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Critical dynamics from the real-time functional renormalization group

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HFHF Theory Retreat
Castiglione della Pescaia, September 12, 2022

Based on
JR, D. Schweitzer, L. J. Sieke, L. von Smekal, Phys. Rev. D **105**, 116017 (2022)
JR, L. von Smekal, in preparation

DFG
CRC-TR 211 Strong-interaction matter under extreme conditions
HFHF Helmholtz Forschungsakademie Hessen für FAIR
HGS-HIRe for FAIR Helmholtz Graduate School for Hadron and Ion Research

Why *real* time?

Performing calculations directly in real time

- avoids the need of analytic continuation in comparison with the imaginary-time formalism
- allows treating phenomena off-equilibrium, e.g. many aspects of heavy-ion collisions, including …

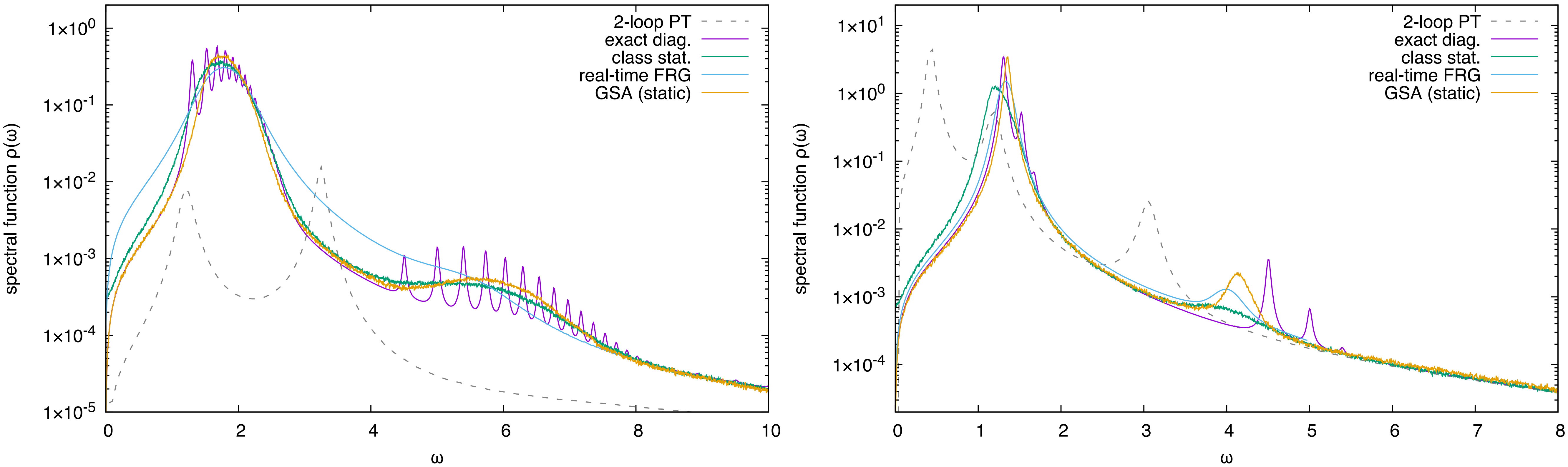


Figure: Spectral functions of the quartic oscillator at finite temperature, stemming from various computational techniques, including the real-time FRG.

JR, D. Schweitzer, L. J. Sieke, L. von Smekal, Phys. Rev. D **105**, 116017 (2022)

Why critical dynamics?

… dynamic universality (classified into 'Models')

- Proper theoretical description needs methods which can handle non-perturbative real-time physics
- Long-term goal: Models G, H
- In this talk: start with somewhat simpler Models A, B, and C

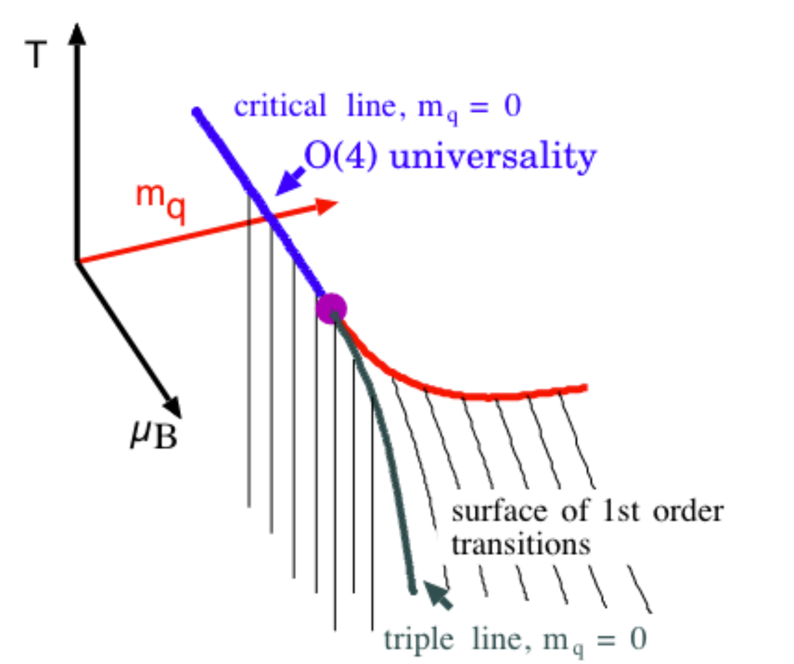


Figure adapted from L. von Smekal, Nucl. Phys. B Proc. Suppl. **228** (2012)

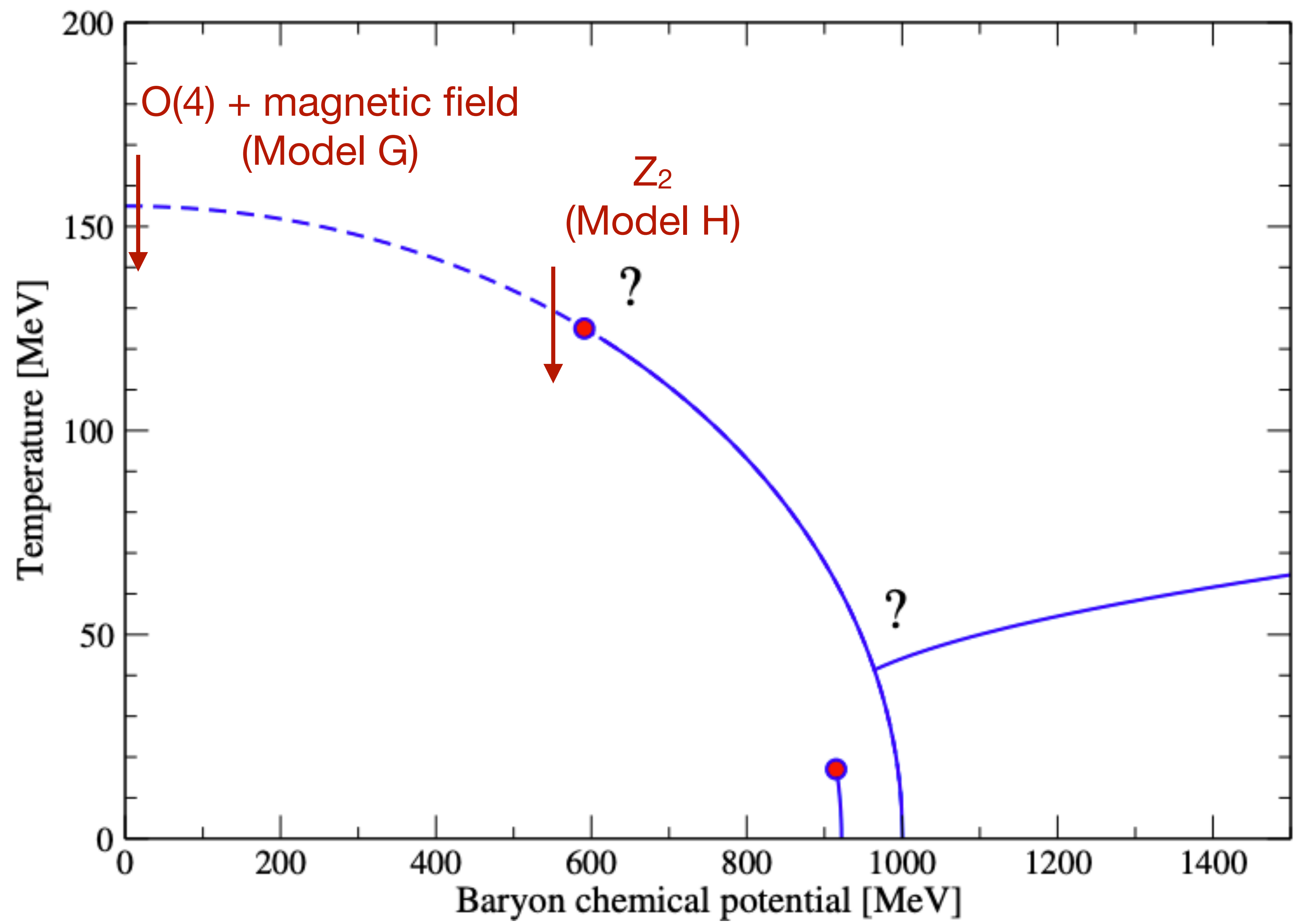
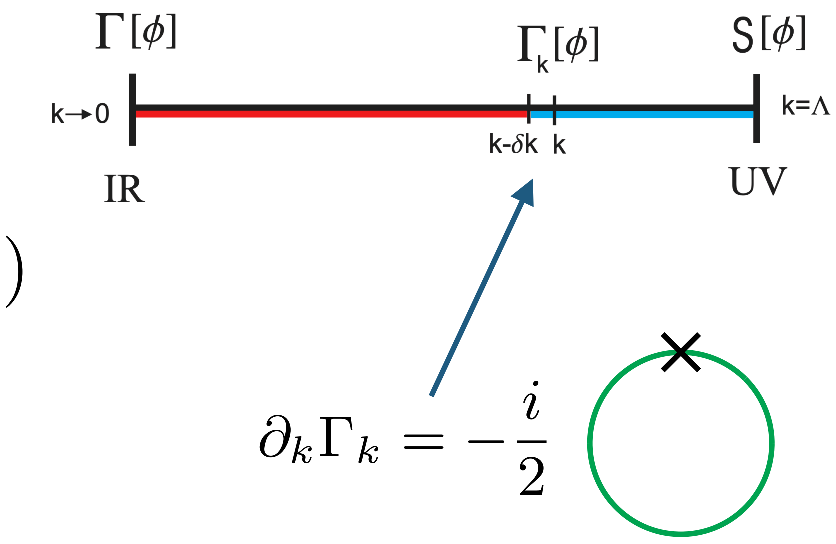


Figure adapted from C. S. Fischer, Prog. Part. Nucl. Phys. **105**, 1 (2019)

Causality in the functional renormalization group

Causality in the FRG

Idea of the FRG is to introduce an *infrared cutoff*



$$\partial_k \Gamma_k = -\frac{i}{2}$$

integrate out fluctuations 'step by step'

$$\Delta S_k[\phi^c, \phi^q] = \frac{1}{2} \int_{xx'} \phi^T(x) R_k(x, x') \phi(x')$$

- with regulator

'anomalous' (zero due to causality) — advanced

$$R_k(x, x') = \begin{pmatrix} R_k^{\widetilde{K}}(x, x') & R_k^A(x, x') \\ R_k^R(x, x') & R_k^K(x, x') \end{pmatrix}$$

retarded — 'Keldysh' (set by FDR in equilibrium)

$$R_k^K(\omega, \boldsymbol{p}) = \coth\left(\frac{\omega}{2T}\right)\left(R_k^R(\omega, \boldsymbol{p}) - R_k^A(\omega, \boldsymbol{p})\right)$$

- which affects analytic structure of propagators

$$G_k^R(\omega, \boldsymbol{p}) = -\frac{1}{\Gamma_k^{qc}(\omega, \boldsymbol{p}) + R_k^R(\omega, \boldsymbol{p})} \qquad G_k^A(\omega, \boldsymbol{p}) = -\frac{1}{\Gamma_k^{cq}(\omega, \boldsymbol{p}) + R_k^A(\omega, \boldsymbol{p})}$$

- What are the consequences?

Causal regulators

Test:

- Observe general property of Keldysh action:

direct consequence of causality structure!

$$S = \frac{1}{2}\int_{xx'} \phi^T(x) \begin{pmatrix} \mathbf{0} & \cdots \\ \cdots & \cdots \end{pmatrix} \phi(x') + \cdots$$

see for example A. Kamenev, *Field Theory of Non-Equilibrium Systems* (Cambridge University Press, 2011)

- Necessary condition for correctness of flow

Find:

- Popular regulators produce such an unphysical component during the flow
- Problem of causality is not trivial

C. Duclut and B. Delamotte, Phys. Rev. E **95**, 012107 (2017)

- An insufficient regulator leads to an incorrect Keldysh action

Causal regulators

What *can* we do?

(Start with 0+1 dimensional case, i.e. quantum mechanics)

Most simple regulator has form of a purely mass-like shift

(Callan-Symanzik regulator)

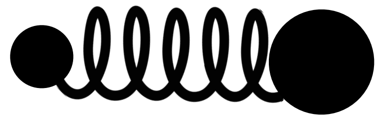
$$R_k^{R/A}(\omega) = -k^2$$

- Trivially causal, only induces mass shift $m^2 \to m^2 + k^2$
- Too simple?
- Flow no longer consistent with Wilson's idea of integrating out momentum (energy) shells?

Causal regulators

Regulator motivated by physics: (Causality guaranteed!)

- Imagine ΔS_k is the result of integrating out an external heat bath (HB), modelled as an ensemble of harmonic oscillators attached to the particle



A. O. Caldeira and A. J. Leggett, Physics A **121**, 587 (1983)

$$H_I + H_B = \sum_s \left(\frac{\pi_s^2}{2} + \frac{\omega_s^2}{2}\varphi_s^2 - g_s \varphi_s x \right)$$

- Integrate out heat bath ≙ Particle acquires self-energy

$$\Sigma^R(\omega) = -\int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega' J(\omega')}{(\omega + i\varepsilon)^2 - \omega'^2}$$

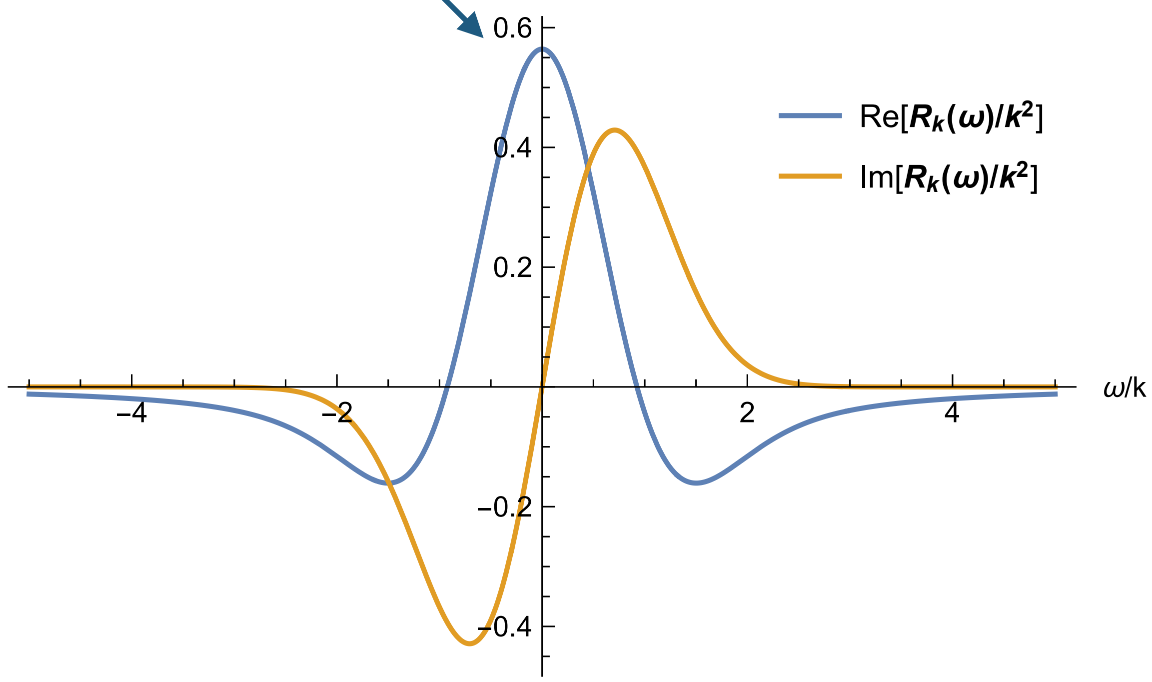
- Fully controlled by spectral density: $J(\omega) = \pi \sum_s \frac{g_s^2}{\omega_s} \delta(\omega - \omega_s) = 2\,\mathrm{Im}\,\Sigma^R(\omega)$
- But self-energy also has a non-vanishing real part!

Causal regulators

Now make the spectral density *k*-dependent, and choose it so as to *damp* infrared modes.

- Resulting self-energy is the regulator,

negative shift in squared mass!



$$m^2 \to m^2 \overset{(!)}{-} \Delta m^2_{\mathrm{HB}}(k)$$

<u>Example:</u>

$$J_k(\omega) = k\omega \exp\left\{-\omega^2/k^2\right\}$$

<u>But:</u> Heat bath induces *negative* (!) shift in the squared mass

$$\Delta m^2_{\mathrm{HB}}(k) = \int_0^\infty \frac{d\omega}{\pi}\frac{J_k(\omega)}{\omega} = \frac{k^2}{\sqrt{4\pi}}$$

which makes the theory *unstable* and *acausal* for sufficiently large values of *k* !

Causal regulators

Way out: Remember that a mass-like shift is causal

- Add mass-like 'counter-term' $-\alpha k^2$ with $\alpha > 0$ to compensate unwanted shift in squared mass

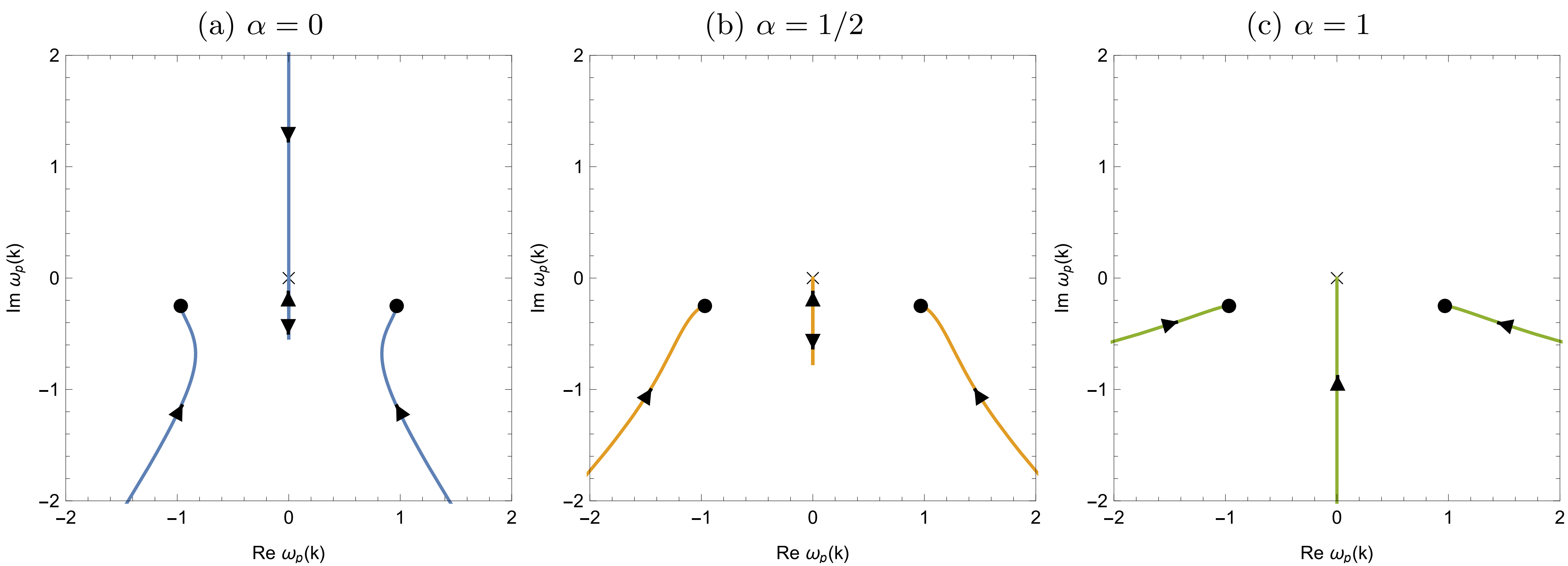
$$R_k^{R/A}(\omega) = -\int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega' J_k(\omega')}{(\omega \pm i\varepsilon)^2 - \omega'^2} - \alpha k^2$$

spectral representation of ω-dependence

mass-like counter-term

- Flow of retarded-propagator poles:

JR, D. Schweitzer, L. J. Sieke, L. von Smekal, Phys. Rev. D **105**, 116017 (2022)



What about a *field* theory?

- Imagine an independent bath of harmonic oscillators 'attached' to every spatial momentum mode $\boldsymbol{p}$
- Spectral representation just acquires an additional $\boldsymbol{p}$-dependence

$$R_k^{R/A}(\omega,\boldsymbol{p}) = -\int_0^\infty \frac{d\omega'}{2\pi}\,\frac{2\omega' J_k(\omega',\boldsymbol{p})}{(\omega \pm i\varepsilon)^2 - \omega'^2} - \alpha_k(\boldsymbol{p})k^2$$

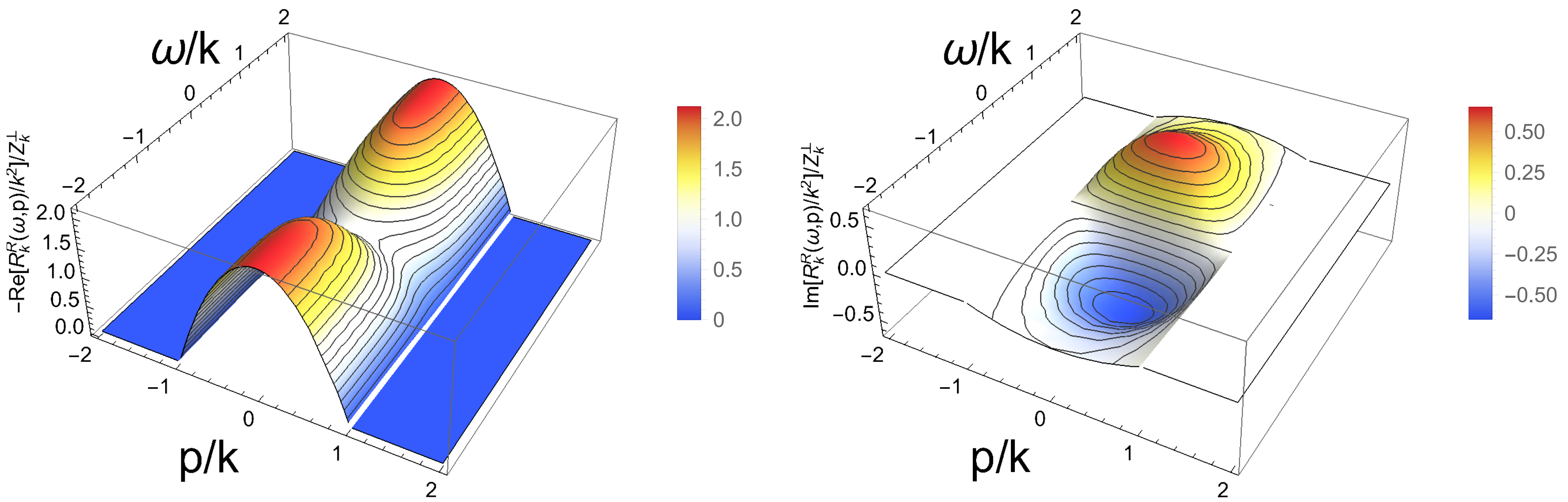


Figure: Real (left, mass shift) and imaginary (right, damping) parts of regulator.

Causal regulators for field theories

And when there is no preferred frame of reference, e.g. no external medium? What about *Lorentz invariance*?

- Regulator like above would break Lorentz symmetry.
- Imagine the heat bath to be an ensemble of Klein-Gordon fields with a relativistic dispersion relation $\omega^2 = m_s^2 + \boldsymbol{p}^2$.
- Our field gains a self-energy
  (Källén-Lehmann representation)

$$\Sigma_k^R(\omega, \boldsymbol{p}) = -\int_0^\infty \frac{d\mu^2}{2\pi} \frac{\widetilde{J}_k(\mu^2)}{(\omega + i\varepsilon)^2 - \boldsymbol{p}^2 - \mu^2}$$

- with *invariant* spectral density: $\widetilde{J}(\mu^2) = 2\pi \sum_s g_s^2 \delta(\mu^2 - m_s^2)$

  connected to the general one above via $J(\omega, \boldsymbol{p}) = \mathrm{sgn}(\omega)\ \theta(p^2)\ \widetilde{J}(p^2)$

- Reintroduce mass-like counter-term $-ak^2$, and then …

Causal regulators for field theories

… find general form of a Lorentz-invariant heat-bath regulator

$$R_k^{R/A}(\omega,\boldsymbol{p}) = -\int_0^\infty \frac{d\mu^2}{2\pi}\, \frac{\widetilde{J}_k(\mu^2)}{(\omega+i\varepsilon)^2 - \boldsymbol{p}^2 - \mu^2} - \alpha k^2$$

intrinsically not UV finite!
(for a positive-definite spectral density)

(special case of general spectral representation from above)

<u>Example:</u> $\widetilde{J}_k(\mu^2) = \dfrac{4k\mu}{(1+\mu^2/k^2)^2}$

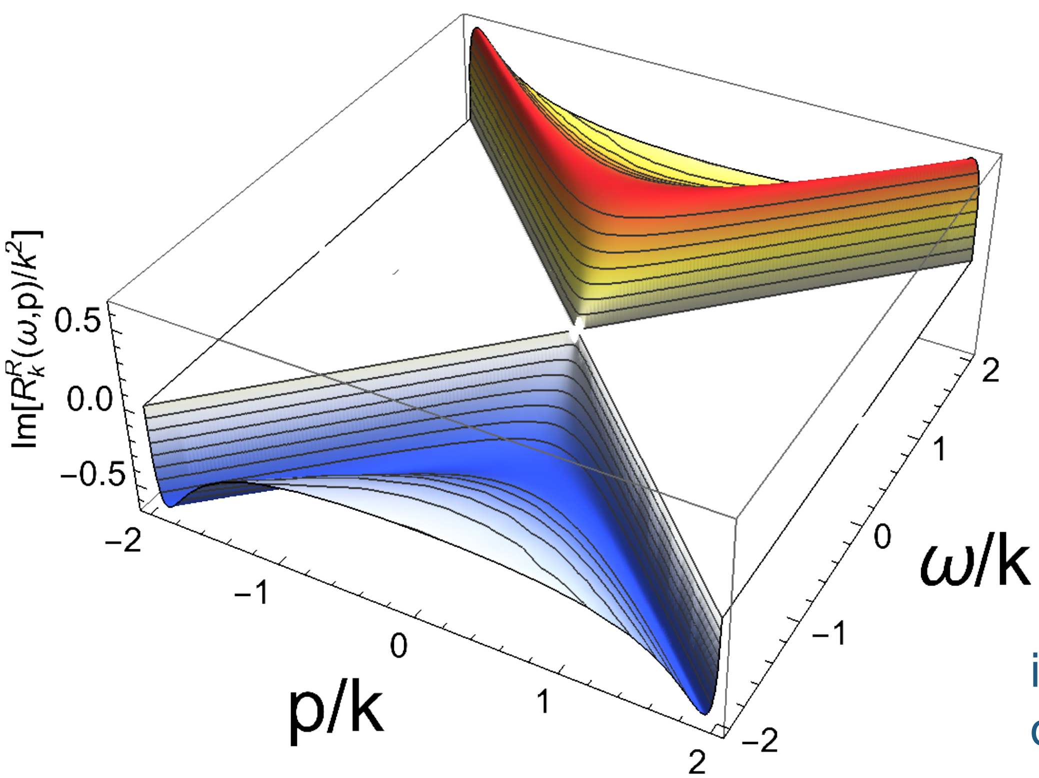
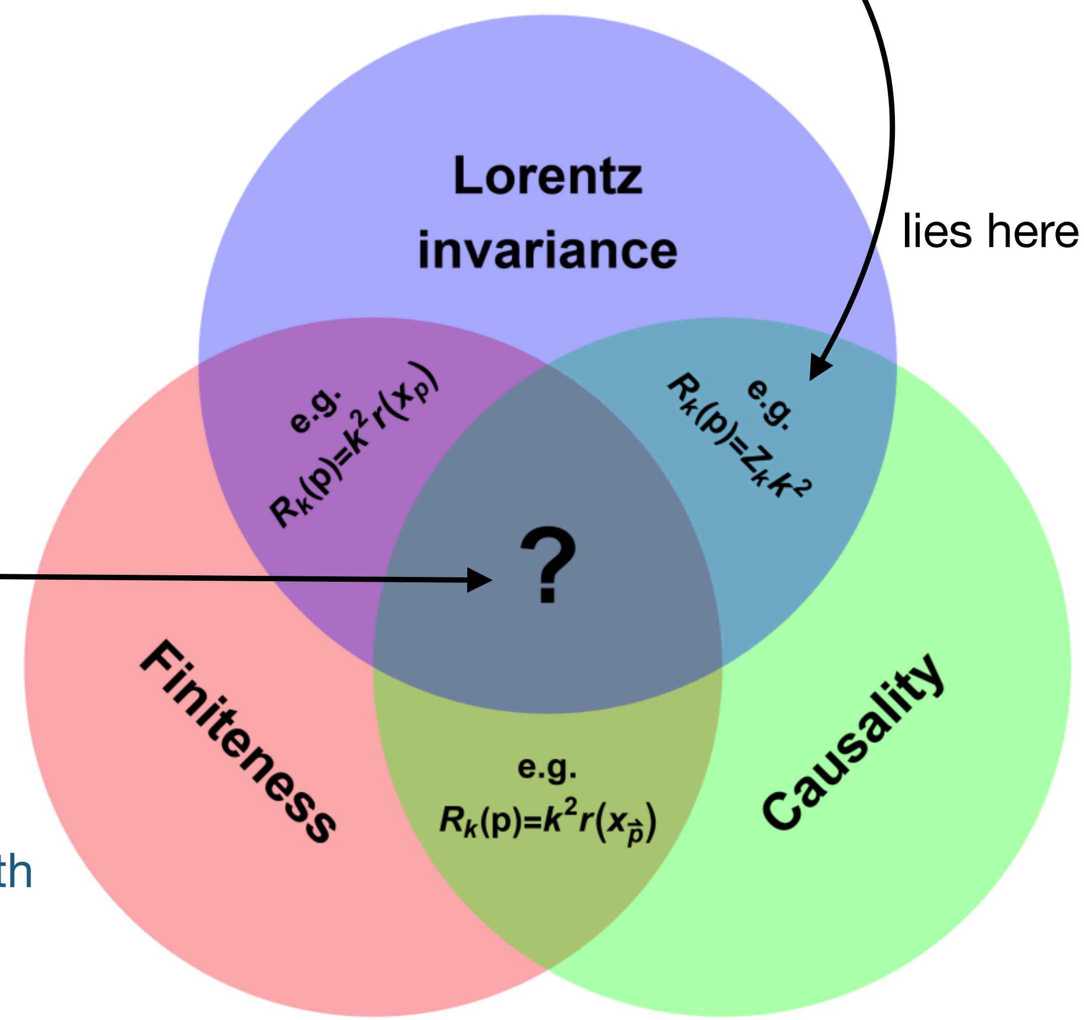


Figure: Imaginary part of the resulting causal, Lorentz invariant, but not-UV-finite, regulator.

lies here

possible at all?



integrating out heat bath defines arrow of time

Figure taken from J. Braun et al., arXiv:2206.10232

Critical dynamics

Spectral function defined as

$$\rho(\omega) = \frac{1}{2\pi i}\int dt\; e^{i\omega t}\int d^d x\; i\langle[\phi(t,\boldsymbol{x}),\phi(0,\boldsymbol{0})]\rangle$$

which

critical scaling: $s^{2-\eta}\rho(s^z\omega, s\boldsymbol{p}, s^{1/\nu}\tau) = \rho(\omega,\boldsymbol{p},\tau)$

- behaves like $\rho(\omega) \sim |\omega|^{-\sigma}$ at critical point, with

['reduced' temperature $\tau = (T - T_c)/T_c$]

- scaling exponent $\sigma = (2 - \eta)/z$, which is related to
- dynamical critical exponent z, defined by $\xi_t \sim \xi^z$

'critical slowing down'

correlation time — correlation length

- z determined by *dynamic* universality class

(called 'Models' in the classification scheme by Halperin and Hohenberg)

Critical dynamics

CRC-TR 211

Consider classical $\lambda\phi^4$-theory with Landau-Ginzburg free energy (statics)

Model A

$z = 2 + c\eta$

$$F = \int d^d x \, \left\{ \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) \right\}$$

equilibrium partition function:

$$Z = \int \mathcal{D}\varphi \quad e^{-\beta F}$$

- equations of motion (dynamics) with dissipative coupling γ to heat bath

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi$$

- Gaussian white noise(s)

$$\langle \xi(x) \rangle_\beta = 0$$
$$\langle \xi(x)\xi(x') \rangle_\beta = 2\gamma T \delta(x - x')$$

- discrete Z_2 symmetry breaks spontaneously for $T < T_c$ when $m^2 < 0$

Critical dynamics

Model B

$z = 4 - \eta$

Consider classical $\lambda\phi^4$-theory with Landau-Ginzburg free energy (statics)

$$F = \int d^d x \, \left\{ \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) + B\varphi n + \frac{n^2}{2\chi_0} \right\}$$

equilibrium partition function:

$$Z = \int \mathcal{D}\varphi \, \mathcal{D}n \, e^{-\beta F}$$

- equations of motion (dynamics) with dissipative coupling γ to heat bath

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi \qquad \tau_R \partial_t^2 n + \partial_t n = \underbrace{\bar{\lambda} \vec{\nabla}^2 \frac{\delta F}{\delta n} + \vec{\nabla} \cdot \vec{\zeta}}_{\text{total divergence } (n \text{ is conserved density})}$$

- Gaussian white noise(s)

$$\langle \xi(x) \rangle_\beta = 0 \qquad \langle \zeta^i(x) \rangle_\beta = 0$$

$$\langle \xi(x)\xi(x') \rangle_\beta = 2\gamma T \delta(x - x') \qquad \langle \zeta^i(x)\zeta^j(x') \rangle_\beta = 2\bar{\lambda} T \delta^{ij} \delta(x - x')$$

- discrete Z_2 symmetry breaks spontaneously for $T < T_c$ when $m^2 < 0$
- e.g. uniaxial ferromagnet

include shear modes of energy-momentum tensor ⇝ Model H

see Son and Stephanov, Phys. Rev. D **70**, 056001 (2004)

Critical dynamics

Consider classical $\lambda\phi^4$-theory with Landau-Ginzburg free energy (statics)

Model C

$z = 2 + \alpha/\nu$

$$F = \int d^d x \left\{ \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) + \frac{n^2}{2\chi_0} + \frac{g}{2}\varphi^2 n \right\}$$

equilibrium partition function:

$$Z = \int \mathcal{D}\varphi\, \mathcal{D}n\, e^{-\beta F}$$

- equations of motion (dynamics) with dissipative coupling γ to heat bath

$$\partial_t^2\varphi + \gamma\partial_t\varphi = -\frac{\delta F}{\delta\varphi} + \xi \qquad \tau_R\partial_t^2 n + \partial_t n = \underbrace{\bar{\lambda}\vec{\nabla}^2\frac{\delta F}{\delta n} + \vec{\nabla}\cdot\vec{\zeta}}$$

total divergence (n is conserved density)

- Gaussian white noise(s)

$$\langle\xi(x)\rangle_\beta = 0 \qquad \langle\zeta^i(x)\rangle_\beta = 0$$

$$\langle\xi(x)\xi(x')\rangle_\beta = 2\gamma T\delta(x-x') \qquad \langle\zeta^i(x)\zeta^j(x')\rangle_\beta = 2\bar{\lambda}T\delta^{ij}\delta(x-x')$$

- discrete Z_2 symmetry breaks spontaneously for $T < T_c$ when $m^2 < 0$

Critical dynamics – truncation

1PI vertex expansion … around scale-dependent minimum $\phi_{0,k}$: use for Models A and B

- effective average action:

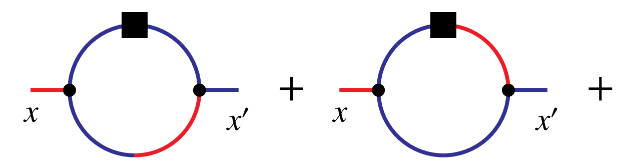
$$\Gamma_k = \frac{1}{2}\int_{xx'} \left(\phi^c - \phi^c_{0,k}, \phi^q\right)_x \begin{pmatrix} 0 & \Gamma^{cq}_k(x,x') \\ \Gamma^{qc}_k(x,x') & \Gamma^{qq}_k(x,x') \end{pmatrix} \begin{pmatrix} \phi^c - \phi^c_{0,k} \\ \phi^q \end{pmatrix}_{x'} - \frac{\kappa_k}{\sqrt{8}}\int_x \left(\phi^c - \phi^c_{0,k}\right)^2 \phi^q - \frac{\lambda_k}{12}\int_x \left(\phi^c - \phi^c_{0,k}\right)^3 \phi^q$$

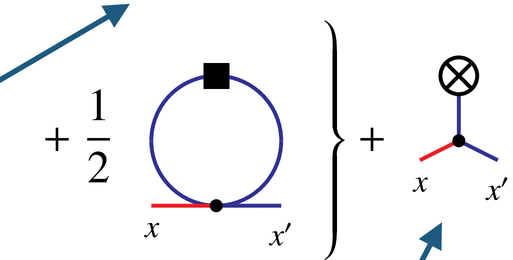
expand 2-point function in spatial gradients, but keep full frequency dependence:

$$\Gamma^{qc}_k(\omega,\boldsymbol{p}) = \Gamma^{qc}_{0,k}(\omega) - Z^{\perp}_k \boldsymbol{p}^2 + \cdots$$
$$\Gamma^{cq}_k(\omega,\boldsymbol{p}) = \Gamma^{cq}_{0,k}(\omega) - Z^{\perp}_k \boldsymbol{p}^2 + \cdots$$
$$\Gamma^{qq}_k(\omega,\boldsymbol{p}) = \frac{2T}{\omega}\left(\Gamma^{qc}_{0,k}(\omega) - \Gamma^{cq}_{0,k}(\omega)\right)$$

- flow of 2-point function:

$$\partial_k \Gamma^{qc}_k(x,x') = -i$$



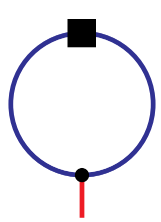


generate non-local power-law behaviour in spectral function

'interaction' with scale-dependent minimum

- flow of effective potential:

$$\partial_k V'_k(\varphi) = -\frac{i}{\sqrt{8}}$$



use for squared mass and quartic coupling

- flow of couplings to density: (Model B)

vanish!
(coupling is linear ⤳ mixing)

for color coding and diagrammatic conventions, see
S. Huelsmann, S. Schlichting, P. Scior, Phys. Rev. D **102**, 096004 (2020)

Critical dynamics – truncation

1PI vertex expansion … around $\phi = 0$: use for Models A and C

- effective average action:

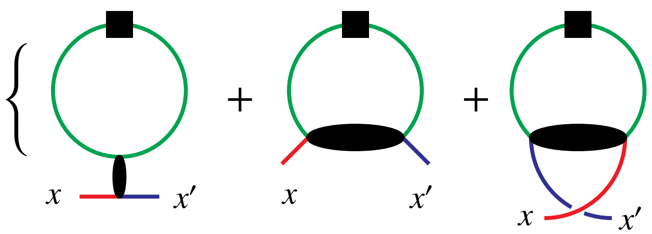
$$\Gamma_k = \frac{1}{2}\int_{xx'} (\phi^c, \phi^q)_x \begin{pmatrix} 0 & \Gamma_k^{cq}(x,x') \\ \Gamma_k^{qc}(x,x') & \Gamma_k^{qq}(x,x') \end{pmatrix} \begin{pmatrix} \phi^c \\ \phi^q \end{pmatrix}_{x'} +$$
$$\frac{3\cdot 2^2}{4!}\int_{xx'} \phi^q(x)\phi^c(x)V_k^{an}(x,x')\phi^q(x')\phi^c(x') +$$
$$\frac{3\cdot 2}{4!}\int_{xx'} \phi^q(x)\phi^c(x)V_k^{cl,R}(x,x')\phi^c(x')\phi^c(x') +$$
$$\frac{3\cdot 2}{4!}\int_{xx'} \phi^c(x)\phi^c(x)V_k^{cl,A}(x,x')\phi^q(x')\phi^c(x')$$

expand 2- and 4-point functions in spatial gradients, but keep full frequency dependence:

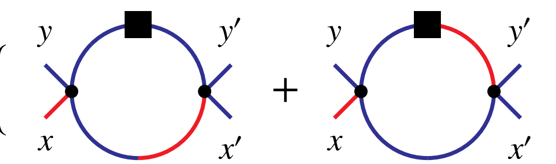
$$\Gamma_k^{qc}(\omega,\boldsymbol{p}) = \Gamma_{0,k}^{qc}(\omega) - Z_k^{\perp}\boldsymbol{p}^2 + \cdots$$
$$\Gamma_k^{cq}(\omega,\boldsymbol{p}) = \Gamma_{0,k}^{cq}(\omega) - Z_k^{\perp}\boldsymbol{p}^2 + \cdots$$
$$\Gamma_k^{qq}(\omega,\boldsymbol{p}) = \frac{2T}{\omega}\left(\Gamma_{0,k}^{qc}(\omega) - \Gamma_{0,k}^{cq}(\omega)\right)$$
$$V_k^{cl,A}(\omega,\boldsymbol{p}) = V_{0,k}^{cl,A}(\omega) + V_{1,k}^{cl,A}(0)\boldsymbol{p}^2 + \cdots$$
$$V_k^{cl,R}(\omega,\boldsymbol{p}) = V_{0,k}^{cl,R}(\omega) + V_{1,k}^{cl,R}(0)\boldsymbol{p}^2 + \cdots$$
$$V_k^{an}(\omega,\boldsymbol{p}) = \frac{2T}{\omega}\left(V_k^{cl,R}(\omega,\boldsymbol{p}) - V_k^{cl,A}(\omega,\boldsymbol{p})\right)$$

for the QM case, see
S. Huelsmann, S. Schlichting, P. Scior, Phys. Rev. D **102**, 096004 (2020)
JR, D. Schweitzer, L. J. Sieke, L. von Smekal, Phys. Rev. D **105**, 116017 (2022)

- flow of 2- and 4-point functions:

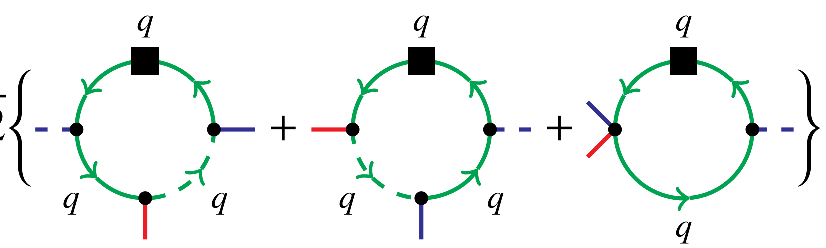
$$\partial_k\Gamma_k^{qc}(x,x') = -\frac{i}{2}\Big\{$$


$\Big\}$

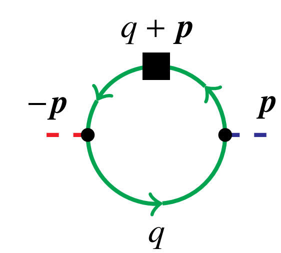
$$\partial_k V_k^{cl,R}(x,x') = -i\int_{x-y,\,x'-y'}\Big\{$$


$\Big\}$

- flow of couplings to density: (Model C)

$$\partial_k g_k = i\sqrt{2}\Big\{$$


$\Big\}$

$$\partial_k\chi_{0,k}^{-1} = \frac{i}{\lambda}\lim_{p\to 0}\frac{1}{\boldsymbol{p}^2}$$


Critical dynamics – results

Model A
$z = 2 + c\eta$

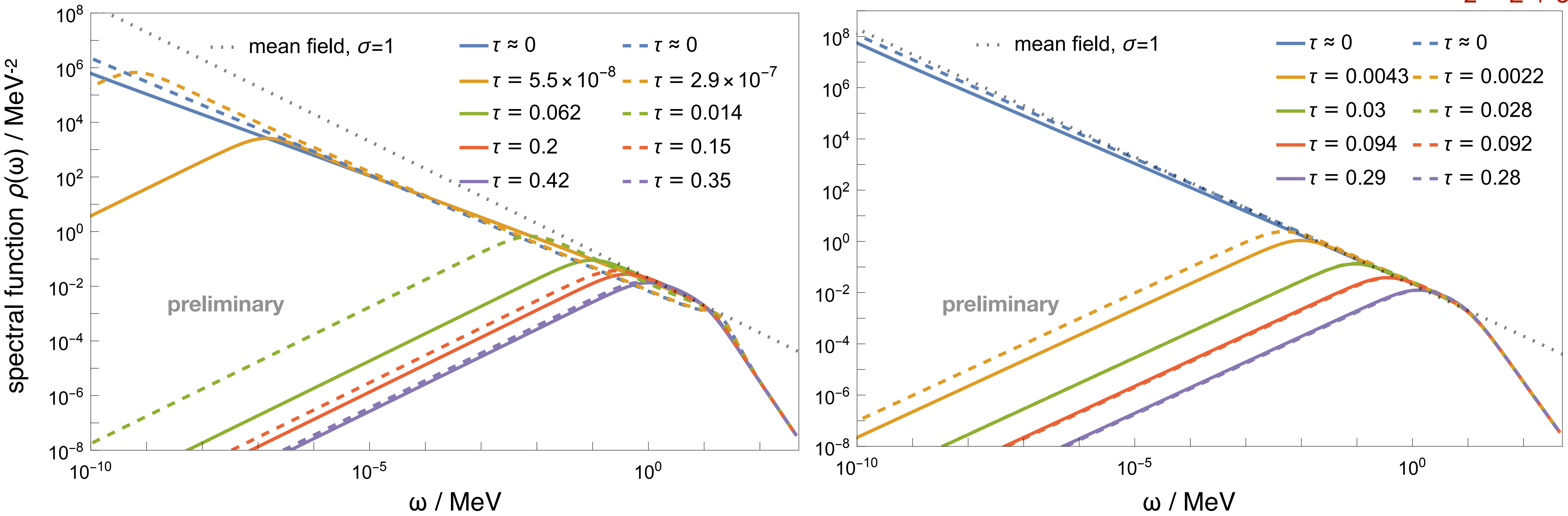


Figure: Critical spectral functions of Model A in 2*d* and 3*d*.

JR, L. von Smekal, in preparation

- visible power-law behaviour building up close to the critical point
- extract dynamical critical exponents $z = 2.09$ in 2d and $z = 2.04$ in 3d

compare $z = 2.1667(5)$ M. P. Nightingale and H. W. J. Blöte, Phys. Rev. B **62**, 1089 (2000)
$z = 2.09(6)$ M. J. Dunlavy and D. Venus, Phys. Rev. B **71**, 144406 (2005)
$z = 2.0842(39)$ A.S. Krinitsyn, V.V. Prudnikov, P.V. Prudnikov, Theor Math Phys **147**, 561–575 (2006)
and more…

compare $z = 2.0245(15)$
see e.g. M. Hasenbusch, Phys. Rev. E **101**, 022126 (2020)

[reduced temperature $\tau = (T - T_c)/T_c$]

Critical dynamics – results

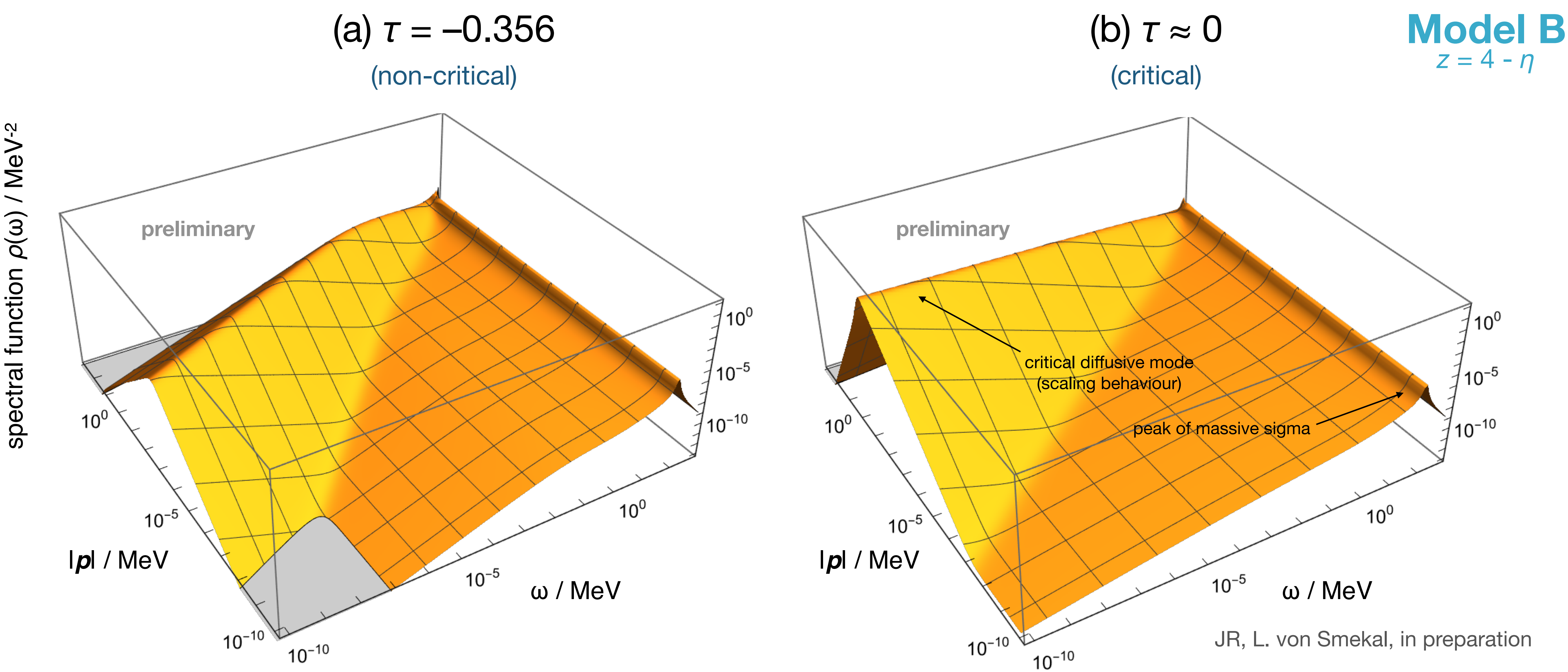


Figure: Non-critical (left) and critical (right) spectral functions of Model B in 3*d*.

- sigma meson stays massive!
- it is the mixed diffusive mode between fluctuations in the sigma meson and the conserved baryon density which becomes critical

[reduced temperature $\tau = (T - T_c)/T_c$]

include shear modes of energy-momentum tensor ⇝ Model H

see Son and Stephanov, Phys. Rev. D **70**, 056001 (2004)

Critical dynamics – results

CRC-TR 211

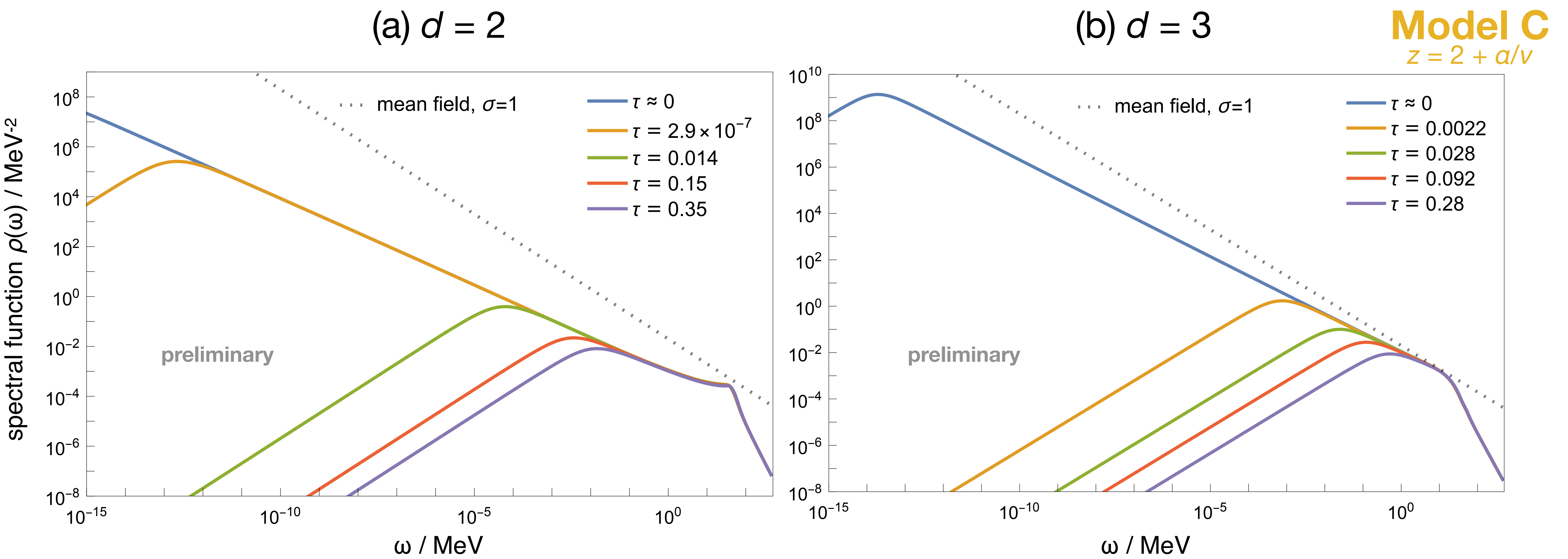


Figure: Critical spectral functions of Model C in 2*d* and 3*d*.

JR, L. von Smekal, in preparation

- exact hyperscaling relation $z = 2 + \alpha/\nu$
- extract dynamical critical exponents $z = 2.56$ in 2*d* and $z = 2.31$ in 3*d*

compare $z = 2$ (exact)

see Onsager's solution of 2d Ising model

compare $z = 2.174749(19)$

using high-precision conformal-bootstrap results of
F. Kos, D. Poland, D. Simmons-Duffin, A. Vichi, JHEP **08**, 036 (2016)

[reduced temperature $\tau = (T - T_c)/T_c$]

Summary & outlook

We have

- constructed regulators in the real-time FRG, and shown that their frequency-dependence (if causal) admits a spectral representation,
- calculated critical spectral functions in Model A, B, and C, using one- and two-loop self-consistent truncation schemes.

For the future, we plan to

- extract universal scaling functions which describe universal behaviour in close vicinity of critical point,
- inspect real-time dynamics of Model G and H,
- include fermions (low-energy effective models of QCD in real time), and
- analyze non-equilibrium phenomena (Kibble-Zurek, …)

Thank you for your attention!