Spin Hydrodynamics II

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in collaboration with

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based mainly on

NW, ES, X.-L. Sheng, Q. Wang, DHR, Phys. Rev. D 104 1, 016022 (2021)
NW, DW, ES, DHR, 2203.04766 (2022)
NW, DW, ES, Phys. Rev. D 105 11,116026 (2022)
DW, NW, ES, 2207.01111 (2022)

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CRC-TR 211 Strong-interaction matter under extreme conditions

DFG

EL EM EN TS

Reminder: Total angular momentum

Decomposition and conservation

$$J^{\lambda\mu\nu} := S^{\lambda\mu\nu} + T^{\lambda[\mu}x^{\nu]} \quad (1)$$

$$\partial_\lambda S^{\lambda\mu\nu} = T^{[\nu\mu]} \quad (2)$$

- Total angular momentum tensor consists of spin and orbital parts
- Usual Hydrodynamics: spinless particles
  - $\rightarrow$ $T^{\mu\nu} = T^{\nu\mu}$ satisfies $\partial_\lambda J^{\lambda\mu\nu} = 0$
  - $\rightarrow$ No need to consider angular momentum further
- Spin Hydrodynamics: Particles have spin
  - $\rightarrow$ $S^{\lambda\mu\nu} \neq 0$!
  - $\rightarrow$ Conservation of total angular momentum has to be explicitly included

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$A^{[\mu}B^{\nu]} := A^\mu B^\nu - A^\nu B^\mu$

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- Spin tensor is **antisymmetric** in the last two indices
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    - **6** equations describe ideal case
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  $\rightarrow$ **Kinetic theory with spin!**
    - Has to respect the relevant conservation laws (in particular allow for $S^{\lambda\mu\nu} \neq 0$)

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  $\rightarrow$ **Kinetic theory with spin!**
    - Has to respect the relevant conservation laws (in particular allow for $S^{\lambda\mu\nu} \neq 0$)
- Rest of the presentation:
    - Construct such a kinetic theory
    - Perform hydrodynamic limit
    - Analyze equations of motion

How to: Quantum kinetic theory

- Spin is a quantum property
  - → Start from quantum field theory
  - → Use **Wigner-function formalism**

Wigner function

$$W^{ab}(x,k) := \frac{\#}{(2\pi\hbar)^4} \int \mathrm{d}^4 v e^{-ik\cdot v/\hbar} \left\langle : \Phi^{\dagger a}(x+v/2)\Phi^b(x-v/2) : \right\rangle \quad (3)$$

- Wigner function: Wigner transform of two-point function
- Determines a **quantum phase-space distribution function**
- Matrix dimension determined by representation of the field Φ
  - → Carries information about spin structure
- Equations of motion follow from field equations
  - Determined by Lagrangian $\mathcal{L}_0 + \mathcal{L}_{\text{int}}$

$A \cdot B := A^\alpha B_\alpha$

More counting

- How many **independent** degrees of freedom (d.o.f.) does W^{ab} have?
  - Formalism is based on the spin-density matrix $\rightarrow$ $\mathbf{(2s+1)^2}$ independent components
  - $\rightarrow$ Some parts of W^{ab} are constrained

Example: Spin-1/2

- Decompose W^{ab} (matrix in Dirac space) according to Clifford algebra

$$W = \frac{1}{4}\left(\mathcal{F} + i\gamma^5\mathcal{P} + \gamma\cdot\mathcal{V} + \gamma^5\gamma\cdot\mathcal{A} + \frac{i}{4}[\gamma^\mu, \gamma^\nu]\mathcal{S}_{\mu\nu}\right) \tag{4}$$

- Only $\mathcal{F}$ (1 d.o.f.) and $\mathcal{A}^\mu$ (3 d.o.f.) are independent
  - $\rightarrow$ Wigner function has $4 = (2\cdot 1/2 + 1)^2$ d.o.f. ✓

Equations of motion

- Wigner function follows three types of equations
  1. Constraint equations (reduce number of independent d.o.f.)
  2. Mass-shell equations
  3. Boltzmann-like equations

Example: Spin 0

- Constraint equations absent (only 1 d.o.f.)

$$\left(k^2 - m^2 - \frac{\hbar^2}{4}\Box\right) W(x,k) = C + C^* \qquad \text{(Mass-shell equation)}$$

$$k \cdot \partial W(x,k) = i(C - C^*) \qquad \text{(Boltzmann equation)}$$

- Collisions determined by interaction $\rho := \partial \mathcal{L}_{\text{int}} / \partial \Phi$

$$C(x,k) := \frac{\#}{(2\pi\hbar)^4} \int \mathrm{d}^4 v e^{-ik\cdot v/\hbar} \left\langle : \Phi_+^\dagger \rho_- : \right\rangle \tag{5}$$

$A_\pm := A(x \pm v/2)$

Approximations and assumptions

- Equations are solved via a (formal) **power series** in $\hbar$
  - Small parameter: $\hbar/m\partial \sim (\hbar/m)/L_{\text{hydro}} \equiv$ (Compton wavelength of the particle)/(macroscopic length scale)
- **Truncate** expansion at first order in $\hbar$

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- **Truncate** expansion at first order in $\hbar$
- Assumptions have to be made about the structure of W^{ab} at zeroth order
- We assume that the system is not prepared in a polarized state
  - $\rightarrow$ Motivation: Want to describe how an unpolarized system **acquires polarization**
- Concrete form of the assumptions:
  - Spin 1/2: $W^{ab}(x,k) = (1/4)\delta^{ab}\mathcal{F}(x,k) + \mathcal{O}(\hbar)$
  - Spin 1: $W^{\mu\nu}(x,k) = (g^{\mu\nu} - k^\mu k^\nu/m^2) f_K(x,k) + \mathcal{O}(\hbar)$

Extending phase space

- Reminder: Wigner function is **matrix-valued**
  - $\rightarrow$ Not one, but $\boldsymbol{(2s+1)^2}$ independent Boltzmann equations
  - Spin 1/2: $k\cdot\partial\mathcal{F}=\mathcal{C}_{\mathcal{F}}$, $k\cdot\partial\mathcal{A}^{\mu}=\mathcal{C}^{\mu}_{\mathcal{A}}$

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  - Idea: Instead of $(2s+1)^2$ equations in 8-dimensional phase space: one equation in 10-dimensional enlarged phase space

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- Measure $\mathrm{d}S:=\#\mathrm{d}^4\mathfrak{s}\delta[\mathfrak{s}^2+\sigma^2]\delta(k\cdot\mathfrak{s})$
  - $\rightarrow$ Spacelike normalized "spin vector" $\mathfrak{s}^\mu$ orthogonal to k^μ

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Boltzmann equation in enlarged phase space

- Only on-shell parts $\mathfrak{f}(x, k, \mathfrak{s}) = \delta(k^2 - m^2) f(x, k, \mathfrak{s})$ contribute

$$k \cdot \partial f(x, k, \mathfrak{s}) = \mathfrak{C}[f] \tag{6}$$

Extending phase space: Spin 1/2

$$\int \mathrm{d}S = 2\,, \; \int \mathrm{d}S\,\mathfrak{s}^\mu \mathfrak{s}^\nu = -2K^{\mu\nu}$$

- Via integration over spin space the individual equations can be recovered

Distribution function: Spin 1/2

$$\mathfrak{f}^{(1/2)}(x,k,\mathfrak{s}) := \frac{1}{2}\left(\mathcal{F} - \mathfrak{s}^\mu \mathcal{A}_\mu\right) \tag{7}$$

$$\Longrightarrow \mathcal{F}(x,k) = \int \mathrm{d}S\,\mathfrak{f}^{(1/2)}(x,k,\mathfrak{s}) \tag{8}$$

$$\Longrightarrow \mathcal{A}^\mu(x,k) = \int \mathrm{d}S\,\mathfrak{s}^\mu \mathfrak{f}^{(1/2)}(x,k,\mathfrak{s}) \tag{9}$$

$K^{\mu\nu} := g^{\mu\nu} - k^\mu k^\nu / m^2$

Extending phase space: Spin 1

$$\int \mathrm{d}S = 3\,, \int \mathrm{d}S \mathfrak{s}^\mu \mathfrak{s}^\nu = -2K^{\mu\nu}\,, \int \mathrm{d}S K^{\mu\nu}_{\rho\sigma} \mathfrak{s}^\rho \mathfrak{s}^\sigma \mathfrak{s}_\alpha \mathfrak{s}_\beta = (8/5) K^{\mu\nu}_{\alpha\beta}$$

- Spin-1 density matrix has a richer structure
  - $\rightarrow$ Have to include more structures in $\mathfrak{f}$

Distribution function: Spin 1

$$\mathfrak{f}^{(1)}(x, k, \mathfrak{s}) := f_K - \mathfrak{s}^\mu G_\mu + \frac{5}{4} \mathfrak{s}^\mu \mathfrak{s}^\nu F_{K,\mu\nu} \quad (10)$$

$$\Longrightarrow \quad f_K(x, k) = \frac{1}{3} \int \mathrm{d}S \mathfrak{f}^{(1)}(x, k, \mathfrak{s}) \quad (11)$$

$$\Longrightarrow \quad G^\mu(x, k) = \frac{1}{2} \int \mathrm{d}S \mathfrak{s}^\mu \mathfrak{f}^{(1)}(x, k, \mathfrak{s}) \quad (12)$$

$$\Longrightarrow \quad F^{\mu\nu}_K(x, k) = \frac{1}{2} \int \mathrm{d}S K^{\mu\nu}_{\alpha\beta} \mathfrak{s}^\alpha \mathfrak{s}^\beta \mathfrak{f}^{(1)}(x, k, \mathfrak{s}) \quad (13)$$

$K^{\mu\nu}_{\alpha\beta} := (K^\mu_\alpha K^\nu_\beta + K^\mu_\beta K^\nu_\alpha)/2 - 1/3 K^{\mu\nu} K_{\alpha\beta}$

$f_K := (1/3) K^{\mu\nu} W_{\mu\nu}\,, \quad G^\mu := -[i/(2m)] \epsilon^{\mu\nu\alpha\beta} k_\nu W_{\alpha\beta}\,, \quad F^{\mu\nu}_K := K^{\mu\nu}_{\alpha\beta} W^{\alpha\beta}$

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  $\rightarrow$ **What do we mean by this?**

Local and nonlocal collisions

- Contributions up to order $\mathcal{O}(\hbar)$ go as

$$f(x,k) + \Delta^\mu \partial_\mu f(x,k) = f(x+\Delta, k) + \mathcal{O}(\hbar^2)$$

- A (momentum- and spin-dependent) **spacetime shift** $\Delta^\mu \sim \mathcal{O}(\hbar)$ enters
  - $\rightarrow$ Particles do not scatter at the same spacetime point!

Collision kernel

$$\begin{aligned} \mathfrak{C}[f] &= \int \mathrm{d}\Gamma_1 \mathrm{d}\Gamma_2 \mathrm{d}\Gamma' \mathrm{d}\bar{S}(k) \mathcal{W} \left[f(x+\Delta_1, k_1, \mathfrak{s}_1) f(x+\Delta_2, k_2, \mathfrak{s}_2) \right. \\ &\quad \left. - f(x+\Delta, k, \bar{\mathfrak{s}}) f(x+\Delta', k', \mathfrak{s}') \right] \\ &+ \int \mathrm{d}\Gamma_2 \mathrm{d}S_1(k) \mathfrak{W} f(x+\Delta, k, \mathfrak{s}_1) f(x+\Delta_2, k_2, \mathfrak{s}_2) \end{aligned} \tag{14}$$

$\mathrm{d}\Gamma := \mathrm{d}^4k\,\delta(k^2 - m^2)\mathrm{d}S$

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Local-equilibrium distribution function

$$f_{\mathrm{eq}}(x,k,\mathfrak{s}) = \exp\left(\alpha_0 - \beta_0 E_{\mathbf{k}} + s\frac{\hbar}{2}\Omega_{\mu\nu}\Sigma_{\mathfrak{s}}^{\mu\nu}\right) \tag{15}$$

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- Necessary conditions on Lagrange multipliers α_0, $\beta_0 u^\mu$, $\Omega^{\mu\nu}$ for a vanishing collision term:
  $\partial^\mu \alpha_0 = 0\ ,\ \partial^{(\mu}(\beta_0 u^{\nu)}) = 0\ ,\ \Omega^{\mu\nu} = -\frac{1}{2}\partial^{[\mu}(\beta_0 u^{\nu]})$
- Same conditions as for **global** equilibrium

→ Rigorously, there is no local equilibrium!

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$\rightarrow$ Rigorously, there is no local equilibrium!

- **However...**

$\Sigma_{\mathfrak{s}}^{\mu\nu} := -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}k_\alpha \mathfrak{s}_\beta,\ E_{\mathbf{k}} := k \cdot u$

Relevant scales and modified equilibrium

- Knudsen number is defined as $\mathrm{Kn} := \lambda_{\mathrm{mfp}}/L_{\mathrm{hydro}}$
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  $\rightarrow$ Much smaller than λ_{mfp}
- The ratio Δ/L_{hydro} is a lot smaller than Kn
- Propose **modified** definition of local equilibrium: $\mathfrak{C}[f_{\text{eq}}] = 0 + \mathcal{O}(\Delta/L_{\text{hydro}})$

Local-equilibrium distribution function

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- Now we can formulate spin hydro in the usual way as an expansion around local equilibrium

Moment expansion

- Split distribution function $f = f_{\mathsf{eq}} + \delta f$

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Irreducible moments

$$\rho_r^{\mu_1\cdots\mu_\ell}(x) \quad := \quad \int \mathrm{d}\Gamma\, E_{\mathbf{k}}^r k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} \delta f(x,k,\mathfrak{s}) \tag{17}$$

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$$\vdots \qquad \vdots \qquad \vdots$$

- Only moments of spin-rank $\leq 2s$ are present

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CRC-TR 211

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- Equations of motion can be derived from Boltzmann equation
- How to truncate this system?

Connecting back to Hydro

- Spin-rank zero moments: $\Pi = m^2\rho_0/3$, $n^\mu = \rho_0^\mu$, $\pi^{\mu\nu} = \rho_0^{\mu\nu}$
- Which moments are contained in the spin tensor?

Spin tensor (HW pseudogauge)

$$S^{\lambda\mu\nu} = s\int \mathrm{d}\Gamma k^\lambda \Sigma_{\mathfrak{s}}^{\mu\nu} + \frac{s\hbar}{m^2(2s+1)}\partial^{[\mu}T^{\nu]\lambda} \tag{20}$$

- Contains τ_0^μ, τ_2^μ, $\tau_1^{\mu,\nu}$, $\tau_0^{\mu,\nu\lambda}$
- Six d.o.f. are removed by matching $u_\lambda J^{\lambda\mu\nu} = u_\lambda J_{\text{eq}}^{\lambda\mu\nu}$

- The independent moments in the spin tensor (after matching) are

$$\mathfrak{p}^\mu := \tau_0^\mu\,, \quad \mathfrak{z}^{\mu\nu} := \tau_1^{(\langle\mu\rangle,\langle\nu\rangle)}\,, \quad \mathfrak{q}^{\lambda\mu\nu} := \tau_0^{\lambda,\mu\nu}$$

- These are the spin-analogues to Π, n^μ and $\pi^{\mu\nu}$

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  - Express all moments outside of the employed basis by the dynamical ones

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  - Read off first- and second-order transport coefficients

$\mathbb{S}_0^{(1)} \coloneqq \{0\}$, $\mathbb{S}_1^{(1)} \coloneqq \{1\}$, $\mathbb{S}_2^{(1)} \coloneqq \{0\}$, $\mathbb{S}_n^{(1)} \coloneqq \emptyset \,\forall\, n > 2$

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  - Read off first- and second-order transport coefficients
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Conservation equations

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- In the linearized limit for a nonrotating background, equations of motion for ω_0^μ, κ_0^μ show wavelike behaviour

  V. E. Ambruș, R. Singh, 2202.03952 (2022)

Results II: Dissipative Spin Hydro

- General (dissipative) case: have to provide evolution equations for dissipative quantities

Dissipative Hydro: Evolution equations

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta + \text{h.o.t.} \quad (22)$$

$$\tau_n \dot{n}^{\langle\mu\rangle} + n^\mu = \kappa \nabla^\mu \alpha_0 + \text{h.o.t.} \quad (23)$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \text{h.o.t.} \quad (24)$$

$$\tau_{\mathfrak{p}} \dot{\mathfrak{p}}^{\langle\mu\rangle} + \mathfrak{p}^{\langle\mu\rangle} = \mathfrak{e}^{(0)} (\tilde{\Omega}^{\mu\nu} - \tilde{\varpi}^{\mu\nu}) u_\nu + \text{h.o.t.} \quad (25)$$

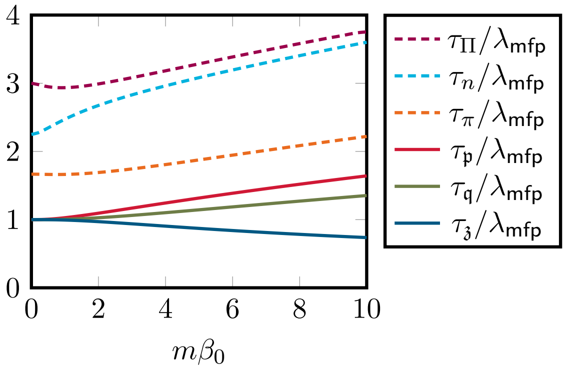
$$\tau_{\mathfrak{z}} \dot{\mathfrak{z}}^{\langle\mu\rangle\langle\nu\rangle} + \mathfrak{z}^{\langle\mu\rangle\langle\nu\rangle} = \text{h.o.t.} \quad (26)$$

$$\tau_{\mathfrak{q}} \dot{\mathfrak{q}}^{\langle\lambda\rangle\langle\mu\nu\rangle} + \mathfrak{q}^{\langle\lambda\rangle\langle\mu\nu\rangle} = \mathfrak{d}^{(2)} \beta_0 \sigma_\alpha{}^{\langle\mu} \epsilon^{\nu\rangle\lambda\alpha\beta} u_\beta + \text{h.o.t.} \quad (27)$$

- Navier-Stokes values of spin-moments are determined by **nonlocal** collisions!
- Relaxation times follow from **local** collisions

$\varpi^{\mu\nu} := -\frac{1}{2}\partial^{[\mu}(\beta_0 u^{\nu]})$, $\tilde{A}^{\mu\nu} := \epsilon^{\mu\nu\alpha\beta} A_{\alpha\beta}$

Relevant time scales: An estimation



- Simplest interaction: constant cross section
- Spin-related relaxation times shorter than standard dissipative time scales, but not much

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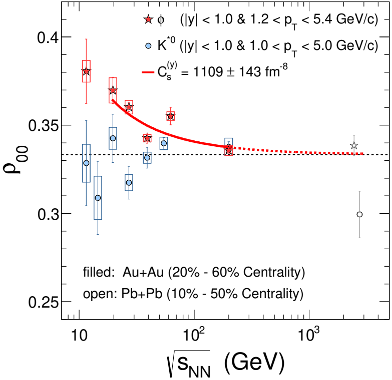

STAR collaboration, arXiv:2204.02302 (2022)

Spin 1: Truncation

Tensor polarization

$$\Theta^{\mu\nu}(k) = \frac{1}{2}\sqrt{\frac{3}{2}}\frac{1}{N(k)}\int \mathrm{d}\Sigma_\gamma k^\gamma \int \mathrm{d}S(k) K^{\mu\nu}_{\alpha\beta}\mathfrak{s}^\alpha \mathfrak{s}^\beta f(x,k,\mathfrak{s}) \tag{28}$$

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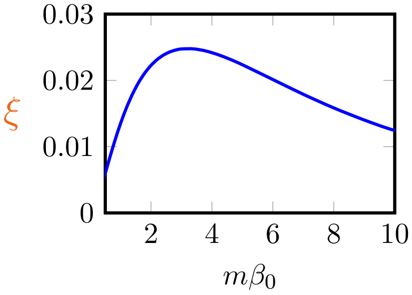

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  - Perform simulations to connect with experimental data
  - Go to higher order in moment expansion

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