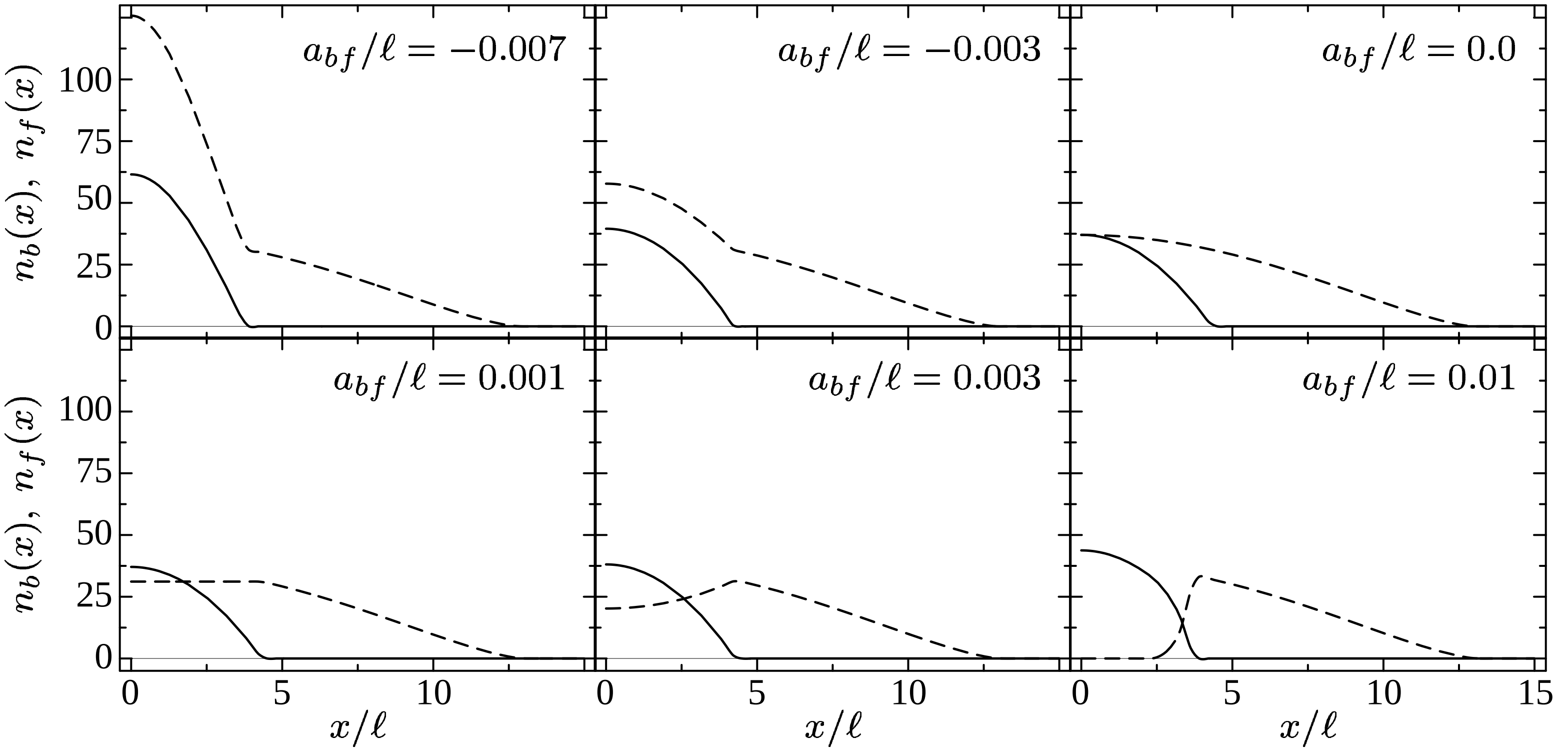
- Functional Minimization

$$\Rightarrow \frac{\partial \mathcal{F}[n_b, n_f]}{\partial n_b} = 0, \quad \frac{\partial \mathcal{F}[n_b, n_f]}{\partial n_f} = 0$$

$$\mathcal{F} = \mathcal{F}[n_b, n_f; \mu_b, \mu_f, \vec{x}]$$

Strength of Interactions

✘ Density profiles for different values of a_{bf}.



$N_f = N_b = 10^5$,

$a_b/\ell = 0.001$.

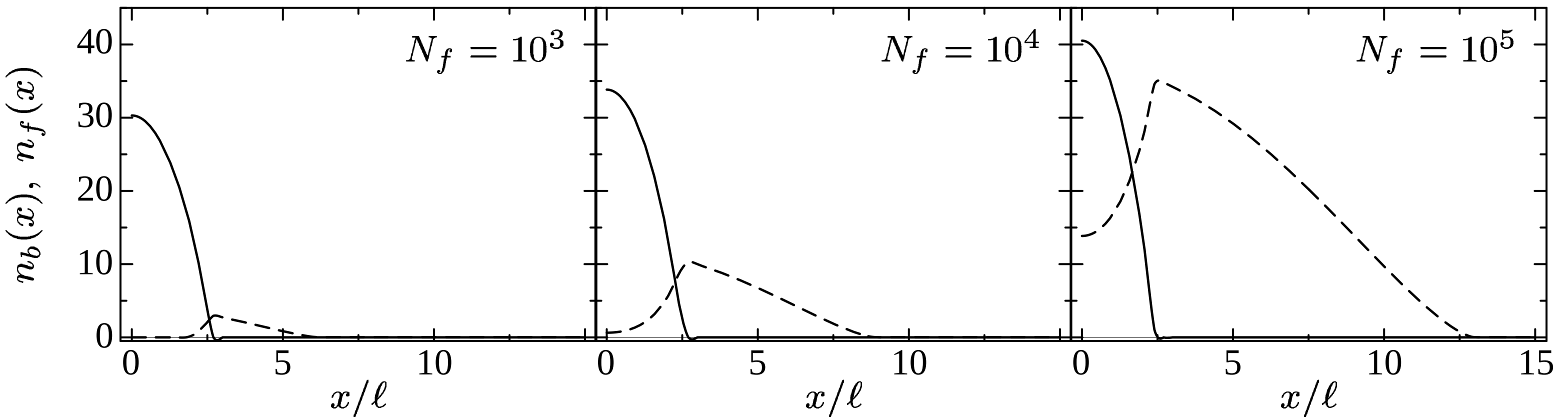
Solid lines $\rightarrow n_b(\vec{x})/20$,

dashed lines $\rightarrow n_f(\vec{x})$.

- $a_{bf} < 0$: increase of n_f and n_b in the overlap region.
- $a_{bf} = a_b$: Constant fermion profile until boson density is zero. Then, it starts to go to zero.
- $a_{bf} = 0$: Same profile as each one alone.
- $a_{bf} > 0$: Fermions expelled out of the center. Bosons form a core.

Number of Particles

✘ Density profiles for different values of N_f



$N_b = 10^4$,

$a_b/\ell = 0.001$,

$a_{bf}/\ell = 0.008$.

Solid lines $\rightarrow\ n_b(x)/10$,

dashed lines $\rightarrow n_f(x)$.

➨ Boson core is compressed if N_f or $/a_{bf}/$ are increased.

Mean Field Collapse

The Collapse

- The attractive mean field due to the boson-fermion interaction cannot be balanced by the kinetic energy of fermions and repulsive boson-boson interaction.
- Dilute gas collapses towards higer densities.

Stability Conditions

- Stability $\rightarrow$ study of Hessian matrix $D(n_b, n_f) = \begin{pmatrix} \frac{\partial^2 \mathcal{F}[n_b,n_f]}{\partial n_b^2} & \frac{\partial^2 \mathcal{F}[n_b,n_f]}{\partial n_b \partial n_f} \\ \frac{\partial^2 \mathcal{F}[n_b,n_f]}{\partial n_f \partial n_b} & \frac{\partial^2 \mathcal{F}[n_b,n_f]}{\partial n_f^2} \end{pmatrix}$
- Saddle point $\Rightarrow det(D) < 0 \Rightarrow$
  extremum $\Rightarrow det(D) > 0 \quad \Rightarrow$

$\Rightarrow det(D) = 0$ begining of unstability

✔ Solution: $n_f^{lim} = \frac{\pi a_b^3}{48 a_{bf}^6}$

Depends only on the interaction

increasing attractive a_{bf} for collapse

4. Conclusions

Summary

- Many-body system described by means of an energy density functional.
  - ➨ Mean-Field
  - ➨ Contact Interaction
  - ➨ Thomas-Fermi Approximation
- Dependence on strength of the interactions.
  - ➨ $a_{bf} < 0$
  - ➨ $a_{bf} = 0$
  - ➨ $a_{bf} > 0$
  - ➨ $a_{bf} = a_b$
- Dependence on number of particles.
  - ➨ Bosons more squeezed increasing N_f or $/a_{bf}/$
- Mean-Field Collapse.
  - ➨ Collapse of n_b and n_f for a_{bf} negative enough

Looking Forward

- Other kinds of instability.
- P-wave: second order in Effective Contact Interaction.
- Kinetic energy of bosons: Gross-Pitaevski equation.
- Attractive boson-boson interaction.
- Beyond Mean-Field Theory.
- Behaviour at finite temperature.

GSI theory