Critical Exponents for $O(N)$ Model via Hydrodynamic approach to FRG

Fabrizio Murgana

In collaboration with D. H. Rischke, A.Koenigstein

Institut für Theoretische Physik, Johann Wolfgang Goethe-Universität Frankfurt am Main

HFHF Theory Retreat
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1. Functional Renormalization Group (FRG)
   - Functional approach to QFT
   - Flow Equation

Outline

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3. Local Potential Approximation (LPA)
4. Critical Behaviour
   - Critical $\rho_0|_{t=0}$
   - Critical exponent ν
   - Critical exponent β
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Functional Renormalization Group (FRG)

Quantum field theory

In quantum field theory (QFT), all physical information is stored in the *generating functional* $Z[J]$

$$Z[J] \equiv \mathcal{N} \int \mathcal{D}\varphi \, e^{-S[\varphi]+\int J\varphi} \tag{1}$$

since all $n-$point functions can be obtained via a functional differentiation

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle := \mathcal{N} \int \mathcal{D}\varphi \, \varphi(x_1) \cdots \varphi(x_n) e^{-S[\varphi]} = \frac{\partial^n}{\partial J^n} Z[J] \bigg|_{J=0} \tag{2}$$

One introduces the *generating functional of connected correlators* $W[J]$:

$$Z[J] \equiv e^{W[J]} = \int \mathcal{D}\varphi \, e^{-S[\varphi]+\int J\varphi} \tag{3}$$

Performing a Legendre transform of $W[J]$ we obtain the *effective action* Γ:

$$\Gamma[\phi] = \sup_J \left(\int J\phi - W[J] \right) \tag{4}$$

At $J = J_{\text{sup}}$, we get

$$\frac{\delta}{\delta J(x)} \left(\int J\phi - W[J] \right) = 0$$

$$\Rightarrow \quad \phi = \frac{\delta W[J]}{\delta J} = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J} = \langle \varphi \rangle_J \tag{5}$$

Flow equation

We are looking for an interpolating action Γ_k, the *effective average action*, such that

$$\Gamma_{k\to\Lambda} = S_{bare}\,, \quad \Gamma_{k\to 0} = \Gamma \tag{6}$$

This can be constructed defining an IR regulated functional

$$e^{W_k[J]} \equiv Z_k[J] := \int_\Lambda \mathcal{D}\varphi\, e^{-S[\varphi]-\Delta S_k[\varphi]-\int J\varphi} \tag{7}$$

where ΔS_k is a regulator term of the form

$$\Delta S_k[\varphi] = \frac{1}{2}\int \frac{d^D p}{(2\pi)^D}\, \varphi(-p)\, R_k(p)\, \varphi(p) \tag{8}$$

The regulator function $R_k(p)$ should satisfy the following properties:

1.
$$\lim_{p^2/k^2 \to 0} R_k(p) > 0$$

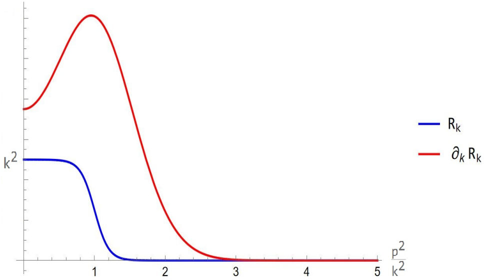
$\Rightarrow$ IR Regularization

2.
$$\lim_{k^2/p^2 \to 0} R_k(p) = 0$$

$\Rightarrow \Gamma_{k \to 0} = \Gamma$

3.
$$\lim_{k^2 \to \Lambda^2 \to \infty} R_k(p) \to \infty$$

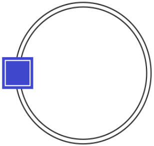
$\Rightarrow \Gamma_{k \to \Lambda} = S_{bare}$



FRG Flow Equation

In this way we can introduce the effective average action:

$$\Gamma_k[\phi] = \sup_J \left(\int J\phi - W_k[J] \right) - \Delta S_k[\phi] \qquad \partial_t \Gamma_k[\phi] = \frac{1}{2}\,\partial_t R_k$$



We can derive the *flow equation* for Γ_k:

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr} \left[\partial_t R_k (\Gamma_k^{(2)}[\phi] + R_k)^{-1} \right] \tag{9}$$

where

$$t = -\ln \frac{k}{\Lambda} \qquad \partial_t = -k \frac{d}{dk}$$

[1] C. Wetterich, Phys. Lett. B 301 (1993) 90-94.
[2] K. G. Wilson, Phys. Rev. B 4, (1971) 3174, Phys. Rev. B 4, (1971) 3184.
[3] J. Berges, N. Tetradis, C. Wetterich, Phys.Rept. 363 (2002) 223-386.

FRG Flow Equation

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\partial_t R_k (\Gamma_k^{(2)}[\phi] + R_k)^{-1}\right]$$

We can notice that:

- The flow equation is a functional differential equation for Γ_k;
- The purpose of the regulator is twofold:
  - IR Regularization;
  - Implements the idea of integrating over momentum shells $p^2 \sim k^2$.
- The solution of the flow equation corresponds to an RG trajectory in *theory space*;
- The trajectory depends on R_k but the final point does not;
- Exact one-loop structure,
- Difficult to solve exactly $\Rightarrow$ we need some ansatz.

We will use a *derivative expansion*

$$\Gamma_k[\phi] = \int d^D x \left[V_k(\phi) + \frac{1}{2} Z_k(\phi)(\partial_\mu \phi)^2 + \mathcal{O}(\partial^2)\right] \tag{10}$$

$O(N)$ Model

$O(N)$ Model

The theory describes N scalar fields $\phi_a(x)$ with $a = 1, \cdots, N$.
The associated bare action at the cutoff scale $k = \Lambda$ is

$$S_{k=\Lambda}[\vec{\phi}\,] = \int d^d x \left\{ \frac{1}{2} (\partial_\mu \phi_a)^2 + V_{k=\Lambda}(\rho) \right\} \tag{11}$$

$$V_{k=\Lambda}(\rho) = \frac{\lambda}{4} (\rho - \rho_0)^2 \,, \qquad\qquad \rho = \frac{1}{2} \phi^a \phi_a \tag{12}$$

The N-component ϕ^4 theory exhibits a spontaneous symmetry breaking of the $O(N)$-group down to $O(N-1)$, which is restored at sufficiently high temperature via a second-order phase transition.

Motivation:

- Universality: close to phase transition UV details do not count.
- Easy to solve, well known results, good framework for numerical tests.

[4] Gell-Mann M and Levy M 1960 Nuovo Cimento 16 705–26.
[5] Adrian Koenigstein, Martin J. Steil, Nicolas Wink, Eduardo Grossi, Jens Braun, Michael Buballa, and Dirk H. Rischke, Numerical fluid dynamics for FRG flow equations: Zero-dimensional QFTs as numerical test cases - Part I: The $O(N)$ model.

Local Potential Approximation (LPA)

As ansatz for Γ_k we will use the LPA:

$$\Gamma_k[\vec{\phi}] = \int_x \left\{ \frac{1}{2}(\partial_\mu \phi_a)^2 + V_k(t,\rho) \right\} \tag{13}$$

where $V_k(t,\rho)$ is the effective potential.
Then we compute the two point functions $\Gamma_k^{(2)}$:

$$\Gamma_{k,ab}^{(2)}(t,\rho,p) = [p^2 + V_k'(t,\rho)]\delta_{ab} + 2\rho V_k''(t,\rho)\delta_{aN}\delta_{bN} \tag{14}$$

with

$$V_k'(t,\rho) = \partial_\rho V_k(t,\rho) \qquad \text{and} \qquad V_k''(t,\rho) = \partial_\rho V_k'(t,\rho)$$

As regulator, we chose the Litim Regulator

$$R_k(p) = (k^2 - p^2)\Theta(k^2 - p^2) \tag{15}$$

[6] Daniel F. Litim, Phys. Rev. D 64, 105007 (2001).

LPA

Inserting Eq.s (14)-(15) into (9) we obtain the flow equation for $V_k(t,\rho)$

$$\partial_t V_k(t,\rho) = -A_d\, k^{d+2} \left(\frac{N-1}{k^2 + V_k'(t,\rho)} + \frac{1}{k^2 + V_k'(t,\rho) + 2\rho V_k''(t,\rho)} \right) \tag{16}$$

with $A_d = \frac{\Omega_d}{(2\pi)^d d}$, and $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$.

It is preferable to formulate the problem in terms of the field $\sigma = \sqrt{2\rho}$.
We thus rewrite the flow equation as:

$$\partial_t V_k(t,\sigma) = -A_d\, k^{d+2} \left(\frac{N-1}{k^2 + \frac{1}{\sigma}\partial_\sigma V_k(t,\sigma)} + \frac{1}{k^2 + \partial^2_{\sigma\sigma} V_k(t,\sigma)} \right) \tag{17}$$

We introduce the derivative of the potential as new variable

$$u(t,\sigma) = \partial_\sigma V_k(t,\sigma), \qquad u'(t,\sigma) = \partial_\sigma u(t,\sigma) \tag{18}$$

We can now introduce the advection flux

$$f(t, \sigma, u) = A_d \, k^{d+2} \frac{N-1}{k^2 + \frac{1}{\sigma} u(t, \sigma)} \tag{19}$$

and the diffusion flux

$$g(t, u') = -A_d \, k^{d+2} \frac{1}{k^2 + u'(t, \sigma)} \tag{20}$$

Taking the derivative of (17) with respect to σ we obtain

$$\partial_t u(t, \sigma) + \partial_\sigma f(t, \sigma, u) = \partial_\sigma g(t, u') \tag{21}$$

which is an *advection-diffusion* equation for the derivative of the potential $u(t, \sigma)$.

[5] Adrian Koenigstein, Martin J. Steil, Nicolas Wink, Eduardo Grossi, Jens Braun, Michael Buballa, and Dirk H. Rischke, Numerical fluid dynamics for FRG flow equations: Zero-dimensional QFTs as numerical test cases - Part I: The $O(N)$ model.
[7]E. Grossi and N. Wink (2019), arXiv:1903.09503.

Advection contribution

$$\partial_t u(t,\sigma) + \partial_u f(t,\sigma,u)\; u'(t,\sigma) + \partial_\sigma f(t,\sigma,u) = 0 \tag{22}$$

Advection coefficient

$$\partial_u f(t,\sigma,u) = -A_d\, k^{d+2} \frac{N-1}{\sigma[k^2 + \frac{1}{\sigma}u(t,\sigma)]^2} < 0 \qquad \forall\sigma > 0 \tag{23}$$

Diffusion contribution:

$$\partial_t u(t,\sigma) = \partial_{u'} g(t,u')\; u''(t,\sigma) \tag{24}$$

Diffusion coefficient

$$\partial_{u'} g(t,\sigma') = A_d\, k^{d+2} \frac{1}{[k^2 + u'(t,\sigma)]^2} > 0 \tag{25}$$

We used Kurganov-Tadmor scheme (FV scheme, second order in Δx... [8]).

[8] Alexander Kurganov and Eitan Tadmor, Journal of Computational Physics 160, 241 (2000).

LPA

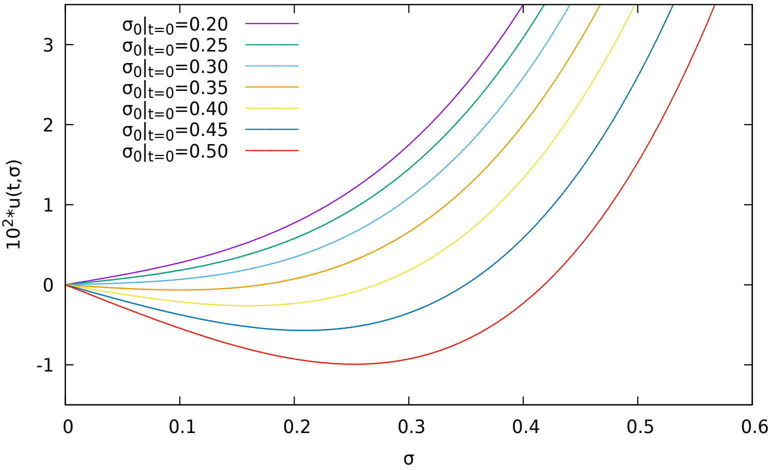


Figure: Effective potential derivative $u(t, \sigma)$ at $t = 1$ for $N = 3$ in LPA.

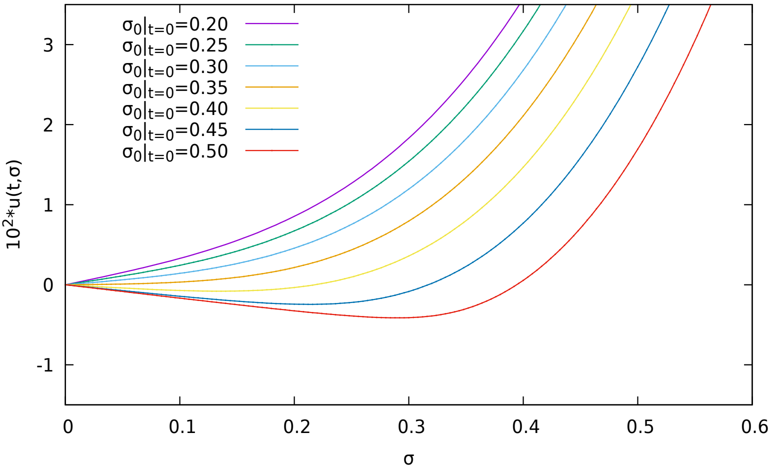
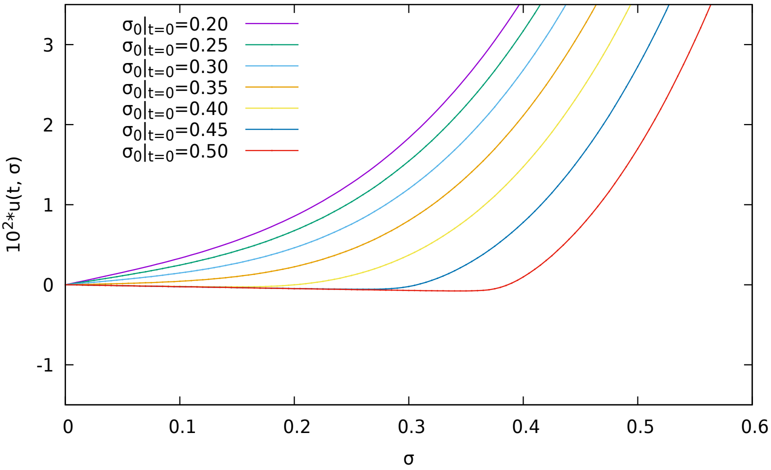


Figure: Effective potential derivative $u(t,\sigma)$ at $t = 2$ for $N = 3$ in LPA.



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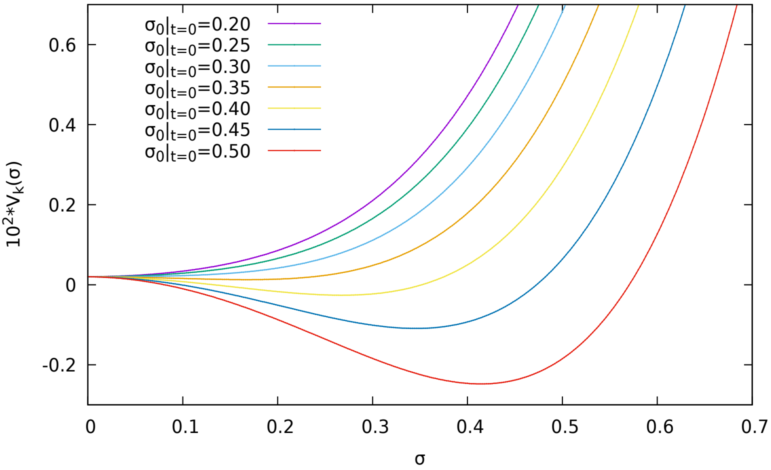


Figure: Effective potential $V_k(\sigma)$ at $t = 1$ for $N = 3$ in LPA.

LPA

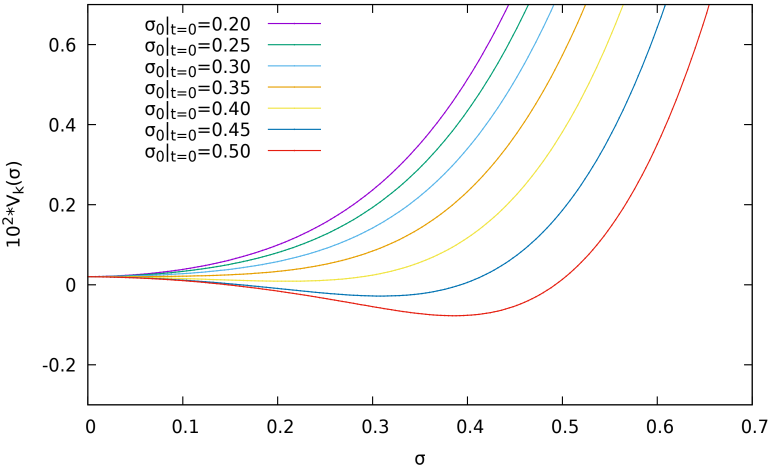


Figure: Effective potential $V_k(\sigma)$ at $t = 2$ for $N = 3$ in LPA.

LPA

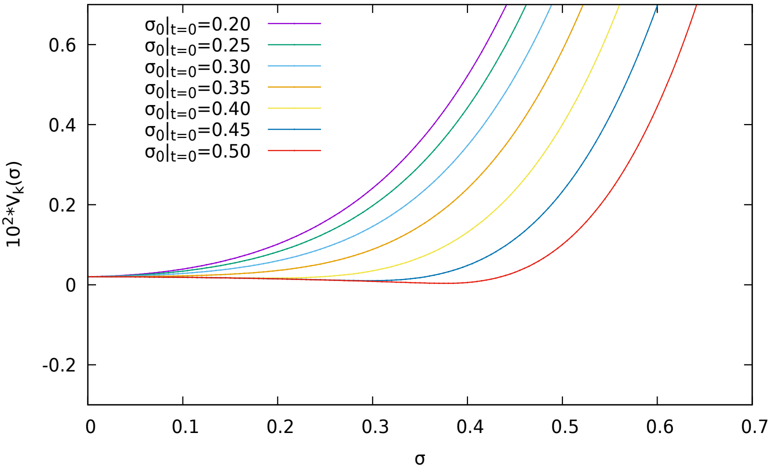


Figure: Effective potential $V_k(\sigma)$ at $t = 3$ for $N = 3$ in LPA.

Critical Behaviour

Critical $\rho_0|_{t=0}$

- Dimensional reduction phenomenon: close to criticality we can use the $T = 0$ $d = 3$ flow equations.

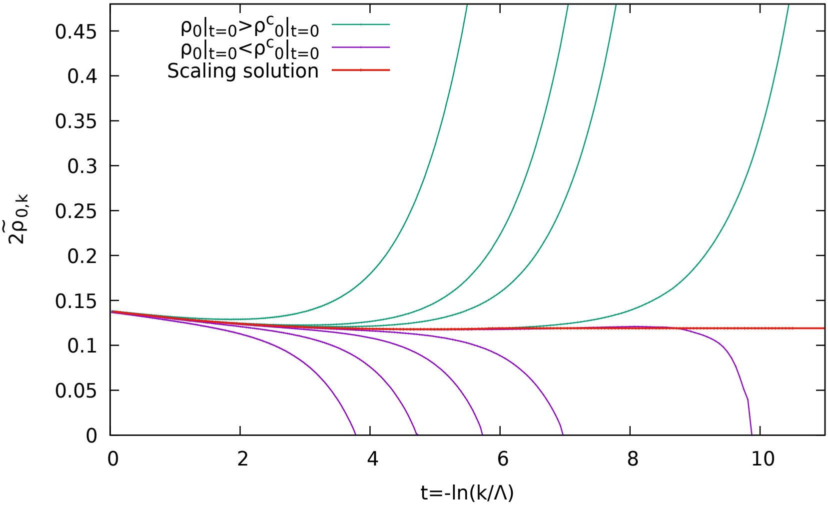
$\sigma_0|_{IR}$ is the *order parameter*, $\rho_0|_{t=0}$ plays the role of the temperature :

- if $\rho_0|_{t=0} > \rho_0^c|_{t=0}$ $(T < T_c)$ $\Rightarrow$ broken phase $\sigma_0|_{IR} > 0$;
- if $\rho_0|_{t=0} < \rho_0^c|_{t=0}$ $(T > T_c)$ $\Rightarrow$ symmetric phase $\sigma_0|_{IR} = 0$.

We fix λ to an arbitrary value $\lambda = 0.5$ and adjust $\rho_0|_{t=0}$ at $t = 0$ in order to find the *scaling solution*.

- broken phase $(\rho_0|_{t=0} > \rho_0^c|_{t=0})$: $\rho_{0,k} \to \rho_0|_{IR} > 0$
  $\Rightarrow$ $\tilde{\rho}_{0,k} = \frac{\rho_0}{k} \to +\infty$ as $k \to 0$;
- symmetric phase $(\rho_0|_{t=0} < \rho_0^c|_{t=0})$: $\rho_{0,k} \to \rho_0|_{IR} = 0$
  $\Rightarrow$ $\tilde{\rho}_{0,k} = \frac{\rho_0}{k} \to 0$ as $k \to 0$;

Critical $\rho_0\big|_{t=0}$



Critical exponent ν

The *correlation length* ξ diverges close to the criticality as

$$\xi(\rho_0|_{t=0}) \sim \left(\left|\rho_0|_{t=0} - \rho_0^c|_{t=0}\right|\right)^{-\nu} \tag{26}$$

We consider the renormalized mass m^2

$$m^2 = \lim_{k\to 0} u'(\sigma = 0, k) = \lim_{k\to 0} V''(\sigma = 0, k) = \frac{1}{\xi^2} \tag{27}$$

Thus

$$m^2 \sim \left(\left|\rho_0|_{t=0} - \rho_0^c|_{t=0}\right|\right)^{2\nu} \tag{28}$$

We can obtain ν from

$$\ln m^2 = 2\nu \ln\left(\left|\rho_0|_{t=0} - \rho_0^c|_{t=0}\right|\right) + const \tag{29}$$

Critical exponent ν

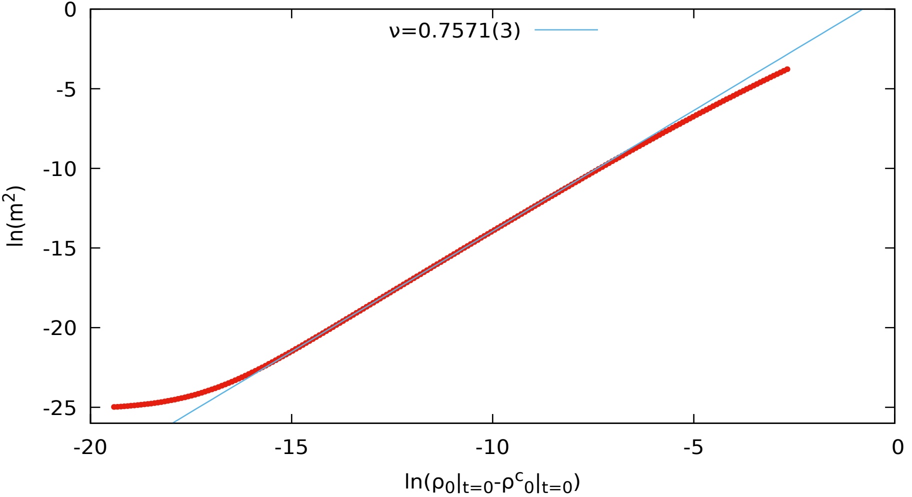


Figure: Critical exponent ν for $N = 3$ in LPA.

Critical exponent β

Close to criticality, the order parameter $\sigma_0|_{IR}$ is described by the following behaviour

$$\sigma_0|_{IR} = \begin{cases} 0 & \rho_0|_{t=0} < \rho_0^c|_{t=0} \\ \sim (\rho_0|_{t=0} - \rho_0^c|_{t=0})^\beta & \rho_0|_{t=0} > \rho_0^c|_{t=0} \end{cases} \tag{30}$$

So we can extract β from

$$\ln \sigma_0|_{IR} = \beta \ln(\rho_0|_{t=0} - \rho_0^c|_{t=0}) + const \tag{31}$$

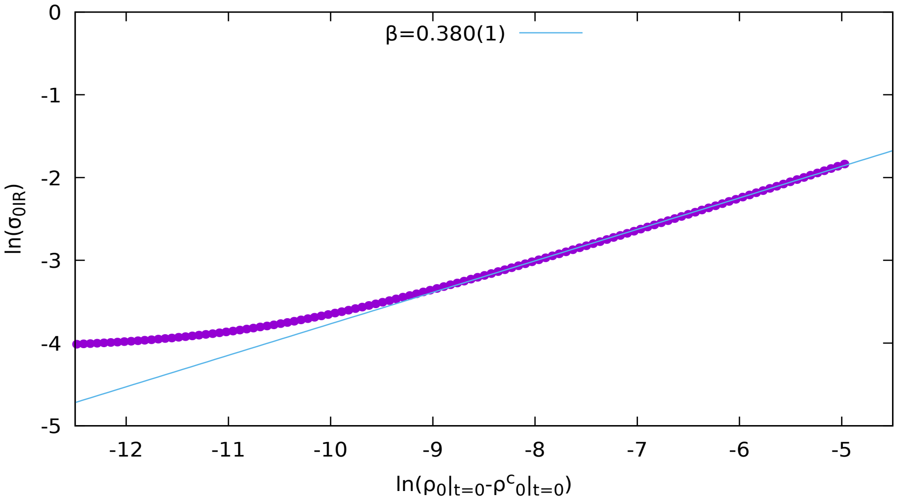
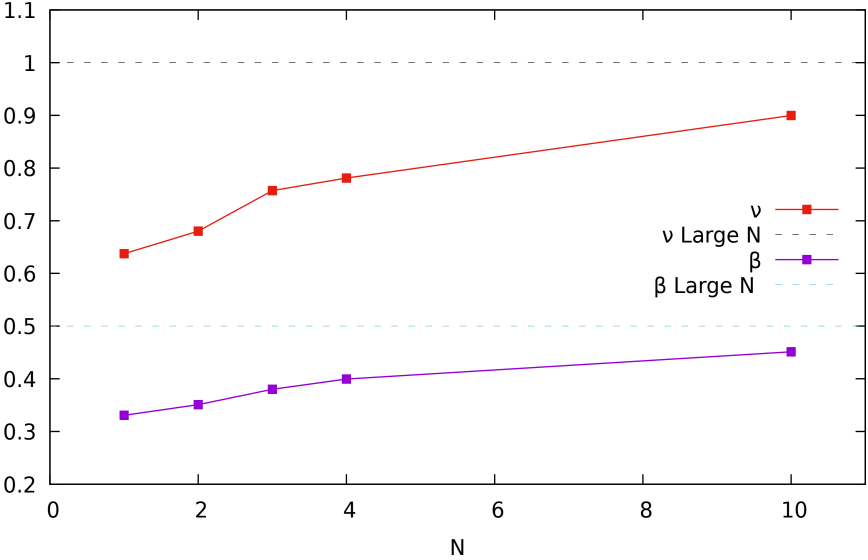
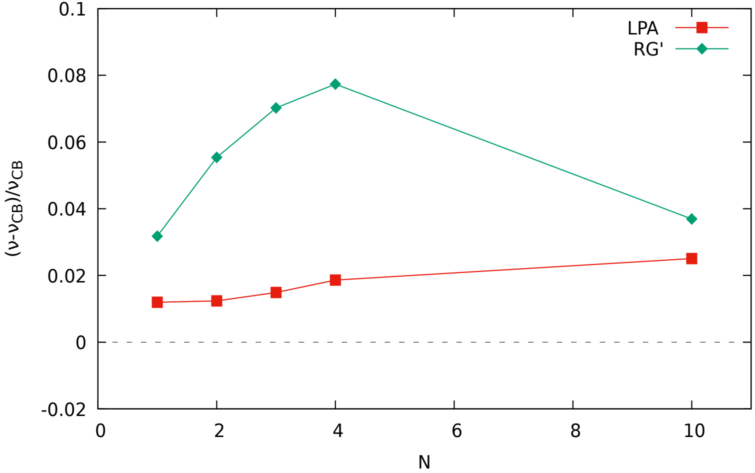


Figure: Critical exponent β for $N = 3$ in LPA.

Results



Results



RG': [9] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J.M. Pawlowski, M. Tissier, and N. Wschebor, Physics Reports 910, 1–114 (2021).
CB : [10] H. Shimada. Fractal dimensions of self-avoiding walks and ising high-temperature graphs in 3d conformal bootstrap. Journal of Statistical Physics 165, 1006-1035 (2016).

Conclusions and Outlook

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What's next?

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What's next?

- Numerical tests for precision;

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What's next?

- Numerical tests for precision;
- Possible generalizations: higher order truncations (LPA', $O(\partial^2)$, ...), extension to different models.

Thanks for your attention!

Appendix

Critical exponent ν

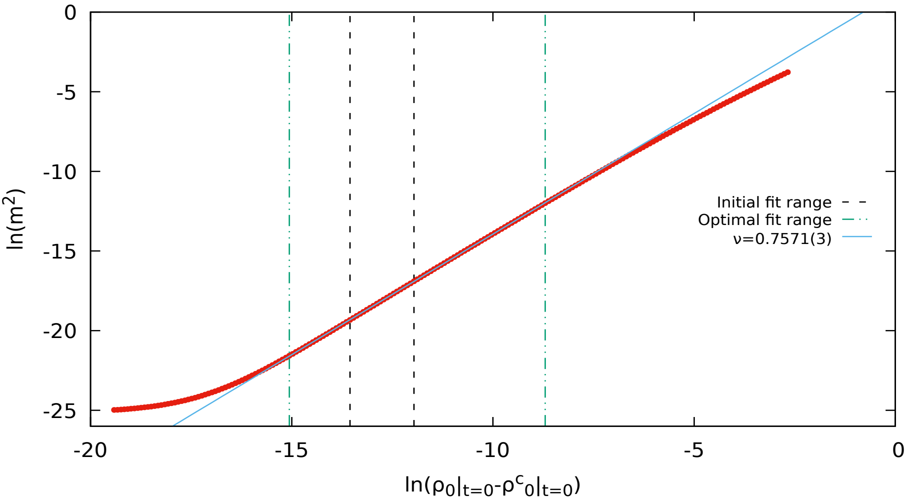
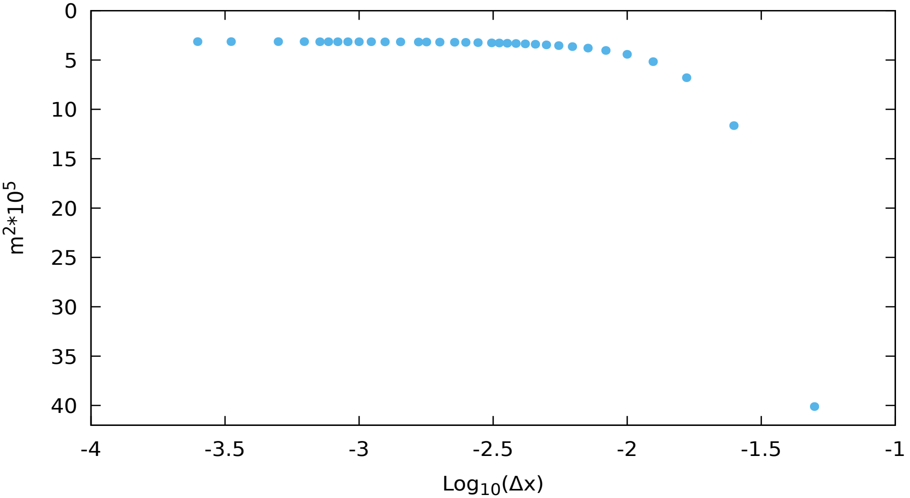
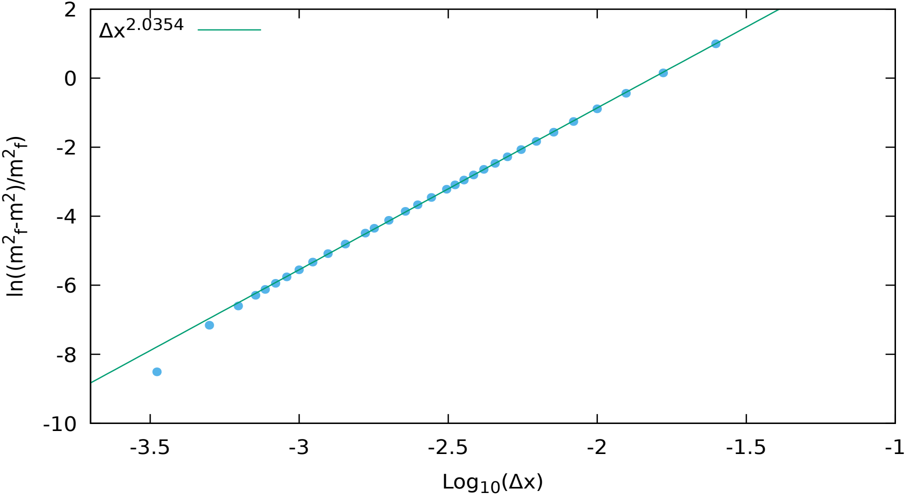


Figure: Fit to obtain critical exponent ν for $N = 3$ in LPA.

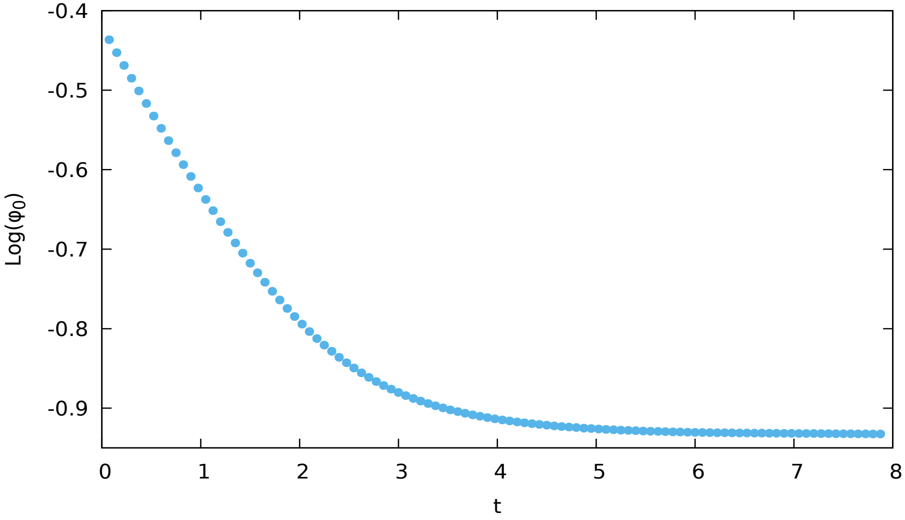
Mass



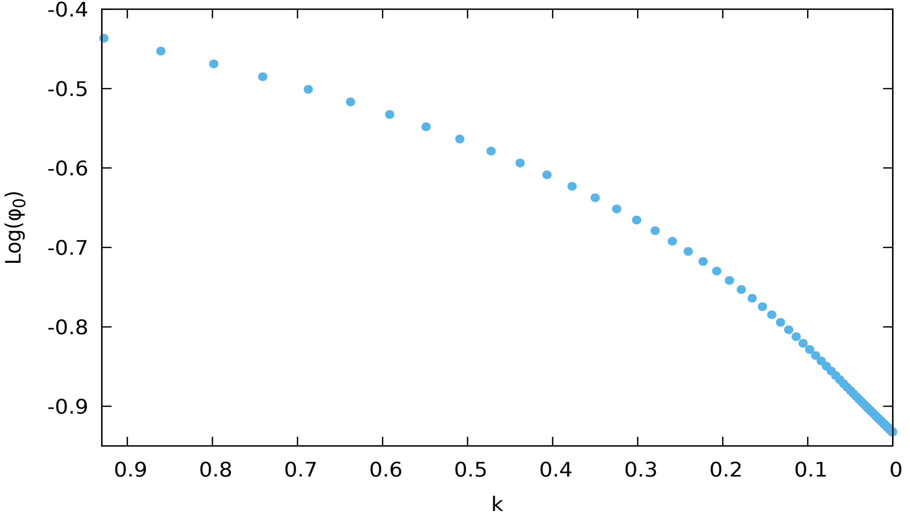


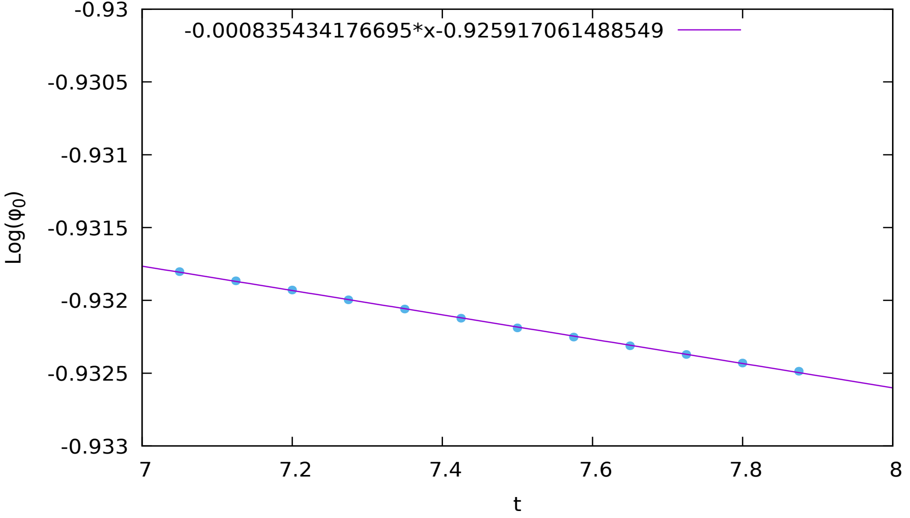
$\Delta x^{2.0354}$
2
0
-2
-4
-6
-8
-10
-3.5
-3
-2.5
-2
-1.5
-1
$\ln((m^2_f - m^2)/m^2_f)$
$\mathrm{Log}_{10}(\Delta x)$

Minimum

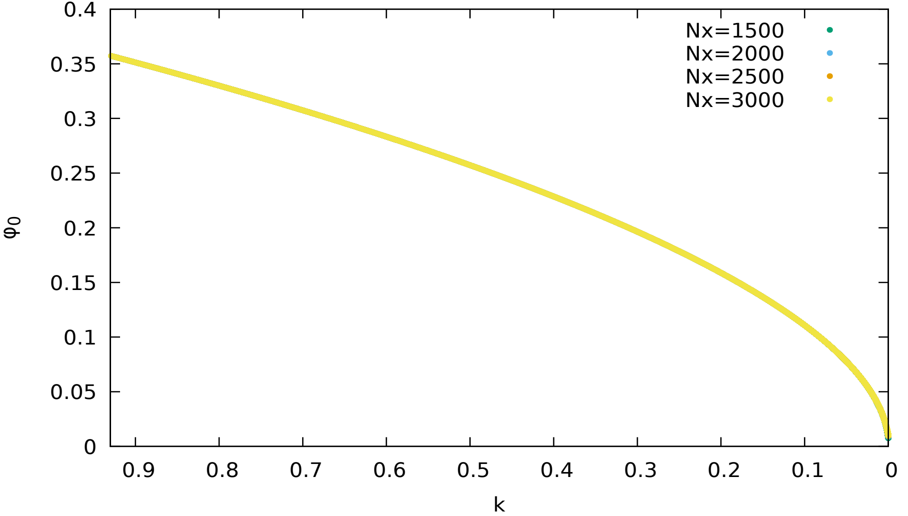


Minimum



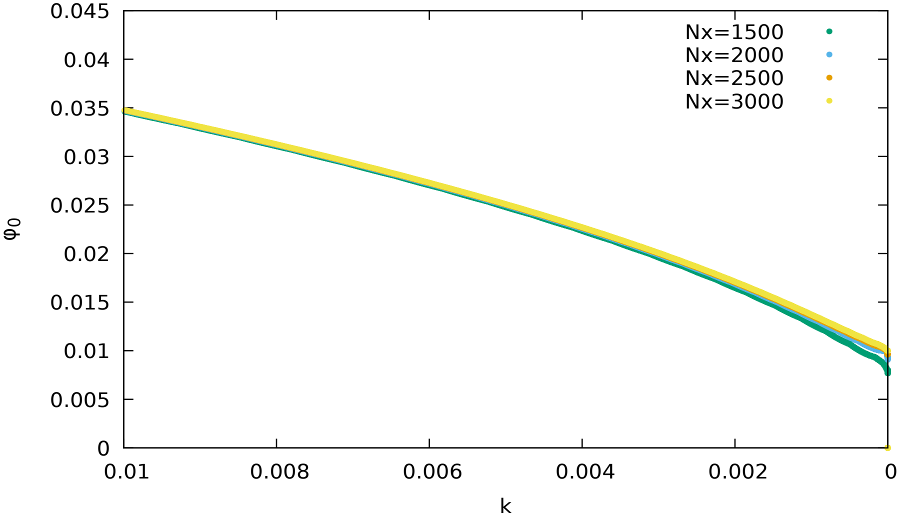

-0.000835434176695*x-0.925917061488549
-0.93
-0.9305
-0.931
-0.9315
-0.932
-0.9325
-0.933
Log(φ0)
7
7.2
7.4
7.6
7.8
8
t

Minimum



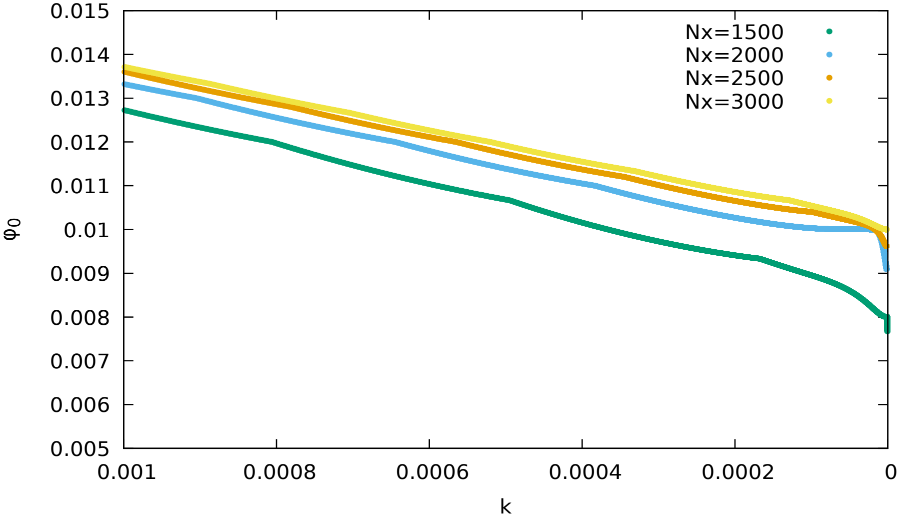
Nx=1500
Nx=2000
Nx=2500
Nx=3000
φ0
k
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0

Minimum



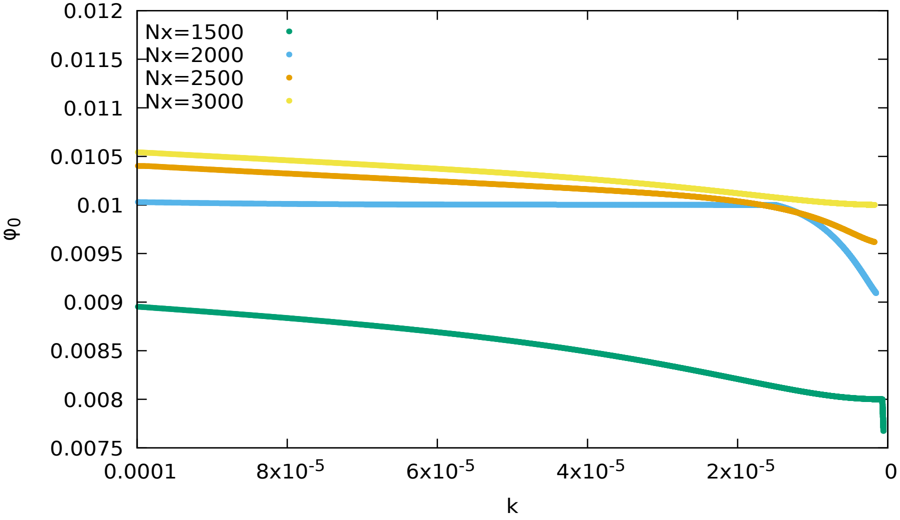
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
φ0
0.01
0.008
0.006
0.004
0.002
0
k
Nx=1500
Nx=2000
Nx=2500
Nx=3000

Minimum



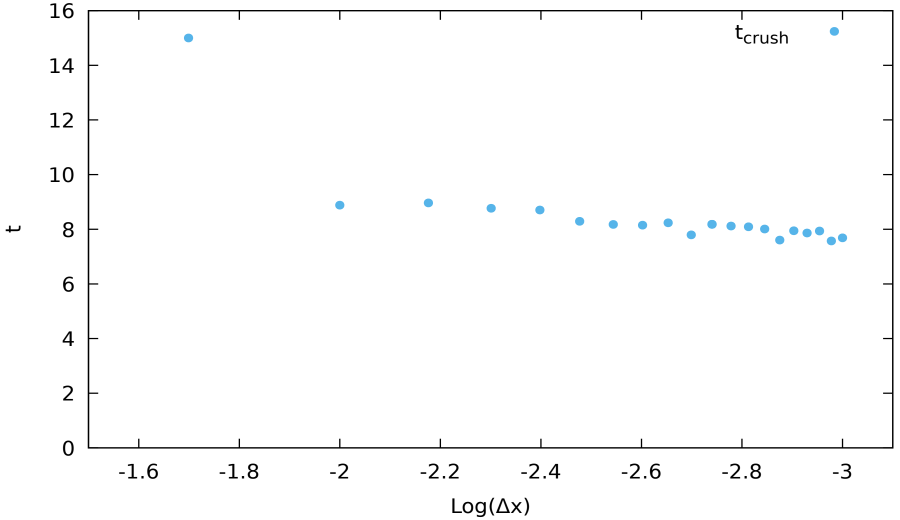
Nx=1500
Nx=2000
Nx=2500
Nx=3000
0.015
0.014
0.013
0.012
0.011
0.01
0.009
0.008
0.007
0.006
0.005
φ_0
0.001
0.0008
0.0006
0.0004
0.0002
0
k

Minimum



Nx=1500
Nx=2000
Nx=2500
Nx=3000
0.012
0.0115
0.011
0.0105
0.01
0.0095
0.009
0.0085
0.008
0.0075
φ0
0.0001
8x10-5
6x10-5
4x10-5
2x10-5
0
k

Crush



Flow equation

A versatile approach to the computation of Γ is based on Renormalization Group RG concepts.
We are looking for an interpolating action Γ_k, the *effective average action*, such that

$$\Gamma_{k\to\Lambda} = S_{bare}\,, \quad \Gamma_{k\to 0} = \Gamma \tag{32}$$

This can be constructed through the definition of the IR regulated functional

$$e^{W_k[J]} \equiv Z_k[J] := \exp\left(-\Delta S_k\left[\frac{\delta}{\delta J}\right]\right) Z[J] =$$

$$= \int_\Lambda \mathcal{D}\varphi\, e^{-S[\varphi]-\Delta S_k[\varphi]-\int J\varphi} \tag{33}$$

where ΔS_K is a regulator therm of the form

$$\Delta S_k[\varphi] = \frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \, \varphi(-q) \, R_k(q) \, \varphi(q) \tag{34}$$

The regulator function $R_k(q)$ should satisfy

$$\lim_{q^2/k^2 \to 0} R_k > 0 \tag{35}$$

$$\lim_{k^2/q^2 \to 0} R_k(q) = 0 \tag{36}$$

$$\lim_{k^2 \to \Lambda^2 \to \infty} R_k(q) \to \infty \tag{37}$$

We introduce the RG-time t, using the abbreviations

$$t = \ln \frac{k}{\Lambda}, \qquad \partial_t = k \frac{d}{dk} \tag{38}$$

Keeping the source J fixed, i.e. k independent, we obtain

$$\partial_t W_k[J] = -\frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \partial_t R_k(q) G_k(q) - \partial_t \Delta S_k[\phi] \tag{39}$$

Here, we have defined the *connected propagator*

$$G_k(p) = \left(\frac{\delta^2 W_k}{\delta J \delta J} \right)(p) = \langle \varphi(-p)\varphi(p) \rangle - \langle \varphi(-p) \rangle \langle \varphi(p) \rangle \tag{40}$$

Functional renormalization group

We define the interpolating effective action Γ_k

$$\Gamma_k[\phi] = \sup_J \left(\int J\phi - W_k[J] \right) - \Delta S_k[\phi] \tag{41}$$

At $J = J_{\text{sup}}$:

$$\phi(x) = \langle \varphi(x) \rangle_J = \frac{\delta W_k[J]}{\delta J(x)} \tag{42}$$

Computing the functional derivative of Eq. (41) with respect to ϕ we get

$$J(x) = \frac{\delta \Gamma_k[\phi]}{\delta \phi(x)} + (R_k \phi)(x) \tag{43}$$

From this, we deduce:

$$\frac{\delta J(x)}{\delta \phi(y)} = \frac{\delta^2 \Gamma_k[\phi]}{\delta \phi(x) \delta \phi(y)} + R_k(x, y) \tag{44}$$

Functional renormalization group

We obtain from Eq. (42):

$$\frac{\delta\phi(y)}{\delta J(x')} = \frac{\delta^2 W_k[J]}{\delta J(x')\delta J(y)} \equiv G_k(y - x') \tag{45}$$

This implies the important identity

$$\mathbb{I} = (\Gamma^{(2)} + R_k)G_k \tag{46}$$

Here, we have introduced the notation

$$\Gamma_k^{(n)}(x_1, \cdots x_n) = \frac{\delta^n \Gamma_k[\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_n)} \tag{47}$$

Finally we can derive the *flow equation* for Γ_k for fixed ϕ and at $J = J_{\text{sup}}$:

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\partial_t R_k(\Gamma^{(2)}[\phi] + R_k)^{-1}\right] \tag{48}$$

We can notice that:

- The flow equation is a functional differential equation for Γ_k;
- We may define QFT based on the flow equation.
- The purpose of the regulator is twofold;
- The solution of the flow equation corresponds to an RG trajectory in *theory space*;
- The variation of the trajectory with respect to R_k reflects the RG scheme dependence of a non-universal quantity, but the final point on the trajectory is independent of R_k;
- Perturbation theory can immediately be re-derived from the flow equation, for instance, imposing the loop expansion on Γ_k, $\Gamma_k = S + \hbar\Gamma_k^{1-loop}$.

Various systematic approximation exist which can be summarized under the label of *method of truncations*.
A first example for such an approximation scheme is the vertex expansion, which now reads

$$\Gamma_k[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^D x_1 \cdots d^D x_n \, \Gamma_k^{(n)}(x_1 \cdots x_n) \, \phi(x_1) \cdots \phi(x_n) \tag{49}$$

As a second example, let us introduce the *operator expansion*

$$\Gamma_k = \int d^D x \left[V_k(\phi) + \frac{1}{2} Z_k(\phi)(\partial_\mu \phi)^2 + \mathcal{O}(\partial^4) \right] \tag{50}$$

where, for instance, $V_k(\phi)$ corresponds to the effective potential.

LPA

The firts ansatz we will use is the LPA:

$$\Gamma_k[\vec{\phi}] = \int_x \left\{ \frac{1}{2}(\partial_\mu \phi_a)^2 + V_k(t, \rho) \right\} \tag{51}$$

where $V_k(t, \rho)$ is the effective potential.
Then we compute the two point functions $\Gamma_k^{(2)}$:

$$\Gamma_{k,ab}^{(2)}(t, \rho, p) = [p^2 + V_k'(t, \rho)]\delta_{ab} + 2\rho V_k''(t, \rho)\delta_{aN}\delta_{bN} \tag{52}$$

with

$$V_k'(t, \rho) = \partial_\rho V_k(t, \rho) \qquad \text{and} \qquad V_k''(t, \rho) = \partial_\rho V_k'(t, \rho)$$

As regulator, we chose the Litim Regulator

$$R_k(p) = (k^2 - p^2)\Theta(k^2 - p^2) \tag{53}$$

Inserting (52) into (9) and performing the integral we obtain the flow equation for the effective potential

$$\partial_t V_k(t,\rho) = -A_d\, k^{d+2}\left(\frac{N-1}{k^2 + V_k'(t,\rho)} + \frac{1}{k^2 + V_k'(t,\rho) + 2\rho V_k''(t,\rho)}\right) \tag{54}$$

with $A_d = \frac{\Omega_d}{(2\pi)^d d}$, and $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$.

It is preferable to formulate the problem in terms of the field $\sigma = \sqrt{2\rho}$. We thus rewrite the flow equation as:

$$\partial_t V_k(t,\sigma) = -A_d\, k^{d+2}\left(\frac{N-1}{k^2 + \frac{1}{\sigma}\partial_\sigma V_k(t,\sigma)} + \frac{1}{k^2 + \partial^2_{\sigma\sigma} V_k(t,\sigma)}\right) \tag{55}$$

We introduce the derivative of the potential as new variable

$$u(t,\sigma) = \partial_\sigma V_k(t,\sigma), \qquad u'(t,\sigma) = \partial_\sigma u(t,\sigma) \tag{56}$$

We can now introduce the convection flux

$$f(t,\sigma,u) = A_d\, k^{d+2} \frac{N-1}{k^2 + \frac{1}{\sigma} u(t,\sigma)} \tag{57}$$

and the diffusion flux

$$g(t,u') = -A_d\, k^{d+2} \frac{1}{k^2 + u'(t,\sigma)} \tag{58}$$

Taking the derivative of (54) with respect to σ we obtain a *convection-diffusion* equation for the derivative of the potential u

$$\partial_t u(t,\sigma) + \partial_\sigma f(t,\sigma,u) = \partial_\sigma g(t,u') \tag{59}$$

LPA'

In addition to the effective potential $V_k(t, \rho)$, the effective action includes a field renormalization factor Z_k (in the LPA $Z_k = 1$) which is field independent:

$$\Gamma_k[\vec{\phi}\,] = \int d^D x \left[\frac{1}{2} Z_k (\partial_\mu \phi_a)^2 + V_k(t, \rho) \right] \tag{60}$$

One can then define a “running” anomalous dimension

$$\eta_k = -k\, \partial_k \ln Z_k = \partial_t \ln Z_k \tag{61}$$

At the critical point one has

$$\lim_{k \to 0} \eta_k \equiv \eta \tag{62}$$

We need to compute the 2-point function. We choose a simple background field

$$\vec{\phi} = (0, \cdots, 0, \sigma)$$

$$\Gamma^{(2)}_{k,ab}(p, \vec{\phi}) = \frac{\phi_a \phi_b}{2\rho} \Gamma^{(2)}_{k,L}(p, \sigma) + \left(\delta_{ab} - \frac{\phi_a \phi_b}{2\rho} \right) \Gamma^{(2)}_{k,T}(p, \sigma) \tag{63}$$

where

$$\Gamma^{(2)}_{k,L}(p, \sigma) = Z_k p^2 + \partial^2_{\sigma\sigma} V_k(t, \sigma) \tag{64}$$

$$\Gamma^{(2)}_{k,T}(p, \sigma) = Z_k p^2 + \frac{1}{\sigma} \partial_\sigma V_k(t, \sigma) \tag{65}$$

Analogously, we can introduce the *longitudinal* and *transverse propagators*:

$$G_{k,\alpha}(p, \sigma) = \left[\Gamma^{(2)}_{k,\alpha}(p, \sigma) + R_k(p) \right]^{-1} \qquad \alpha = L, T \tag{66}$$

Now we just need to plug Eq.s (**??**)-(**??**) into the flow equation :

$$\partial_t V_k(t,\sigma) = \frac{1}{2}\int_q \partial_t R_k(q)\left[G_{k,L}(q,\sigma) + (N-1)G_{k,T}(q,\sigma)\right] =$$

$$= \frac{1}{2}\int_q \partial_t R_k(q)\left[\frac{1}{Z_k q^2 + \frac{1}{\sigma}\partial_\sigma V_k(t,\sigma) + R_k(q)} + \frac{N-1}{Z_k q^2 + \partial^2_{\sigma\sigma} V_k(t,\sigma) + R_k(q)}\right] \tag{67}$$

As regulator we choose:

$$R_k(p) = \alpha Z_k(k^2 - p^2)\Theta(k^2 - p^2) \tag{68}$$

with $\alpha \sim 1$ as a free parameter.

We obtain the flow equation for the effective potential:

$$\partial_t V_k(t,\sigma,\eta_k) = \frac{\alpha}{4\pi^2(1-\alpha)} k^3 \Bigg\{ (N-1)\Big\{ \left[\eta_k\,(1+m_1)-2\right] \times \left(1-\sqrt{m_1}\arctan\frac{1}{\sqrt{m_1}}\right) - \frac{1}{3}\eta_k \Big\} + \Big\{ \left[\eta_k\,(1+m_2)-2\right] \times \left(1-\sqrt{m_2}\arctan\frac{1}{\sqrt{m_2}}\right) - \frac{1}{3}\eta_k \Big\} \Bigg\} \tag{69}$$

where:

$$m_1(Z_k,\partial_\sigma V_k,\sigma) = \frac{\alpha}{1-\alpha} + \frac{\partial_\sigma V_k}{Z_k\,\sigma\,k^2\,(1-\alpha)} \tag{70}$$

$$m_2(Z_k,\partial^2_{\sigma\sigma} V_k) = \frac{\alpha}{1-\alpha} + \frac{\partial^2_{\sigma\sigma} V}{Z_k\,k^2\,(1-\alpha)} \tag{71}$$

We introduce the conserved quantity

$$u(t, \sigma, \eta_k) = \partial_\sigma V_k(t, \sigma, \eta_k) \tag{72}$$

and

$$u'(t, \sigma, \eta_k) = \partial_\sigma u(t, \sigma, \eta_k) = \partial^2_{\sigma\sigma} V_k(t, \sigma, \eta_k) \tag{73}$$

Thus we can rewrite

$$m_1(Z_k, u, \sigma) = \frac{\alpha}{1-\alpha} + \frac{u}{Z_k\, \sigma\, k^2\, (1-\alpha)} \tag{74}$$

$$m_2(Z_k, u') = \frac{\alpha}{1-\alpha} + \frac{u'}{Z_k\, k^2\, (1-\alpha)} \tag{75}$$

We can now define the convection flux

$$f(k,u,\sigma,\eta_k) = \frac{\alpha}{4\pi^2(1-\alpha)}(N-1)k^3 \times$$

$$\times \left\{ \left[2-\eta_k\left(1+m_1\right)\right]\left(1-\sqrt{m_1}\arctan\frac{1}{\sqrt{m_1}}\right)+\frac{1}{3}\eta_k \right\} \tag{76}$$

and the diffusion flux

$$g(k,u',\eta_k) = \frac{\alpha}{4\pi(1-\alpha)}k^3 \times$$

$$\times \left\{ \left[\eta_k\left(1+m_2\right)-2\right]\left(1-\sqrt{m_2}\arctan\frac{1}{\sqrt{m_2}}\right)-\frac{1}{3}\eta_k \right\} \tag{77}$$

We take a derivative of (**??**) with respect to σ:

$$\partial_t u(t,\sigma,\eta_k) + \partial_\sigma f(t,u,\sigma,\eta_k) = \partial_\sigma g(t,u',\eta_k) \tag{78}$$

From the definition of the effective action (**??**) we derive that

$$Z_k = \lim_{p \to 0} \frac{\partial}{\partial p^2} \Gamma^{(2)}_{k,T}(p, \sigma_{0,k}) \tag{79}$$

Trivially we can obtain an equation for $\partial_t Z_k$ just taking the derivative of Eq.(**??**) w.r.t RG time t:

$$\partial_t Z_k = \lim_{p \to 0} \frac{\partial}{\partial p^2} \partial_t \Gamma^{(2)}_{k,T}(p, \sigma_{0,k}) \tag{80}$$

Now we can easily exploit the flow equation for Γ_k to obtain a flow equation for $\Gamma^{(2)}_k$:

$$\partial_t \Gamma^{(2)}_k[\vec{\phi}] = \frac{1}{2} \mathrm{Tr} \left[\partial_t R_k \frac{\delta^2}{\delta \phi_a \delta \phi_b} (\Gamma^{(2)}_k[\vec{\phi}] + R_k)^{-1} \right] \tag{81}$$

Computing the derivatives, projecting (**??**) onto the transverse direction and summing over the indices one obtains

$$\partial_t Z_k = 4A_d \partial^2_{\rho\rho} V_k(\rho_{0,k}, t) \tilde{\partial}_t \int_0^\infty dq \; q^{d+1} \partial_{q^2} G_{k,L}(q, \rho_{0,k}) \, \partial_{q^2} G_{k,T}(q, \rho_{0,k}) \tag{82}$$

where the symbol $\tilde{\partial}_t$ indicates that the time derivative acts only on the t dependence of R_k.
It is convenient to introduce the following dimensionless quantities:

$$\tilde{\rho} = Z_k k^{2-d} \rho \qquad \tilde{V}_k(\tilde{\rho}, t) = k^{-d} V_k(\rho, t) \tag{83}$$

We will also use the dimensionless form for the regulator:

$$R_k(p^2) = Z_k p^2 r(y) \tag{84}$$

with $y = p^2/k^2$. We will also use the shorthand notations

$$r' = \frac{dr(y)}{dy} \qquad r'' = \frac{d^2 r(y)}{dy^2}$$

In particular we will use the regulator (**??**), thus

$$regulator2 r(y) = \alpha \left(\frac{1-y}{y} \right) \theta(1-y) \tag{85}$$

In this way, we can easily put Eq.(**??**) in a dimensionless form:

$$\eta_k = 4A_d\tilde{\rho}_{0,k}\,(\partial^2_{\tilde{\rho}\tilde{\rho}}\tilde{V}_k(\tilde{\rho}_{0,k}))^2\,m^d_{22}(2\tilde{\rho}_{0,k}\partial^2_{\tilde{\rho}\tilde{\rho}}\tilde{V}_k(\tilde{\rho}_{0,k}),\eta_k) \tag{86}$$

where we have defined the threshold function

$$m^d_{22}(w,\eta) = \int_0^\infty dy\,y^{d/2}\frac{1+r+yr'}{P(w)^2P(0)^2}\left\{y(\eta r+2yr')(1+r+yr')\times\right.$$

$$\left.\times\left[\frac{1}{P(w)}+\frac{1}{P(0)}\right]-\eta r-(\eta+4)yr'-2y^2r''\right\} \tag{87}$$

and

$$P(w) = y(1+r)+w$$

In the particular choice we made for the regulator (**??**) we get:

$$m_{22}^d(w,\eta) = \int_0^1 dy\, y^{d/2} \frac{(1-\alpha)\alpha}{P(w)^2 P(0)^2} \Big\{ \big[\eta(1-y) - 2\big] \times$$
$$eta3 \times\ (1-\alpha) \left[\frac{1}{P(w)} + \frac{1}{P(0)}\right] + \eta \Bigg\} \tag{88}$$

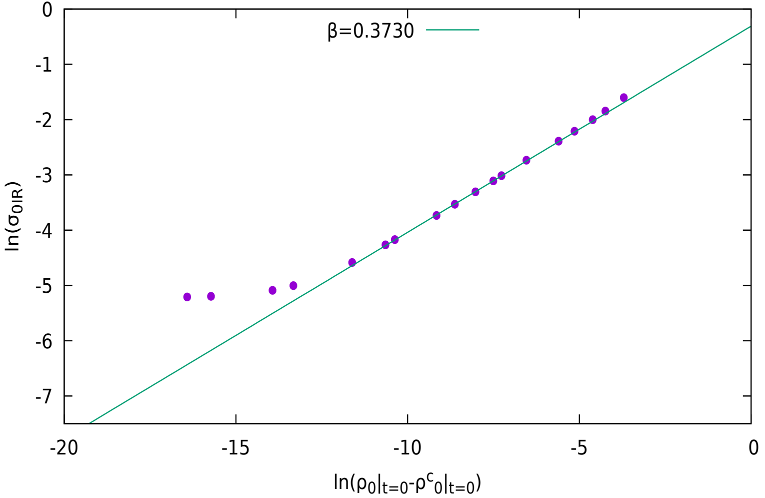


Figure: Critical exponent β for $N = 3$ in LPA.

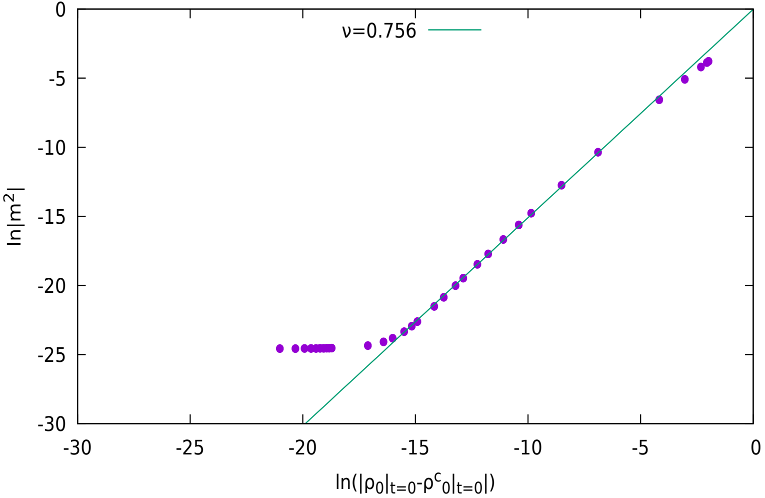
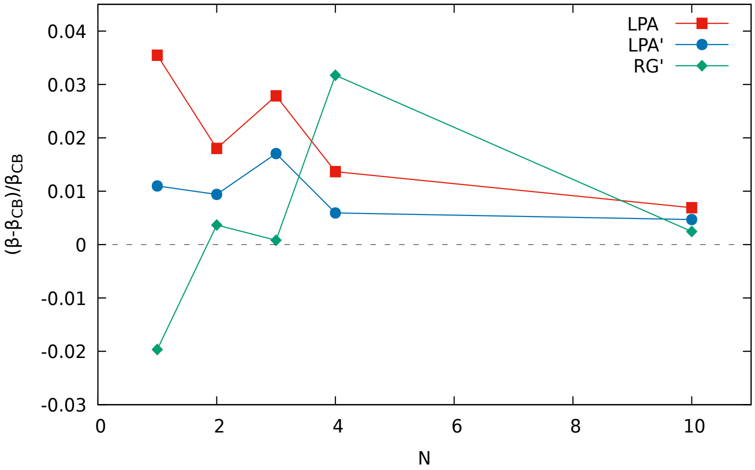
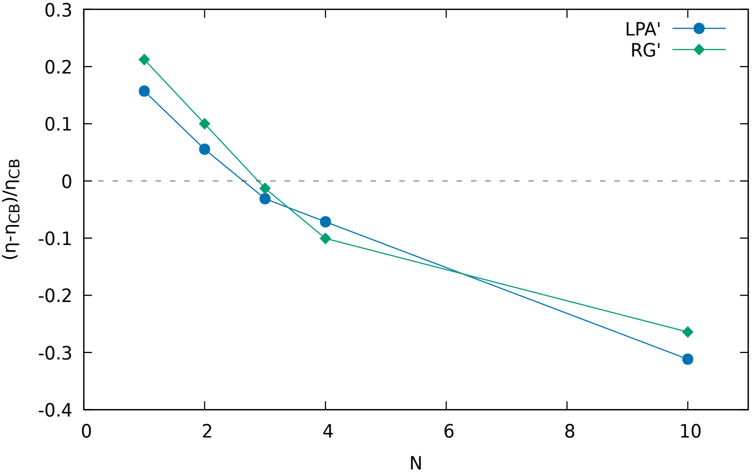


Figure: Critical exponent ν for $N = 3$ in LPA.



LPA
LPA'
RG'
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
0
2
4
6
8
10
N
(β-β_CB)/β_CB



LPA'
RG'
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
(η-ηCB)/ηCB
0
2
4
6
8
10
N

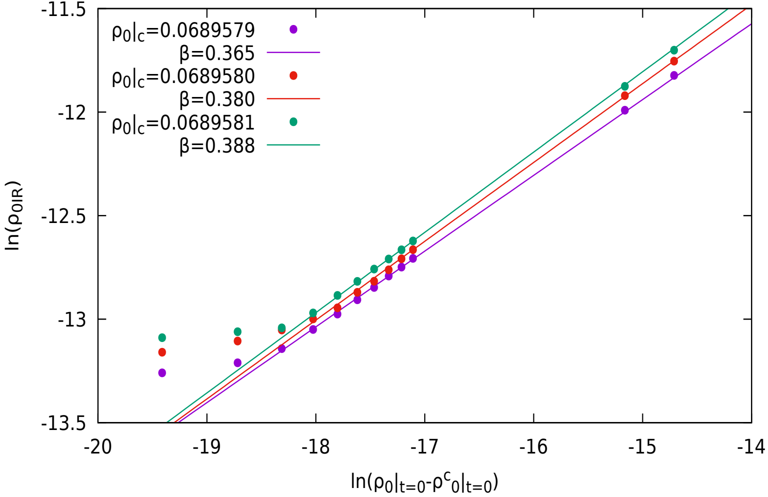


Figure: Error on ritical exponent β for $N = 3$ in LPA.

Results Tables

N	LPA	LPA'	RG'	MC	$\varepsilon-$exp.	CB	DE_4
1	0.338(14)	0.330(16)	0.32	0.32643(6)	0.32599(32)	0.326419(2)	0.3263(4)
2	0.355(16)	0.352(20)	0.35	0.34864(5)	0.3472(6)	0.34872(5)	0.3485(5)
3	0.380(1)	0.376(17)	0.37	0.3689(3)	0.366(1)	0.3697(12)	0.3691(7)
4	0.393(16)	0.390(17)	0.40	0.3873(4)	0.3834(18)	0.3877(47)	0, 3874(6)
10	0.452(12)	0.451(15)	0.45		0.4398(7)		0.4489(6)

Table: Order parameter exponent β.

N	LPA	LPA'	RG'	MC	$\varepsilon-$exp.	CB	DE_4
1	0.655(17)	0.639(19)	0.64	0.63002(10)	0.6292(5)	0.629971(4)	0.6299(3)
2	0.687(15)	0.683(18)	0.69	0.67169(7)	0.6690(10)	0.6718(1)	0.6716(6)
3	0.7571(3)	0.732(18)	0.74	0.7112(5)	0.7059(20)	0.7120(23)	0.7114(9)
4	0.788(15)	0.773(15)	0.78	0.7477(8)	0.7397(35)	0.7472(87)	0.7478(9)
10	0.911(10)	0.910(12)	0.91		0.859(1)		0.8776(10)

Table: Correlation length exponent ν.

N	LPA	LPA'	RG'	MC	$\varepsilon-$exp.	CB	DE$_4$
1	0	0.0420(13)	0.044	0.03627(10)	0.0362(6)	0.0362978(20)	0.0362(12)
2	0	0.0403(14)	0.042	0.03810(8)	0.0380(6)	0.03818(4)	0.0380(13)
3	0	0.0373(11)	0.038	0.0375(5)	0.0378(5)	0.0385(13)	0.0376(13)
4	0	0.0351(11)	0.034	0.0360(4)	0.0366(4)	0.0378(32)	0.0360(12)
10	0	0.0159(10)	0.017		0.024(1)		0.0231(6)

Table: Anomalous dimension η.

Central schemes

Central schemes are finite-volume methods for solving non-linear advection–diffusion equations. In particular conservation laws are expressed in the form

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}f(u(x,t)) = 0 \tag{89}$$

while convection-diffusion equations are represented by

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}f(u(x,t)) = \frac{\partial}{\partial x}Q[u(x,t), u_x(x,t)] \tag{90}$$

where:

- $u(x,t) = (u_1(x,t), \cdots, u_N(x,t))$ is an N-component vector of conserved quantities in the d spatial variables $x = (x_1, \cdots, x_d)$;
- $f(u) = (f_1, \cdots, f_d)$ is the advection flux;
- $Q(u, u_x)$, or in general $Q(u, \nabla_x u) = (Q_1, \cdots, Q_d)$, is a diffusion flux.

Kurganov and Tadmor scheme

The main idea in the Kurganov and Tadmor central scheme is to average the nonsmooth parts of the computed solution over smaller cells of variable size.
We need to estimate the local speed of propagation at the cell boundaries: the upper bound is denoted by $a^n_{j+1/2}$ and given by

$$a^n_{j+1/2} = \max_{u \in \mathcal{C}(u^-_{j+1/2}, u^+_{j+1/2})} \rho\left(\frac{\partial f}{\partial u}(u)\right) \tag{91}$$

where

- ρ denotes the spectral radius of the flux Jacobian,
- $u^-_{j+1/2} = u^n_{j+1} - \frac{\Delta x}{2}(u_x)^n_{j+1}$ and $u^+_{j+1/2} = u^n_j + \frac{\Delta x}{2}(u_x)^n_j$ are the correspondent left and right intermediate values of $\hat{u}(x,t)$ at $x_{j+1/2} = x_j + \Delta x/2$,
- $\mathcal{C}(u^-_{j+1/2}, u^+_{j+1/2})$ is a curve in phase space connecting $u^-_{j+1/2}$ and $u^+_{j+1/2}$.

The derivatives are reconstructed through the *minmod limiter*:

$$(u_x)^n_j = \text{minmod}\left(\frac{\bar{u}^n_j - \bar{u}^n_{j-1}}{\Delta x}, \frac{\bar{u}^n_{j+1} - \bar{u}^n_j}{\Delta x}\right), \tag{92}$$

with $\text{minmod}(a,b) = 1/2[\text{sgn}(a) + \text{sgn}(b)] \cdot \text{min}(|a|,|b|)$

Kurganov and Tadmor scheme

Kurganov and Tadmor scheme is then constructed in the following steps.

- We assume we have already computed the piecewise-linear solution at time level t^n, based on the cell averages $\bar{u}_j^n$

$$u(x, t^n) \approx \hat{u}(x, t^n) = \sum_j \bar{u}_j^n + (u_x)_j^n (x - x_j) \mathbb{I}_{[x_{j-1/2},\, x_{j+1/2}]} \tag{93}$$

- We compute the new cell averages $w_{j+1/2}^{n+1}$ and w_j^{n+1} at t^{n+1} in the following way:

$$w_{j+1/2}^{n+1} = \frac{1}{\Delta x_{j+1/2}} \int_{x_{j+1/2,l}^n}^{x_{j+1/2,r}^n} u(\xi, t^{n+1}) = \frac{u_j^n + u_{j+1}^n}{2} + \frac{\Delta x - a_{j+1/2}^n \Delta t}{4} \times$$

$$((u_x)_j^n - (u_x)_{j+1}^n) - \frac{1}{2a_{j+1/2}^n} [f(u_{j+1/2,r}^{n+1/2}) - f(u_{j+1/2,l}^{n+1/2})] \tag{94}$$

$$w_j^{n+1} = \frac{1}{\Delta x_j}\int_{x^n_{j-1/2,r}}^{x^n_{j+1/2,l}} u(\xi, t^{n+1}) = u_j^n + \frac{\Delta t}{2}(a^n_{j-1/2} - a^n_{j+1/2})(u_x)^n_j$$

$$- \frac{\lambda}{1-\lambda(a^n_{j-1/2} + a^n_{j+1/2})}[f(u^{n+1/2}_{j+1/2,r}) - f(u^{n+1/2}_{j+1/2,l})] \qquad (95)$$

with

$$x^n_{j+1/2,r} = x_{j+1/2} + a^n_{j+1/2}\Delta t \qquad x^n_{j+1/2,l} = x_{j+1/2} - a^n_{j+1/2}\Delta t \qquad (96)$$

$$\Delta x_{j+1/2} = x^n_{j+1/2,r} - x^n_{j+1/2,l} \qquad \Delta x_j = x^n_{j+1/2,l} - x^n_{j-1/2,r} \qquad (97)$$

Kurganov and Tadmor scheme

$$u_{j+1/2,l}^{n+1/2} = u_{j+1/2,l}^{n} - \frac{\Delta t}{2} f(u_{j+1/2,l}^{n})_x \tag{98}$$

$$u_{j+1/2,l}^{n} = u_j^n + \Delta x (u_x)_j^n \left(\frac{1}{2} - \lambda a_{j+1/2}^n\right) \tag{99}$$

$$u_{j+1/2,r}^{n+1/2} = u_{j+1/2,r}^{n} - \frac{\Delta t}{2} f(u_{j+1/2,r}^{n})_x \tag{100}$$

$$u_{j+1/2,r}^{n} = u_j^n - \Delta x (u_x)_{j+1}^n \left(\frac{1}{2} - \lambda a_{j+1/2}^n\right) \tag{101}$$

$$\lambda = \frac{\Delta t}{\Delta x} \tag{102}$$

Kurganov and Tadmor scheme

- We consider the piecewise-linear reconstruction over the nonuniform cells at $t = t^{n+1}$

$$\hat{w}(x, t^{n+1}) = \sum_j \{[w^{n+1}_{j+1/2} + (u_x)^{n+1}_{j+1/2}(x - x_{j+1/2})]\mathbb{I}_{[x^n_{j+1/2,l}, x^n_{j+1/2,r}]} + w^{n+1}_j \mathbb{I}_{[x^n_{j-1/2,r}, x^n_{j+1/2,l}]}\} \tag{103}$$

- We project its averages back onto the original uniform grid.

$$u^{n+1}_j = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \hat{w}(\xi, t^{n+1})d\xi = \lambda a^n_{j-1/2} w^{n+1}_{j-1/2} + [1 - \lambda(a^n_{j-1/2} + a^n_{j+1/2})]w^{n+1}_j + \lambda a^n_{j+1/2} w^{n+1}_{j+1/2} + \frac{\Delta x}{2}[(\lambda a^n_{j-1/2})^2 (u_x)^{n+1}_{j-1/2} - (\lambda a^n_{j+1/2})^2 (u_x)^{n+1}_{j+1/2}] \tag{104}$$

Semi-discrete reduction: $\Delta t \to 0, \lambda \to 0$.

$$\frac{d}{dt}u_j(t) = -\frac{(f(u^+_{j+1/2}(t) + f(u^-_{j+1/2}(t)) - (f(u^+_{j-1/2}(t) + f(u^-_{j-1/2}(t))}{2\Delta x}$$
$$+ \frac{1}{2\Delta x}\{a_{j+1/2}[u^+_{j+1/2}(t) - u^-_{j+1/2}(t)] - a_{j-1/2}[u^+_{j-1/2}(t) - u^-_{j-1/2}(t)]\} \quad (105)$$

In this reduction the maximal local speed $a_{j+1/2}(t)$ takes the form

$$a^n_{j+1/2}(t) = \max\left\{\rho\left(\frac{\partial f}{\partial u}(u^-_{j+1/2}(t))\right), \rho\left(\frac{\partial f}{\partial u}(u^+_{j+1/2}(t))\right)\right\} \quad (106)$$

Kurganov and Tadmor scheme

Conservative form:

$$\frac{d}{dt}u_j(t) = -\frac{H_{j+1/2}(t) - H_{j-1/2}(t)}{\Delta x} \tag{107}$$

with the numerical flux

$$H_{j+1/2}(t) = \frac{f(u^+_{j+1/2}(t)) + f(u^-_{j+1/2}(t))}{2} - \frac{a_{j+1/2}(t)}{2}[u^+_{j+1/2}(t) - u^-_{j+1/2}(t)] \tag{108}$$

Kurganov and Tadmor second-order semi-discrete scheme, can be easily applied to one-dimensional convection–diffusion equations

$$\frac{d}{dt}u_j(t) = -\frac{H_{j+1/2}(t) - H_{j-1/2}(t)}{\Delta x} + \frac{P_{j+1/2}(t) - P_{j-1/2}(t)}{\Delta x} \tag{109}$$

with $P_{j+1/2}(t)$ is a reasonable approximation of the diffusion flux

$$P_{j+1/2}(t) = \frac{1}{2}\left[Q\left(u_j(t), \frac{u_{j+1}(t)-u_j(t)}{\Delta x}\right) + Q\left(u_{j+1}(t), \frac{u_{j+1}(t)-u_j(t)}{\Delta x}\right)\right] \quad (110)$$

Key features of Kurganov and Tadmor scheme

- semplicity, since no spectral decomposition of the flux f is needed;
- second order precision in Δx;
- sharp resolution of discontinuities;
- stability, since semi-discrete formulation allows to treat easily small Δt.

Example 1: linear steady shock.
Let us consider the following problem

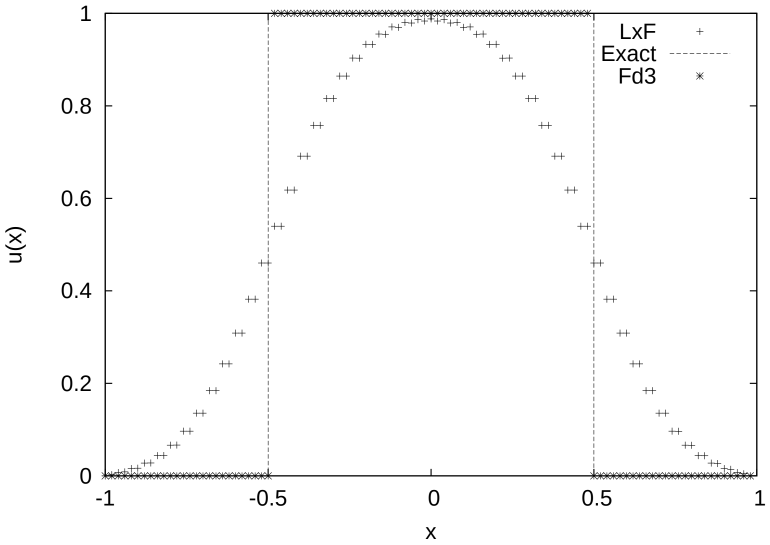
$$f(u) = 0 \tag{111}$$

$$u_t = 0 \tag{112}$$

subject to the discontinuous initial data

$$u(x,0) = \begin{cases} 1 & -0.5 < x < 0.5 \\ 0 & \text{otherwise} \end{cases} \tag{113}$$

Kurganov and Tadmor scheme



Example 2: inviscid Burgers' equation
Let us consider the following problem

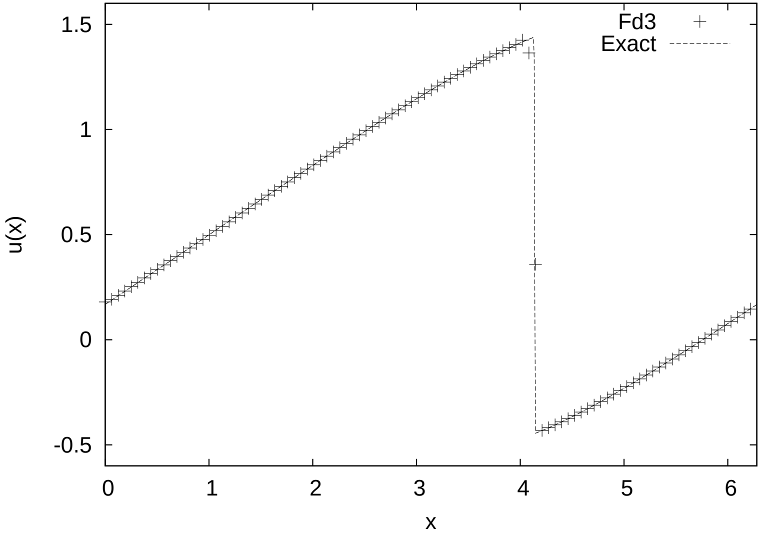
$$f(u) = \frac{u^2}{2} \tag{114}$$

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{115}$$

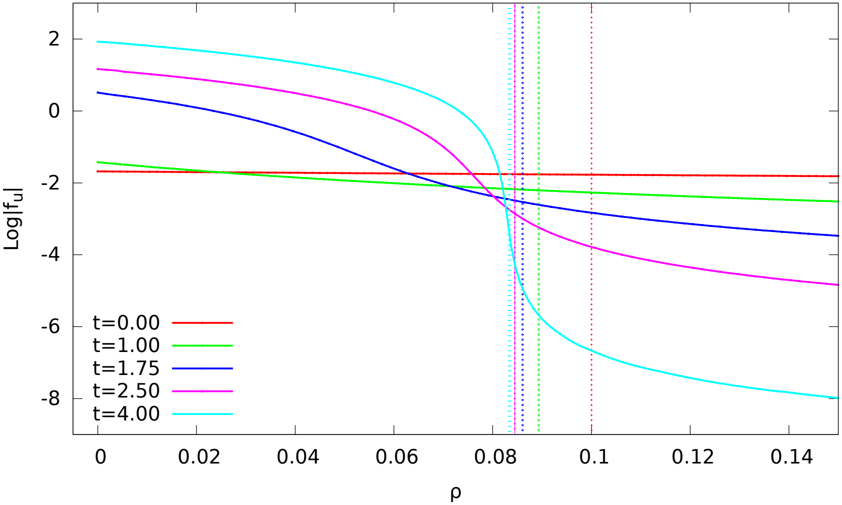
with a smooth periodic initial data

$$u(x, 0) = 0.5 + \sin x. \tag{116}$$

Kurganov and Tadmor scheme



Local speed



2
0
-2
-4
-6
-8
Log|f_u|
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
ρ
t=0.00
t=1.00
t=1.75
t=2.50
t=4.00