# The absence of symmetry breaking in the $(1+1)$-dimensional Gross-Neveu model with bosonic fluctuations at non-zero $T$ and $\mu$


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HFHF Theory Retreat 2022
September 13, 2022, Castiglione della Pescaia, Italy

# Motivation

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The Gross-Neveu model:

- It shares some properties with the QCD like asymptotic freedom (without being a gauge theory) and chiral symmetry.
- It's low dimensional and is well studied.
- In the original version, it is formulated in a purely fermionic theory with only one four-fermion interaction term:

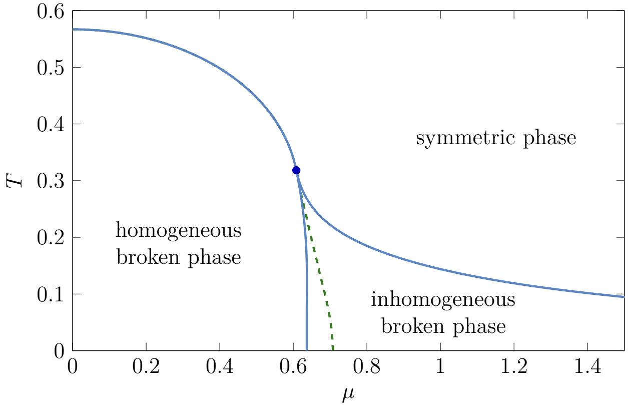
$$S_{\mathrm{GN}}(\beta,\mu)=\int_{-\infty}^{\infty}\mathrm{d}x\int_{0}^{\beta}\mathrm{d}\tau\left[\bar{\psi}_f(\partial\!\!\!/-\mu\gamma^0)\psi^f-\frac{g^2}{2N}(\bar{\psi}_f\psi^f)^2\right],$$

  where $f=1,2,\ldots,N$, $N$ is the number of fermions, $\mu$ the chemical potential and $\beta=1/T$ the inverse temperature.

- Discrete chiral symmetry: $\psi\mapsto\gamma_{\mathrm{ch}}\psi$, $\bar{\psi}\mapsto-\gamma_{\mathrm{ch}}\bar{\psi}$.

D. J. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974).

# Motivation: The large-$N$ approximation



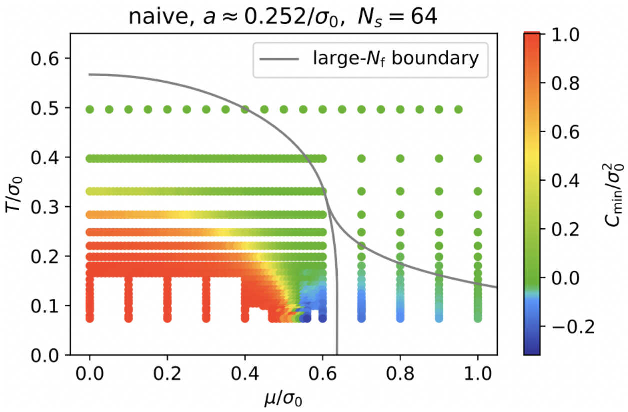

- This model is analytically solved in the large-$N$ approximation where the number of fermions is send to infinity.
- The phase diagram (PD) has three different phases w.r.t. the order parameter $\langle\bar{\psi}\psi\rangle$.


D. J. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974).

U. Wolff, Phys. Lett. B 157, 303-308 (1985).

M. Thies, J. Phys. A: Math. Gen. 39 12707, (2006).

# Motivation: Finite $N$ Results




- There are some lattice Monte-Carlo simulations which see indications that there is a **non-trivial PD** similar to the large-$N$ diagram.

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Fig.: J. Lenz, L. Pannullo, M. Wagner, B. Wellegehausen, A. Wipf, Phys. Rev. D 101, 094512 (2020).

F. Karsch, J. B. Kogut, H. W. Wyld, Nucl. Phys. B 280, 289 (1987).

Y. Cohen, S. Elitzur, E. Rabinovici, Phys. Lett. B 104, 289 (1981).

# How does the PD look like for finite $N$?



- What about the theorem of **Mermin-Wagner**?
  *Holds for continuous symmetries...*

- What about **Landau's argument** which forbids phase coexistence in $1+1$ dimensions? This should be applicable for finite $N$!

  $\Rightarrow$ Expectation: The phase diagram should be **trivial** for $T > 0$!

This talk: Crosscheck via the FRG.

N. D. Mermin, H. Wagner, Phys. Rev. Lett. 17, 1133 (1966).
L. D. Landau, E. M. Lifshitz, Statistical Physics, Part 1 & 2.

# In this talk ...

- **Part I**: we will use the FRG to study SSB.
- **Part II**: we artificially put the FRG calculation on a space-time lattice such that we have the same parameters as in lattice MC simulations.

**Part I**: J. Stoll, **N. Z.**, A. Koenigstein, M. J. Steil, et al. arXiv:2108.10616 (2021).
**Part II**: J. Braun, L. Pannullo, **N. Z.**, in preparation.

# Part I

## Theoretical background

# Functional Renormalization Group

- **Main idea**: There exists a one-parameter family of actions, the so-called scale dependent effective average action $\bar{\Gamma}_k$, which interpolates between the bare action $\bar{\Gamma}_{k=\Lambda} = \mathcal{S}$ (UV) and the full quantum effective action $\bar{\Gamma}_{k\to 0} = \Gamma$ (IR).

- The scale dependent effective average action is a solution of a initial value problem given by the **Wetterich equation** (WE)

$$\partial_k \bar{\Gamma}_k[\Phi] = \frac{1}{2}\mathrm{STr}\left\{\partial_k R_k \left(\bar{\Gamma}_k^{(2)}[\Phi] + R_k\right)^{-1}\right\}$$

and the initial condition $\bar{\Gamma}_{k=\Lambda} = \mathcal{S}$.

C. Wetterich, Phys. Lett. B 301, 90-94 (1993).

# Truncation and regulator choice



- Original fermionic model $\Rightarrow$ Bosonized GN model via a **Hubbard-Stratonovich transformation**. The action becomes

$$S_{\mathrm{bGN}}(\beta,\mu)=\int_{-\infty}^{\infty}\mathrm{d}x\int_{0}^{\beta}\mathrm{d}\tau\left[\bar{\psi}_f(\partial\!\!\!/-\mu\gamma^0+h\phi)\psi^f-N\frac{h^2}{2g^2}\phi^2\right],$$

  where $\phi$ is an auxiliary (real) scalar field.

- Since WE is to complicated, we have to truncate $\bar{\Gamma}_k[\Phi]$. We choose the **local potential approximation (LPA)**:

$$\bar{\Gamma}_k[\Phi]=\int_{-\infty}^{\infty}\mathrm{d}x\int_{0}^{\beta}\mathrm{d}\tau\left[\bar{\psi}_f(\partial\!\!\!/-\mu\gamma^0+h\phi)\psi^f+NU_k(\phi)-\frac{1}{2}N\phi\Box\phi\right]$$

  - $U_k(\phi)$ is the only flowing term.
  - LPA enforces an artificial kinetic boson term in the UV action.

- Litim regulators: those are optimized within the LPA.

# Scale dependent effective potential $U_k(\sigma)$



- Inserting the ansatz into the WE and evaluating it at $\Phi = [\phi = \sigma, \bar{\psi} = 0, \psi = 0]$, we obtain the flow equation for $U_k(\sigma)$:

$$-k\partial_k U_k(\sigma) = \text{bosonic}(k, \partial_\sigma^2 U_k) + \text{fermionic}(k, \sigma)$$

- It is a PDE in **two** variables: $\sigma$ and $k$!

**Comment on the symmetry:**

- The **discrete chiral symmetry** of the original model is now inherited into the **mirror symmetry** of $U_k(\sigma)$:

$$U_k(-\sigma) = U_k(\sigma)\,.$$

- This symmetry is broken if $U_k(\sigma)$ has non-trivial minima for $k \to 0$.

# LPA flow eq. as a continuity equation

... this PDE can be reformulated into a continuity equation by applying $\frac{\mathrm{d}}{\mathrm{d}\sigma}$:

$$-k\partial_k u_k(\sigma) = \frac{\mathrm{d}}{\mathrm{d}\sigma} Q(k, \partial_\sigma u) + S(k, \sigma),$$

where $u \equiv \partial_\sigma U$.

- $Q(k, \partial_\sigma u)$ is a highly non-linear **diffusion** term (bosonic) and
- $S(k, \sigma)$ is the **source** term (fermionic).

This gives us the opportunity to use **finite volume methods**.

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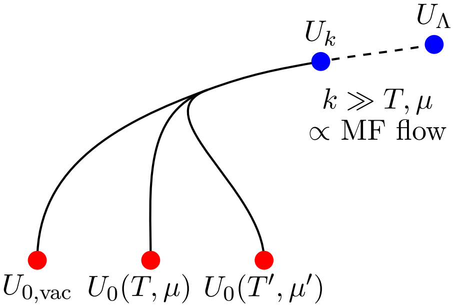
A. Koenigstein, M. J. Steil, N. Wink, et al. arXiv:2108.02504 (2021)

E. Grossi, N. Wink arXiv:1903.09503 (2019)

Finite volume method: A. Kurganov, E. Tadmor, J. Comput. Phys. 160, 241 – 282 (2000).

# Initial condition and the early RG flow






- Goal: RG-consistent initial condition, i.e., $\partial_\Lambda U_{k=0} = 0$.
- Observation: For very large RG scales $k$ the **bosonic term is negligible**.
  $\Rightarrow$ Choose the initial condition via the MF flow:

$$\frac{1}{g^2} = \frac{1}{\pi}\Big\{\operatorname{arsinh}\big(\tfrac{\Lambda}{h}\big) - \Big[1 + \big(\tfrac{h}{\Lambda}\big)^2\Big]^{-1/2}\Big\}, \text{ s.t. } \sigma_{\text{MF,min}} = 1 \text{ for } T = \mu = 0$$

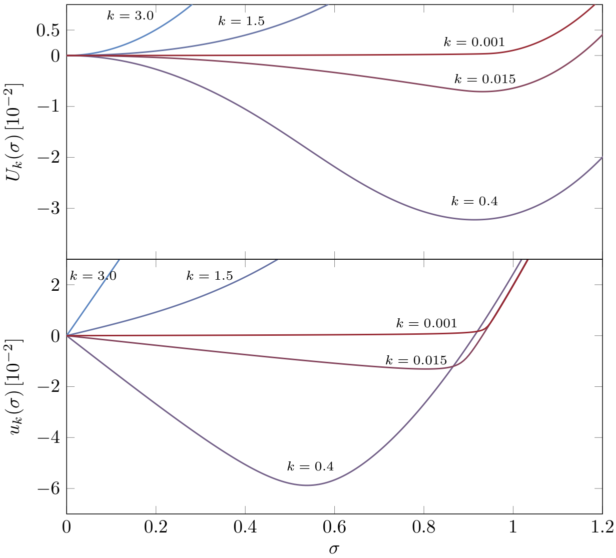
- **This makes the IR $\Lambda$-independent.**
- All dimensionful quantities are given in units of $h\sigma_{\text{MF,min}}$ for $T = \mu = 0$.

# Part I

## Numerical results

# Sample RG flow

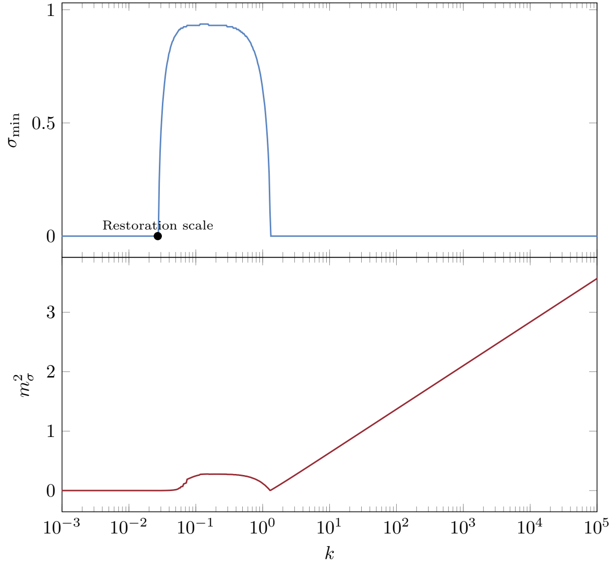
$T = 0.1,\ \mu = 0.1,\ N = 2,\ \Lambda = 10^5$




- At high $k$, the **fermionic** part dominates and acts like a sink $\Rightarrow$ The minimum of $U_k(\sigma)$ becomes nonzero.
- At low $k$, the **bosonic** part dominates and flattens $u_k(\sigma)$ $\Rightarrow$ The minimum of $U_k(\sigma)$ becomes zero again.

# Sample RG flow

$T = 0.1,\ \mu = 0.1,\ N = 2,\ \Lambda = 10^5$




- $\sigma_{\min}$ is the global minimum of $U_k(\sigma)$ and $m_\sigma^2$ is its curvature at $\sigma_{\min}$.
- The **restoration scale** $k_{\text{res}}$ is the scale where the broken symmetry restores.
- We find a similar behavior for $T > 0$, $N < \infty$ and arbitrary $\mu$.

# Phase diagram

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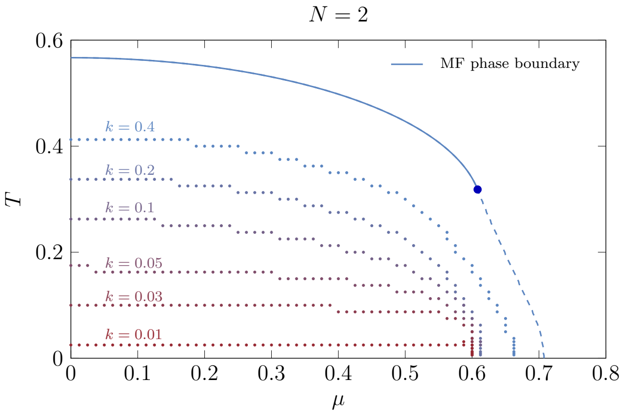
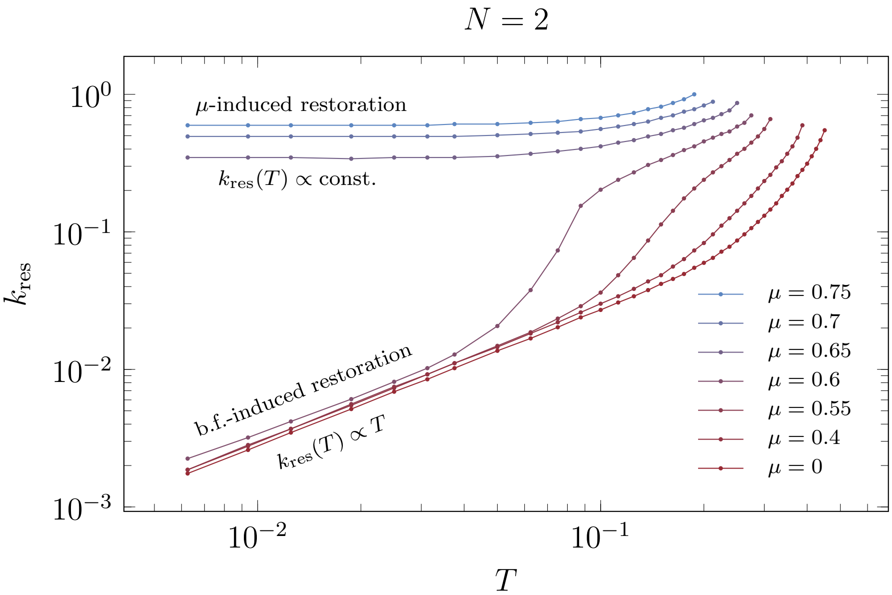


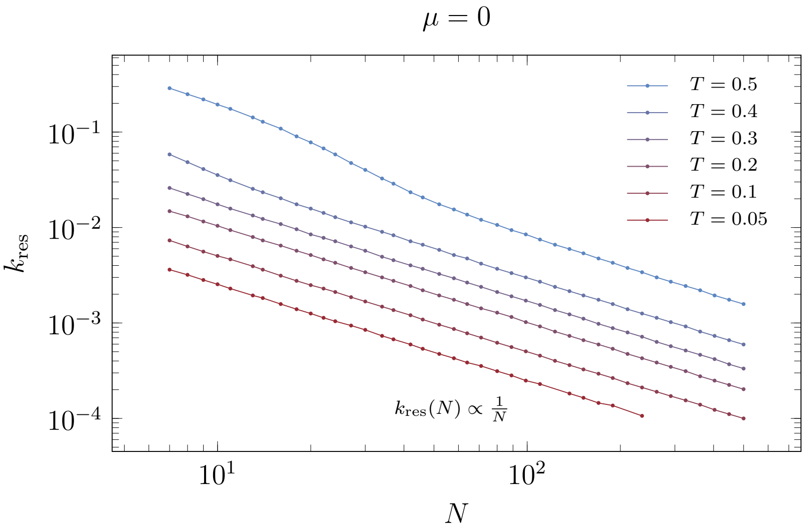

- For small RG scales, the **broken phase** shrinks towards lower temperatures and **vanishes** for all our test points with $T > 0$.
- Indications for SSB at $T = 0$ and small $\mu$.

- To be more quantitative, we analyze the $T$-, $\mu$- and $N$-dependence of the restoration scale and extrapolate it w.r.t. $N$ and $T$.

# Restoration scale: $T$- & $\mu$-dependence



# Restoration scale: $N$-dependence




- The bosonic part is suppressed by the factor $N$, hence we have to flow to deeper energies to see restoration.

# Conclusions of Part I



- Our results suggest that there is no SSB in the IR for finite $N$ and $T > 0$.
- For $T = 0$ and small $\mu$ we see indications for SSB.

This supports Landau's argument and disagrees with the observations from the lattice calculations.

J. Stoll, **N. Z.**, A. Koenigstein, M. J. Steil, et al. arXiv:2108.10616 (2021).

# Part II

## Continuum to finite lattice

# Momentum spaces



The momentum/reciprocal space $\tilde{V}$ depends on the position space $V$ and on the boundary conditions of the fields on it.

- Finite $V = L^d$ corresponds to the discretization of $\tilde{V} = (\frac{2\pi}{L}(\mathbb{Z} + s_\mu))^d$, where $s_\mu$ denotes whether we have periodic or antiperiodic boundary conditions in the $\mu$-th direction.
- Discrete $V = (a\mathbb{Z})^d$ corresponds to the restriction to the first Brillouin zone, $\tilde{V} = (-\frac{\pi}{a}, \frac{\pi}{a}]^d$.
- Finite lattice, i.e., finite and discrete $V$, is the combination.

# LPA flow equation

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Using the two-dimensional Litim regulator and momenta spaces $\tilde{V}_{\mathrm{b}}$ and $\tilde{V}_{\mathrm{f}}$, the flow equation reads

$$-k\partial_k U_k(\sigma) = \frac{1}{V}\sum_{\boldsymbol{q}\in\tilde{V}_{\mathrm{b}}} \frac{1}{N}\frac{-\epsilon_k\Theta(1-\epsilon^{\mathrm{b}}_{\boldsymbol{q}}/\epsilon_k)}{\epsilon_k+\partial^2_\sigma U_k(\sigma)} - \frac{1}{V}\sum_{\boldsymbol{q}\in\tilde{V}_{\mathrm{f}}} \frac{-d_\gamma\epsilon_k\Theta(1-\epsilon^{\mathrm{f}}_{\boldsymbol{q}}/\epsilon_k)}{\epsilon_k+\sigma^2}\,,$$

where $\epsilon_k = \epsilon_0 k^2$.

- In the Heaviside function we compare **energies**, not the distance of two momenta.
- We can use the same numerics to solve this flow equation.

# Dispersion relations

For a discrete $V$ we have to choose some discretization for the bosonic and fermionic fields. In the flow equation this is encoded through the dispersion relations $\epsilon_{\boldsymbol{q}}^{\mathrm{b/f}}$:

- NAIVE discretization:

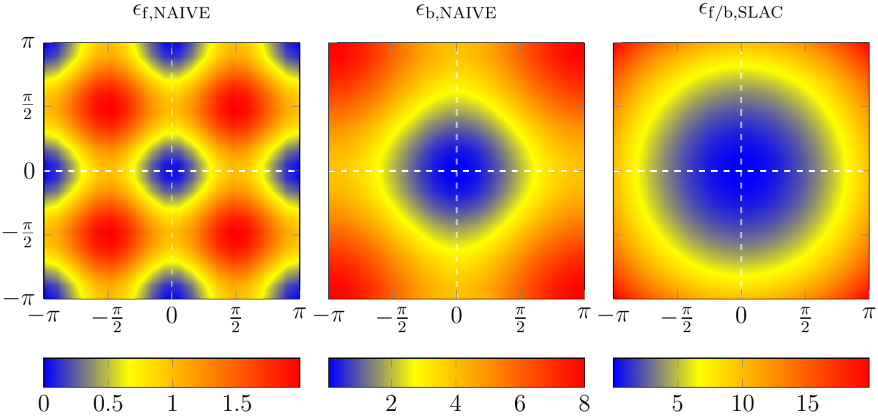
$$\epsilon_{\boldsymbol{q}}^{\mathrm{b}} = \sum_{\mu} \left[ \frac{2}{a} \sin(\frac{1}{2} a \boldsymbol{q} \cdot \boldsymbol{e}_{\mu}) \right]^2 , \quad \epsilon_{\boldsymbol{q}}^{\mathrm{f}} = \sum_{\mu} \left[ \frac{1}{a} \sin(a \boldsymbol{q} \cdot \boldsymbol{e}_{\mu}) \right]^2 .$$

- SLAC discretization:

$$\epsilon_{\boldsymbol{q}}^{\mathrm{b}} = \epsilon_{\boldsymbol{q}}^{\mathrm{f}} = \sum_{\mu} \left[ \boldsymbol{q} \cdot \boldsymbol{e}_{\mu} \right]^2 .$$

# Dispersion relations





# Scales in the system

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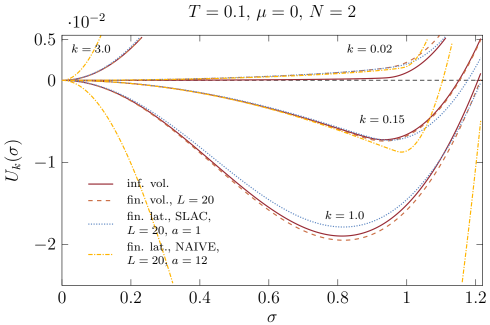
- In total we have five dimensionful quantities: $T$, $L$, $a$, $\Lambda$ and $k_{\text{IR}}$.
- How can we **remove** $\Lambda$ and $k_{\text{IR}}$?
  - The UV-cutoff can be removed by choosing, e.g., $\Lambda = 100\frac{\pi}{a}$.
  - For fermions we find a minimal energy $\min_{\boldsymbol{q}\in\tilde{V}_{\text{f}}} \epsilon^{\text{f}}_{\boldsymbol{q}} > 0$, this means that there exists a RG scale $k^{\star}$ such that for $k < k^{\star}$ the fermions give no further contribution to the flow.
  - This argument is not applicable for the boson, since it has a zero energy mode meaning that we have to take $k_{\text{IR}} \to 0$ even if we are on a finite lattice.
- We are now left with only $T$, $L$ and $a$ as in lattice calculations.

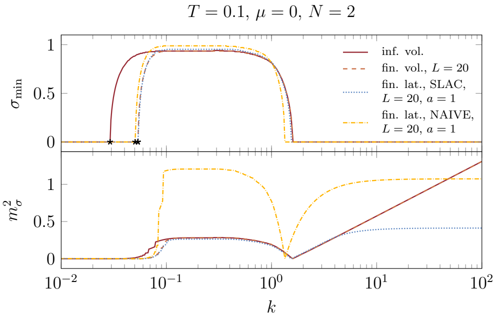
# Part II

## Numerical results

# Sample RG flows






- **Conclusion**: Still no SSB detected neither in infinite volume nor in finite volume nor on a finite lattice for our various discretizations.
- However, it is now possible to directly compare both methods: Lattice and FRG calculations. (This is also possible for other models like NJL-type models, e.g., the QM model.)

# Summary & Outlook

## Summary:

- Within the LPA, we find a trivial PD for $T > 0$ and $N < \infty$.
- Our results suggest that there could be SSB for $T = 0$ and small $\mu$.
- Finite lattice spacing and lattice extent do not alter these results (for $\mu = 0$).

This supports Landau's argument and disagrees with the lattice MC simulations.

## Outlook:

- The reason for this discrepancy is not clear yet.
- But we are currently working on it in a FRG-lattice collaboration.

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J. Stoll, **N. Z.**, A. Koenigstein, M. J. Steil, et al. arXiv:2108.10616 (2021).

FRG-lattice collaboration: J. Braun, L. Pannullo, **N. Z.**, in preparation.

# Thank you for your attention!

CRC-TR 211
Strong-interaction matter under extreme conditions

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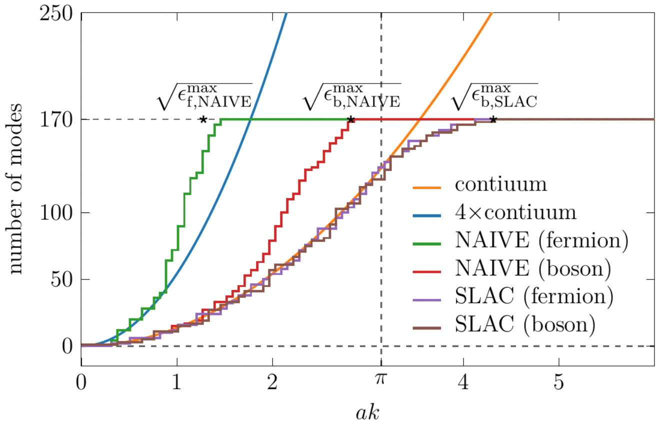
# Appendix

# Modes count



250
170
100
50
0
number of modes
0
1
2
π
4
5
ak
$\sqrt{\epsilon^{\max}_{\mathrm{f,NAIVE}}}$
$\sqrt{\epsilon^{\max}_{\mathrm{b,NAIVE}}}$
$\sqrt{\epsilon^{\max}_{\mathrm{b,SLAC}}}$
contiuum
4×contiuum
NAIVE (fermion)
NAIVE (boson)
SLAC (fermion)
SLAC (boson)

# Initial condition