Kinetic model with arbitrary transport coefficients

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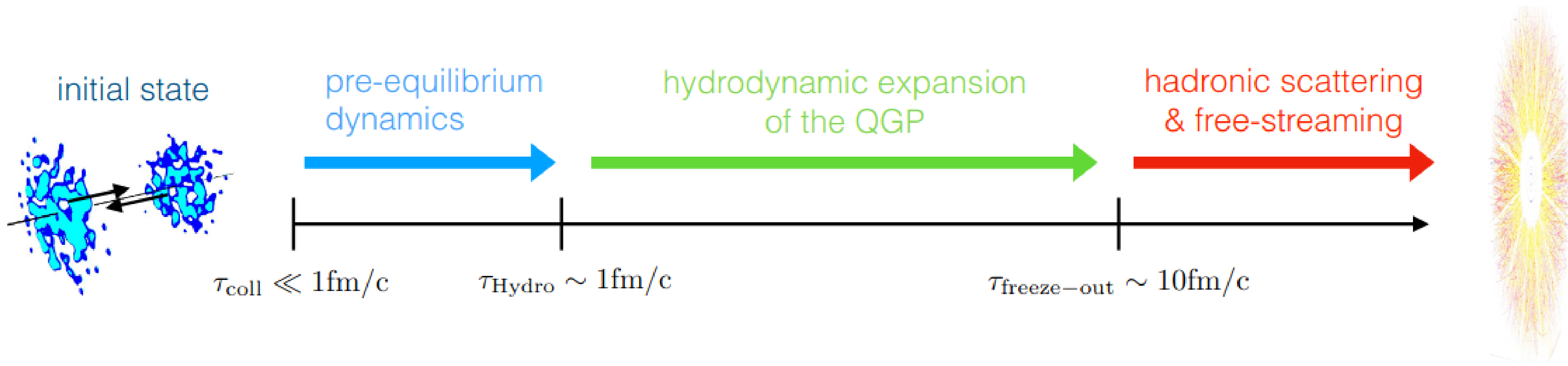
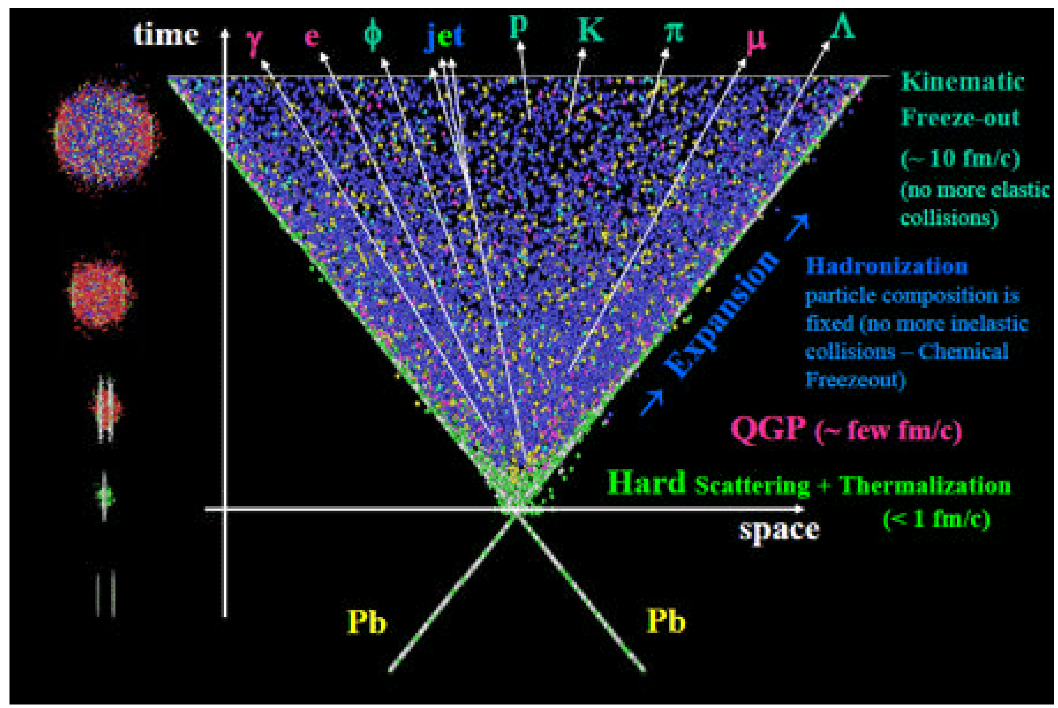
Outline

Relativistic hydro playground: Heavy-ion collisions

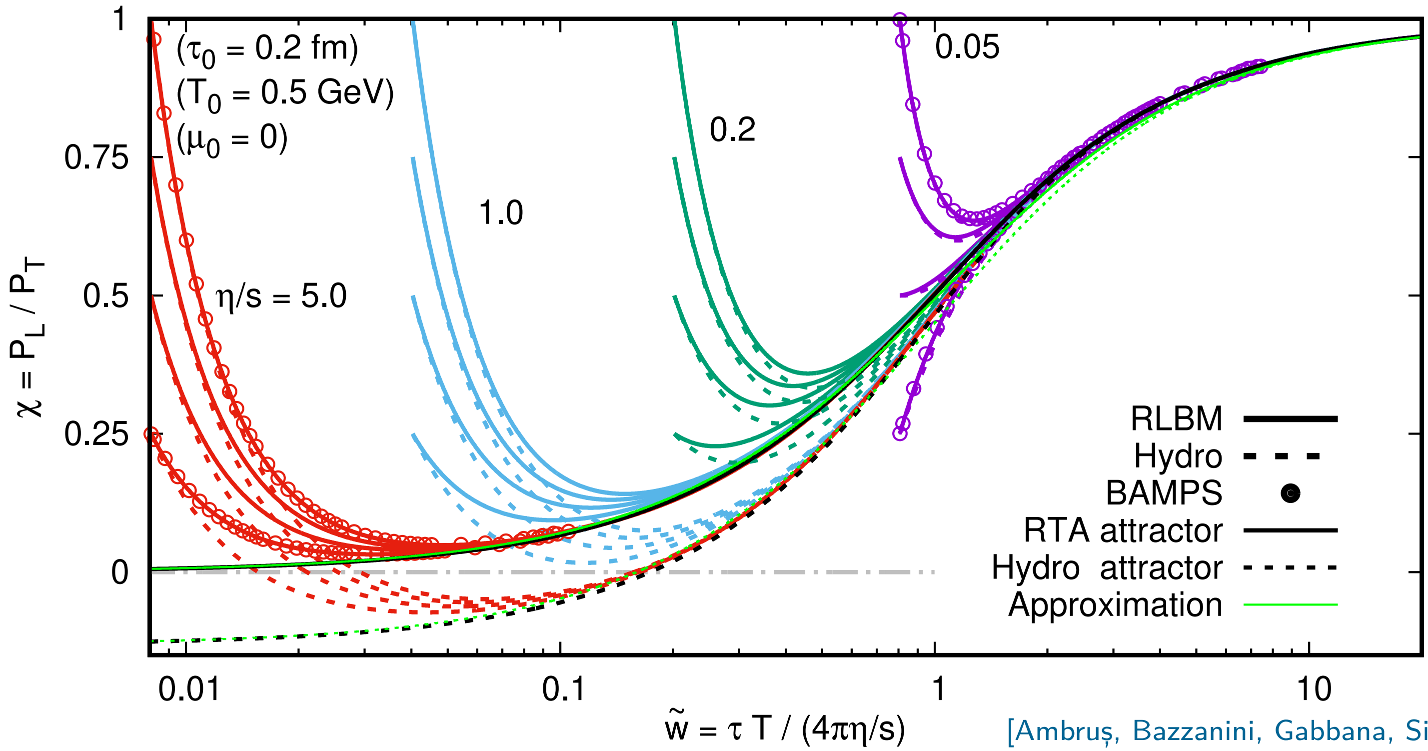


- Shortly after the collision, the system is far-from-equilibrium.
- Pre-eq. dynamics require a non-eq. description.
- Strongly-interacting QGP leaves imprints of thermalization and collectivity in final-state observables.



[Venaruzzo, PhD Thesis, 2011]

Hydro vs Kinetic theory



[Ambruș, Bazzanini, Gabbana, Simeoni, Succi, Nature Comput. Sci. **2** (2022) 641]

- Hydro employed in HIC modelling, but it breaks down far from eq.
- Kinetic theory overcomes this limitation, but realistic simulations are expensive due to $C[f]$.

  AMPT: He, Edmonds, Lin, Liu, Molnar, Wang [PLB 753 (2016) 506]
  BAMPS: Greif, Greiner, Schenke, Schlichting, Xu [PRD 96 (2017) 091504]
- RTA: $C[f] = -\frac{E_{\mathbf{k}}}{\tau_R}(f_{\mathbf{k}} - f_{0\mathbf{k}}) \Rightarrow 1-2$ o.m. faster than BAMPS.

  VEA, Busuioc, Fotakis, Gallmeister, Greiner [PRD **104** (2021) 094022]
- τ_R fixes the IR limit of RTA by matching e.g. η to that of $C[f] \Rightarrow$ good agreement with BAMPS.

Anderson-Witting model

- The Anderson & Witting RTA reads [Anderson, Witting, Physica **74** (1974) 466]

$$k^\mu \partial_\mu f_{\mathbf{k}} = C_{\rm AW}[f], \quad C_{\rm AW}[f] = -\frac{E_{\mathbf{k}}}{\tau_R}(f_{\mathbf{k}} - f_{0\mathbf{k}}), \tag{1}$$

  where $E_{\mathbf{k}} = k^\mu u_\mu$, and τ_R is the relaxation time.
- The macroscopic quantities N^μ and $T^{\mu\nu}$ are obtained from $f_{\mathbf{k}}$ via

$$N^\mu = \int dK\, k^\mu\, f_{\mathbf{k}}, \quad T^{\mu\nu} = \int dK\, k^\mu k^\nu f_{\mathbf{k}}, \tag{2}$$

  where $dK = g\, d^3k/[k_0(2\pi)^3]$ and g is the degeneracy factor.
- $f_{0\mathbf{k}}$ describes a fictitious local thermodynamic equilibrium, for which

$$N_0^\mu = n_0 u^\mu, \quad T_0^{\mu\nu} = \epsilon_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu}, \tag{3}$$

  with $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$.
- Imposing $\partial_\mu N^\mu = \partial_\nu T^{\mu\nu} = 0$ requires Landau matching:

$$n = n_0, \quad \epsilon = \epsilon_0, \quad T^\mu{}_\nu u^\nu = \epsilon u^\mu. \tag{4}$$

- The AW model retains from $C[f]$ the property of driving $f_{\mathbf{k}}$ towards $f_{0\mathbf{k}}$, on a timescale τ_R.

Chapman-Enskog expansion

- We are now interested to obtain constitutive relations for the non-equilibrium quantities

$$N^\mu - N_0^\mu = V^\mu, \quad T^{\mu\nu} - T_0^{\mu\nu} = -\Pi\Delta^{\mu\nu} + \pi^{\mu\nu}. \tag{5}$$

- Employing the Chapman-Enskog procedure gives

$$\delta f_{\mathbf{k}} \equiv f_{\mathbf{k}} - f_{0\mathbf{k}} \simeq -\frac{\tau_R}{E_{\mathbf{k}}} k^\mu \partial_\mu f_{0\mathbf{k}} = -f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}\big[E_{\mathbf{k}}^2\dot{\beta} - E_{\mathbf{k}}\dot{\alpha}$$
$$+\tfrac{\beta}{3}(m^2 - E_{\mathbf{k}}^2)\theta + k^{\langle\mu\rangle}(\beta E_{\mathbf{k}}\dot{u}_\mu + E_{\mathbf{k}}\nabla_\mu\beta - \nabla_\mu\alpha) + \beta k^{\langle\mu}k^{\nu\rangle}\sigma_{\mu\nu}\big],$$

with $\tilde{f}_{0\mathbf{k}} = 1 - af_{0\mathbf{k}}$, $\alpha = \beta\mu$, $\theta = \partial_\mu u^\mu$ and $\sigma^{\mu\nu} = \nabla^{\langle\mu}u^{\nu\rangle}$.

- Taking appropriate moments gives

$$\Pi = -\zeta_R\theta, \quad V^\mu = \kappa_R\nabla^\mu\alpha, \quad \pi^{\mu\nu} = 2\eta_R\sigma^{\mu\nu}, \tag{6}$$

where ζ_R, κ_R and η_R are given by

$$\zeta_R = \frac{m^2}{3}\tau_R\alpha_0^{(0)}, \quad \kappa_R = \tau_R\alpha_0^{(1)}, \quad \eta_R = \tau_R\alpha_0^{(2)}. \tag{7}$$

where $\alpha_0^{(\ell)}$ are τ_R-independent thermodynamic functions.

QGP Transport coefficients

- Bayesian estimation shows that η/s and ζ/s can be parametrized as

J. E. Bernhard, J. S. Moreland, S. A. Bass, Nature Phys. **15** (2019) 1113

$$\frac{\eta}{s} = (\eta/s)_{\rm min} + (\eta/s)_{\rm slope}(T - T_c)\left(\frac{T}{T_c}\right)^{(\eta/s)_{\rm crv}}, \quad (8)$$

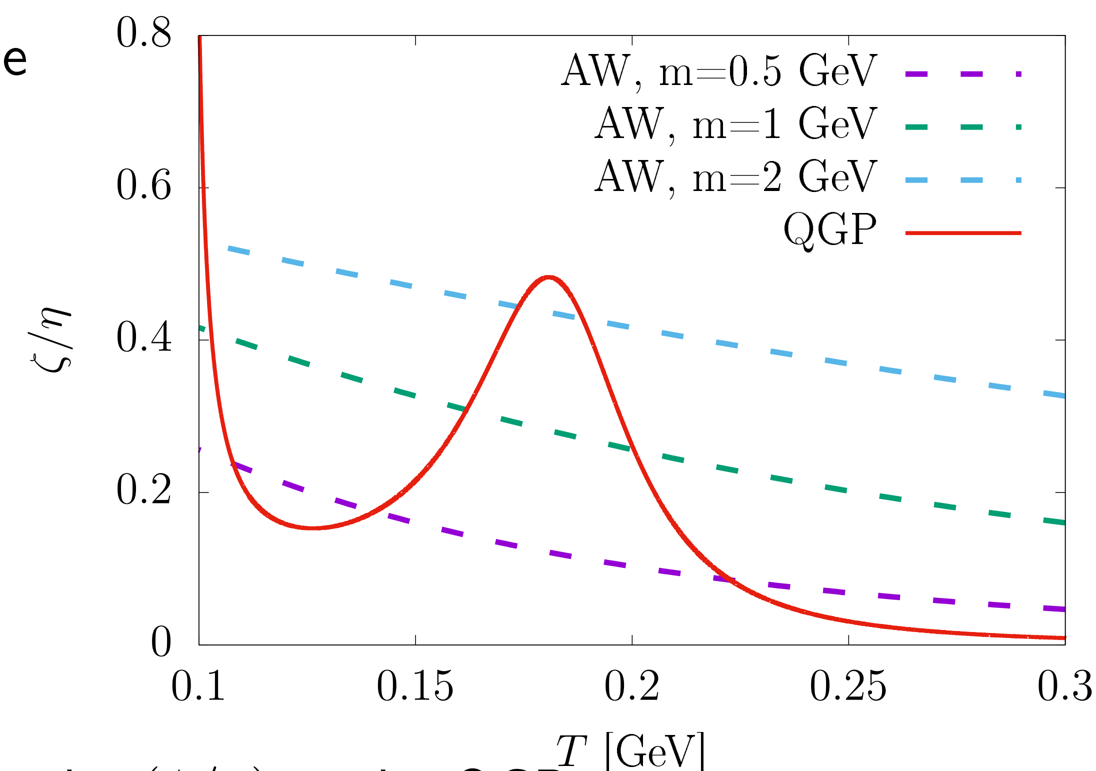
$$\frac{\zeta}{s} = (\zeta/s)_{\rm max} \times \left[1 + \left(\frac{T - T_{\rm peak}}{(\zeta/s)_{\rm width}}\right)^2\right]^{-1}. \quad (9)$$

- RTA allows, e.g. η to be specified by setting

$$\tau_R = \frac{\eta}{\alpha_0^{(2)}},$$

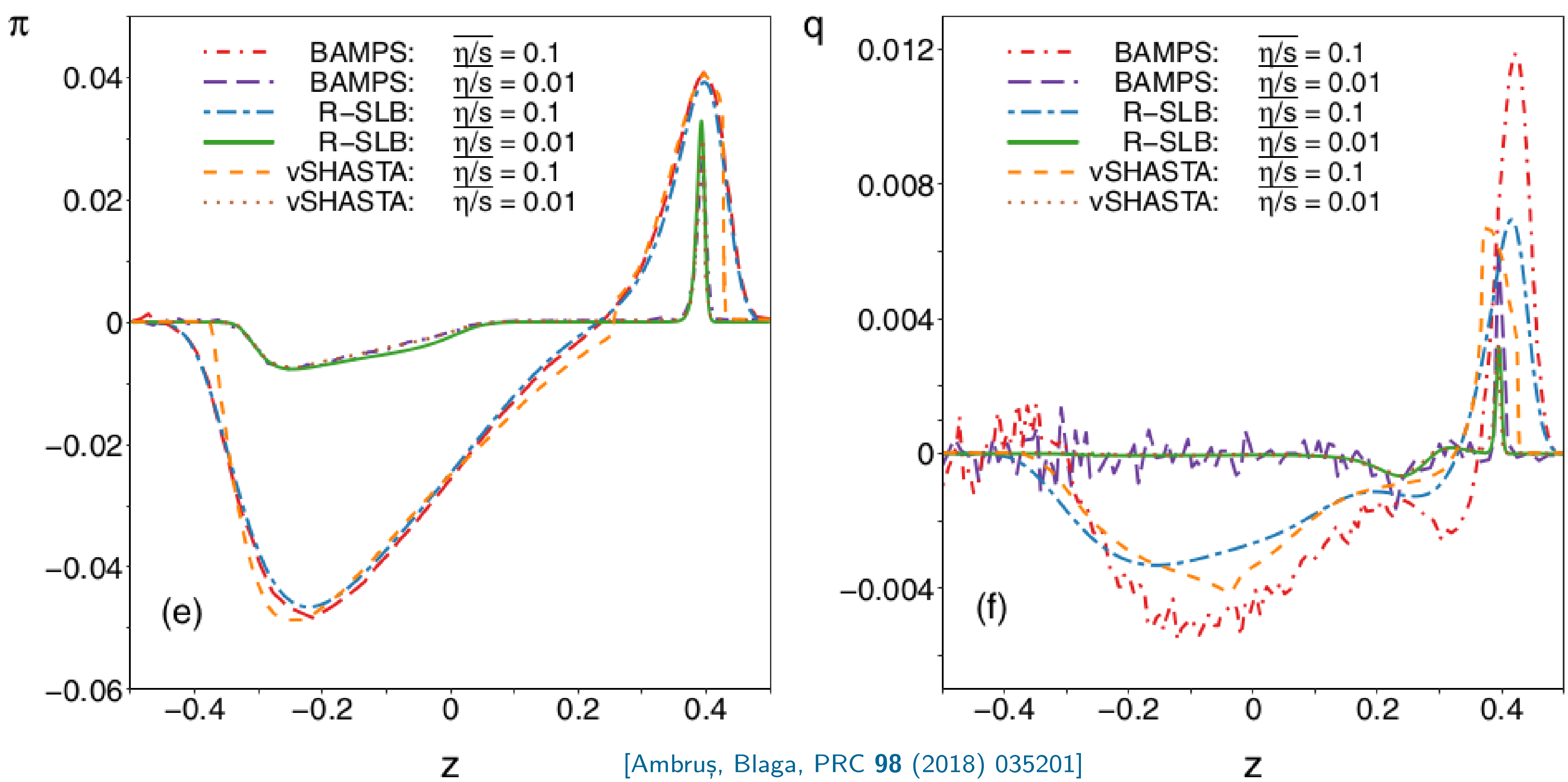
however, ζ/η is fixed uniquely by

$$\frac{\zeta}{\eta} = \frac{m^2\alpha_0^{(0)}}{3\alpha_0^{(2)}},$$



which does not resemble the (ζ/η) in the QGP.

RTA vs BAMPS



[Ambruș, Blaga, PRC **98** (2018) 035201]

- Also for UR hard spheres, $(\kappa T/\eta)_{\rm HS} \simeq 0.125$, whereas $(\kappa T/\eta)_{\rm AW} = 5/48 \simeq 0.104$. DNMR, PRD **85** (2012) 114047
- Fixing η via τ_R gives good agreement with BAMPS for $\pi^{\mu\nu}$ but q^μ is not captured correctly.
- **Aim of this work:** Extend RTA with extra parameters allowing multiple transport coefficients to be controlled independently.

Shakhov-like extension

[Ambruș, Molnár, under review]

- We consider a Shakhov-like extension: [Shakhov, Fluid Dyn. 3 (1968) 112]

$$C_{\mathrm{S}}[f] = -\frac{E_{\mathbf{k}}}{\tau_R}(f_{\mathbf{k}} - f_{\mathrm{S}\mathbf{k}}), \tag{10}$$

  where $f_{\mathrm{S}\mathbf{k}} \to f_{0\mathbf{k}}$ as $\delta f_{\mathbf{k}} = f_{\mathbf{k}} - f_{0\mathbf{k}} \to 0$.

- In the Shakhov model, $f_{\mathbf{k}}$ relaxes towards $f_{0\mathbf{k}}$ on a modified path compared to AW.

- The cons. eqs. $\partial_\mu N^\mu = \partial_\nu T^{\mu\nu} = 0$ imply:

$$u_\mu N^\mu = u_\mu N^\mu_{\mathrm{S}}, \quad u_\nu T^{\mu\nu} = u_\nu T^{\mu\nu}_{\mathrm{S}}, \tag{11}$$

  which allows for plenty of degrees of freedom (δn, $\delta\epsilon$, W^μ, etc).

- For simplicity, we stick to the Landau matching conditions:

$$\delta n = \delta\epsilon = 0, \qquad T^{\mu\nu} u_\nu = \epsilon u^\mu. \tag{12}$$

Shakohv-like extension

- Employing the Chapman-Enskog procedure gives

$$\delta f_{\mathbf{k}} - \delta f_{\mathrm{S}\mathbf{k}} = -\frac{\tau_R}{E_{\mathbf{k}}} k^\mu \partial_\mu f_{0\mathbf{k}}, \tag{13}$$

leading to

$$\Pi - \Pi_{\mathrm{S}} = -\zeta_R \theta, \quad V^\mu - V^\mu_{\mathrm{S}} = \kappa_R \nabla^\mu \alpha, \quad \pi^{\mu\nu} - \pi^{\mu\nu}_{\mathrm{S}} = 2\eta_R \sigma^{\mu\nu}. \tag{14}$$

- We seek to replace ζ_{R} etc by independent transport coefficients:

$$\begin{aligned} \Pi &\simeq -\zeta_{\mathrm{S}}\theta, & V^\mu &\simeq \kappa_{\mathrm{S}} \nabla^\mu \alpha, & \pi^{\mu\nu} &\simeq 2\eta_{\mathrm{S}} \sigma^{\mu\nu}, \\ \zeta_{\mathrm{S}} &= \frac{\tau_\Pi}{\tau_R} \zeta_R, & \kappa_{\mathrm{S}} &= \frac{\tau_V}{\tau_R} \kappa_R, & \eta_{\mathrm{S}} &= \frac{\tau_\pi}{\tau_R} \eta_R. \end{aligned} \tag{15}$$

- Eq. (15) can be obtained from Eq. (14) when

$$\Pi_{\mathrm{S}} = \Pi \left(1 - \frac{\tau_\Pi}{\tau_R}\right), \quad V^\mu_{\mathrm{S}} = V^\mu \left(1 - \frac{\tau_V}{\tau_R}\right), \quad \pi^{\mu\nu}_{\mathrm{S}} = \pi^{\mu\nu} \left(1 - \frac{\tau_\pi}{\tau_R}\right). \tag{16}$$

Minimal $\delta f_{\mathrm{S}\mathbf{k}}$

- Writing $f_{\mathrm{S}\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathrm{S}\mathbf{k}}$, we require:

$$\begin{array}{l} \text{Bulk visc. p.} \\ \text{Particle cons.} \\ \text{Energy cons.} \end{array} \Rightarrow \int dK \begin{pmatrix} 1 \\ E_{\mathbf{k}} \\ E_{\mathbf{k}}^2 \end{pmatrix} \delta f_{\mathrm{S}\mathbf{k}} \equiv \begin{pmatrix} \rho_{\mathrm{S};0} \\ \rho_{\mathrm{S};1} \\ \rho_{\mathrm{S};2} \end{pmatrix} = \begin{pmatrix} -3\Pi_{\mathrm{S}}/m^2 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{array}{l} \text{Diff. current} \\ \text{Mom. cons.} \end{array} \Rightarrow \int dK \begin{pmatrix} 1 \\ E_{\mathbf{k}} \end{pmatrix} k^{\langle\mu\rangle} \delta f_{\mathrm{S}\mathbf{k}} \equiv \begin{pmatrix} \rho^{\mu}_{\mathrm{S};0} \\ \rho^{\mu}_{\mathrm{S};1} \end{pmatrix} = \begin{pmatrix} V^{\mu}_{\mathrm{S}} \\ 0 \end{pmatrix},$$

$$\text{SS tens.} \Rightarrow \int dK k^{\langle\mu} k^{\nu\rangle} \delta f_{\mathbf{k}} \equiv \rho^{\mu\nu}_{\mathrm{S};0} = \pi^{\mu\nu}_{\mathrm{S}}, \quad (17)$$

with $k^{\langle\mu\rangle} = \Delta^{\mu}_{\alpha} k^{\alpha}$ and $k^{\langle\mu} k^{\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} k^{\alpha} k^{\beta}$ *irreducible* tensors.

- The solution can be written as $\delta f_{\mathrm{S}\mathbf{k}} = f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \mathbb{S}_{\mathbf{k}}$, where

$$\mathbb{S}_{\mathbf{k}} = -\frac{3\Pi}{m^2}\left(1 - \frac{\tau_R}{\tau_\Pi}\right)\mathcal{H}^{(0)}_{\mathbf{k}0} + k_{\langle\mu\rangle} V^{\mu}\left(1 - \frac{\tau_R}{\tau_V}\right)\mathcal{H}^{(1)}_{\mathbf{k}0} + k_{\langle\mu} k_{\nu\rangle} \pi^{\mu\nu}\left(1 - \frac{\tau_R}{\tau_\pi}\right)\mathcal{H}^{(2)}_{\mathbf{k}0}. \quad (18)$$

- $\mathcal{H}^{(\ell)}_{\mathbf{k}0}$ are polynomials in $E_{\mathbf{k}}$ satisfying (17).

[DNMR, PRD **85** (2012) 114047]

Entropy production

- The entropy current is given by [classical stat. used for simplicity]

$$S^\mu = -\int dK\, k^\mu \left(f_{\mathbf{k}} \ln f_{\mathbf{k}} - f_{\mathbf{k}}\right). \tag{19}$$

- In the Shakhov model, $k^\mu \partial_\mu f = C_{\rm S}[f]$ and

$$\partial_\mu S^\mu = -\int dK\, C_{\rm S}[f] \ln f_{\mathbf{k}} = \frac{1}{\tau_R} \int dK\, E_{\mathbf{k}} (\delta f_{\mathbf{k}} - \delta f_{{\rm S}\mathbf{k}}) \ln f_{\mathbf{k}}. \tag{20}$$

- $\partial_\mu S^\mu$ difficult for generic $f_{\mathbf{k}}$.
- When $\phi_{\mathbf{k}} = \delta f_{\mathbf{k}} / f_{0\mathbf{k}}$ is small, detailed manipulations lead to

$$\partial_\mu S^\mu \simeq \frac{\beta}{\zeta_{\rm S}} \Pi^2 - \frac{1}{\kappa_{\rm S}} V_\mu V^\mu + \frac{\beta}{2\eta_{\rm S}} \pi_{\mu\nu} \pi^{\mu\nu} \geq 0. \tag{21}$$

- Close to eq., the S-model satisfies the $2^{\rm nd}$ law of thermodynamics.
- Proof far from eq. unavailable even for non-rel. Shakhov!

Application: Bjorken flow

- Bjorken model: flow invariant under longitudinal boosts:

$$u^\mu \partial_\mu = \frac{t}{\tau}\partial_t + \frac{z}{\tau}\partial_z, \quad \tau = \sqrt{t^2 - z^2}, \quad \eta_s = \tanh^{-1}(z/t). \tag{22}$$

- In Bjorken coordinates $(\tau, \mathbf{x}_\perp, \eta_s)$,

$$T^{\mu\nu} = \mathrm{diag}(e, P_T, P_T, \tau^{-2} P_L),$$
$$P_T = P + \Pi - \frac{\pi_d}{2}, \qquad P_L = P + \Pi + \pi_d. \tag{23}$$

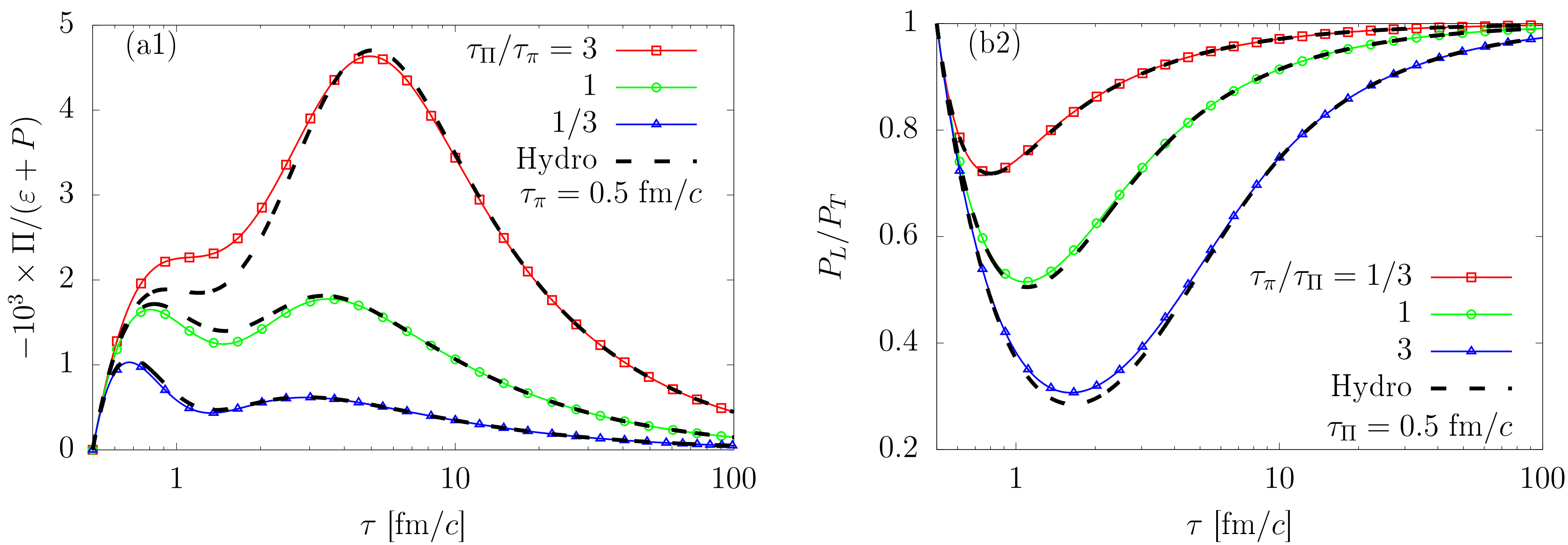
- In 2^{nd}-order hydro, we have: [Denicol, Florkowski, Ryblewski, Strickland, PRC **90** (2014) 044905]

$$\tau\dot{\epsilon} + \epsilon + P_L = 0, \tag{24a}$$

$$\tau\dot{\Pi} + \left(\frac{\delta_{\Pi\Pi}}{\tau_\Pi} + \frac{\tau}{\tau_\Pi}\right)\Pi + \frac{\lambda_{\Pi\pi}}{\tau_\Pi}\pi_d = -\frac{\zeta}{\tau_\Pi},$$
$$\tau\dot{\pi}_d + \left(\frac{\delta_{\pi\pi}}{\tau_\pi} + \frac{\tau_{\pi\pi}}{3\tau_\pi} + \frac{\tau}{\tau_\pi}\right)\pi_d + \frac{2\lambda_{\pi\Pi}}{3\tau_\pi}\Pi = -\frac{4\eta}{3\tau_\pi}. \tag{24b}$$

- We employ the Shakhov model to control ζ independently from η.

Shakhov model: ζ vs. η



- Choosing $\tau_R = \tau_\Pi$, the Shakhov distribution becomes

$$f_{S\mathbf{k}} = f_{0\mathbf{k}} \left[1 + \frac{\beta^2 k_\mu k_\nu \pi^{\mu\nu}}{2(e+P)} \left(1 - \frac{\tau_\Pi}{\tau_\pi} \right) \right]. \tag{25}$$

- Left panel: τ_π is fixed and τ_Π is varied using the Shakhov model.
- Right panel: τ_Π is fixed and τ_π is varied using the Shakhov model.
- $m = 1$ GeV; $\tau_0 = 0.5$ fm; $\beta_0^{-1} = 0.6$ GeV; For $\tau_\pi = 0.5$ fm, $4\pi\eta/s \simeq 3.3$ at $\tau = \tau_0$.

Application: Sound waves

- We now consider an infinitesimal perturbation propagating in an ultrarelativistic fluid at rest.
- Writing $u^\mu \simeq (1,0,0,\delta v)$, $\epsilon = \epsilon_0 + \delta\epsilon$ and $n = n_0 + \delta n$, we have

$$
\begin{aligned}
\partial_t \delta n + n_0 \partial_z \delta v + \partial_z \delta V =& 0, \\
\partial_t \delta\epsilon + (\epsilon_0 + P_0)\partial_z \delta v =& 0, \\
(\epsilon_0 + P_0)\partial_t \delta v + \partial_z \delta P + \partial_z \delta\pi =& 0, \\
\tau_V \partial_t \delta V + \delta V + \kappa \partial_z \delta\alpha - \ell_{V\pi}\partial_z \delta\pi =& 0, \\
\tau_\pi \partial_t \delta\pi + \delta\pi + \frac{4\eta}{3}\partial_z \delta v + \frac{2}{3}\ell_{\pi V}\partial_z \delta V =& 0,
\end{aligned}
\qquad (26)
$$

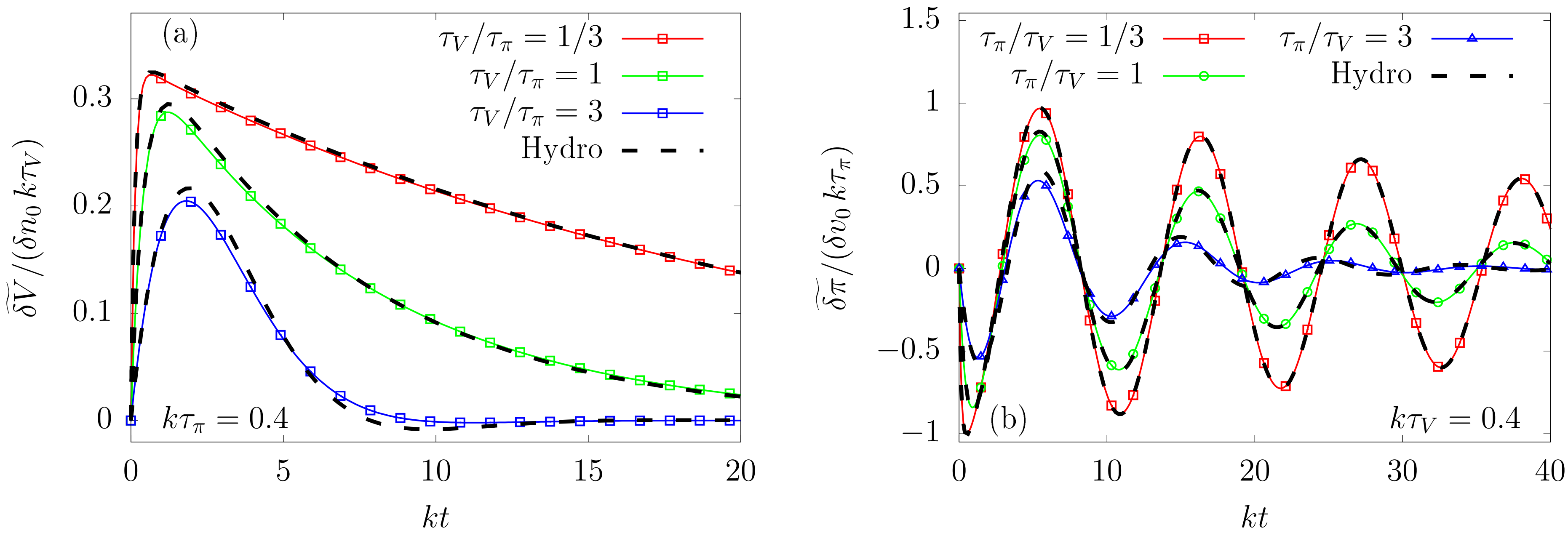
where $\delta V = V^z$ and $\delta\pi = \pi^{zz}/\gamma^2$.

- **In RTA, $\ell_{V\pi} = \ell_{\pi V} = 0$.** [Ambruș, Molnár, Rischke, PRD **106** (2022) 076005]
- We track the time evolution of the amplitudes

$$
\widetilde{\delta V} = \frac{2}{L}\int_0^L dz\, \delta V \cos(kz), \quad \widetilde{\delta\pi} = \frac{2}{L}\int_0^L dz\, \delta\pi \sin(kz). \qquad (27)
$$

- We employ the Shakhov model to control κ independently from η.

Shakhov model: κ vs. η



- Setting $\tau_R = \tau_\pi$, the Shakhov distribution becomes

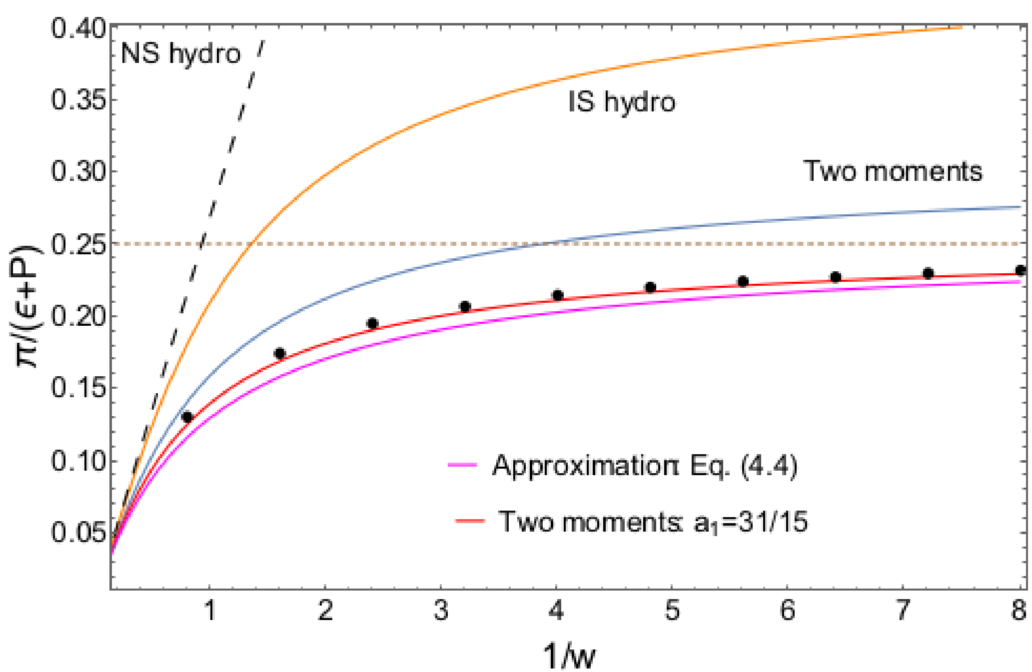
$$f_{S\mathbf{k}} = f_{0\mathbf{k}}\left[1 + \frac{k_\mu V^\mu}{P}(\beta E_\mathbf{k} - 5)\left(1 - \frac{\tau_\pi}{\tau_V}\right)\right]. \qquad (28)$$

Beyond first order: second-order transport coefficients?

- Relativistic hydrodynamics must obey causality ⇒ first-order theories are excluded.
- One example is the Israel-Stewart-type hydro, by which e.g. $\pi^{\mu\nu}$ evolves according to $\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu}$, with

$$\begin{aligned}\mathcal{J}^{\mu\nu} &= 2\tau_\pi \pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} \\ &\quad - \tau_{\pi V} V^{\langle\mu}\dot{u}^{\nu\rangle} + \ell_{\pi V}\nabla^{\langle\mu}V^{\nu\rangle} + \lambda_{\pi V} V^{\langle\mu}\nabla^{\nu\rangle}\alpha, \\ \mathcal{R}^{\mu\nu} &= \varphi_6 \Pi\pi^{\mu\nu} + \varphi_7 \pi^{\lambda\langle\mu}\pi_\lambda^{\nu\rangle} + \varphi_8 V^{\langle\mu}V^{\nu\rangle}. \end{aligned} \tag{29}$$

- In RTA, $\mathcal{R}^{\mu\nu} = 0$.
- 2^{nd}-order t.c. are important e.g. in preeq!
- In conformal RTA, $\delta_{\pi\pi} + \tau_{\pi\pi}/3 = 38/21$.
- Solving hydro with $\delta_{\pi\pi} + \tau_{\pi\pi}/3 = 31/15$ gives much better agreement with RTA!



[J.-P. Blaizot, L. Yan, PRC **104** (2021) 055201]

- Etc...

Second-order hydro from KT

- In the method of moments, second-order hydro can be derived using:
  - Irreducible moments of $\delta f_{\mathbf{k}}$: $\rho_r^{\mu_1\cdots\mu_\ell} = \int dK E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f_{\mathbf{k}}$.
  - Irreducible moments of $C[f]$: $C_r^{\mu_1\cdots\mu_\ell} = \int dK E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} C[f]$.
  - Define collision matrix via $C_{r-1}^{\mu_1\cdots\mu_\ell} = -\sum_n \mathcal{A}_{rn}^{(\ell)}\rho_n^{\mu_1\cdots\mu_\ell}$.
  - Define inverse matrix $\tau_{rn}^{(\ell)}$ via $\sum_n \tau_{rn}^{(\ell)}\mathcal{A}_{nm}^{(\ell)} = \delta_{rm}$.
- For example, the first-order transport coeffs. are

$$\zeta_r = \frac{m^2}{3}\sum_n \tau_{rn}^{(0)}\alpha_n^{(0)}, \quad \kappa_r = \sum_n \tau_{rn}^{(1)}\alpha_n^{(1)}, \quad \eta_r = \sum_n \tau_{rn}^{(2)}\alpha_n^{(2)}.$$

- The relaxation times can be obtained via

$$\tau_\Pi = \sum_n \tau_{0n}^{(0)}\mathcal{C}_n^{(0)}, \quad \tau_V = \sum_n \tau_{0n}^{(1)}\mathcal{C}_n^{(1)}, \quad \tau_\pi = \sum_n \tau_{0n}^{(2)}\mathcal{C}_n^{(2)}. \quad (30)$$

- ...all other 2nd-order t.c. are computed using $\tau_{0n}^{(\ell)}$ and $\mathcal{C}_n^{(\ell)}$.
- Idea: Use Shakhov model to “manipulate” $\mathcal{A}_{rn}^{(\ell)}$.

From RTA to Shakhov

- In RTA, $C[f] = -\frac{E_{\mathbf{k}}}{\tau_R}\delta f_{\mathbf{k}}$ and [Ambruș, Molnár, Rischke, PRD **106** (2022) 076005]

$$C_{r-1}^{\mu_1\cdots\mu_\ell} = -\frac{1}{\tau_R}\rho_r^{\mu_1\cdots\mu_\ell} \Rightarrow \mathcal{A}_{rn}^{(\ell)} = \frac{\delta_{rn}}{\tau_R} \Rightarrow \tau_{rn}^{(\ell)} = \tau_R\delta_{rn}. \tag{31}$$

- In the Shakhov model, $C_{\rm S} = -\frac{E_{\mathbf{k}}}{\tau_R}[\delta f_{\mathbf{k}} - \delta f_{{\rm S}\mathbf{k}}]$ and

$$C_{r-1}^{\mu_1\cdots\mu_\ell} = -\frac{1}{\tau_R}[\rho_r^{\mu_1\cdots\mu_\ell} - \rho_{{\rm S};r}^{\mu_1\cdots\mu_\ell}], \tag{32}$$

  where $\rho_{{\rm S};r}^{\mu_1\cdots\mu_\ell}$ are essentially arbitrary.

- Imposing $C_{r-1}^{\mu_1\cdots\mu_\ell} = -\sum_n \mathcal{A}_{rn}^{(\ell)}\rho_n^{\mu_1\cdots\mu_\ell}$ suggests taking

$$\rho_{{\rm S};r}^{\mu_1\cdots\mu_\ell} = \sum_n [\delta_{rn} - \tau_R\mathcal{A}_{rn}^{(\ell)}]\rho_n^{\mu_1\cdots\mu_\ell}, \tag{33}$$

  where $\mathcal{A}_{rn}^{(\ell)}$ is the desired collision matrix and $\rho_n^{\mu_1\cdots\mu_\ell}$ is extracted from $f_{\mathbf{k}}$.

- Problem: For a generic $C[f]$, $\mathcal{A}_{rn}^{(\ell)}$ is infinite!

Constructing $\mathbb{S}_{\mathbf{k}}$

[VEA, D. Wagner, in prep.]

- Our approach is to fix a subset of $\rho_{\mathrm{S};r}^{\mu_1\cdots\mu_\ell}$ with:

$$0 \le \ell \le L = 2, \qquad -s_\ell \le r \le N_\ell, \tag{34}$$

where $s_\ell \equiv$ "shift" and $N_\ell \ge \{2, 1, 0\}$.
- This can be achieved using the Method of Moments for $\delta f_{\mathrm{S}\mathbf{k}} \equiv f_{\mathrm{S}\mathbf{k}} - f_{0\mathbf{k}} \equiv= f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \mathbb{S}_{\mathbf{k}}$, by setting:

$$\mathbb{S}_{\mathbf{k}} = \sum_{\ell=0}^{L} \sum_{n=-s_\ell}^{N_\ell} \rho_{\mathrm{S};n}^{\mu_1\cdots\mu_\ell} E_{\mathbf{k}}^{-s_\ell} k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle} \widetilde{\mathcal{H}}_{\mathbf{k},n+s_\ell}^{(\ell)}, \tag{35}$$

with $\widetilde{\mathcal{H}}_{\mathbf{k}n}^{(\ell)}$ to be determined.
- Inverting the logic, $\rho_{\mathrm{S};r}^{\mu_1\cdots\mu_\ell}$ are obtained from $\delta f_{\mathrm{S}\mathbf{k}}$ through

$$\rho_{\mathrm{S};r}^{\mu_1\cdots\mu_\ell} = \sum_{n=-s_\ell}^{N_\ell} \rho_{\mathrm{S};n}^{\mu_1\cdots\mu_\ell} \widetilde{\mathcal{F}}_{-(r+s_\ell),n+s_\ell}^{(\ell)},$$

$$\widetilde{\mathcal{F}}_{rn}^{(\ell)} \equiv \frac{\ell!}{(2\ell+1)!!} \int dK\, f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} E_{\mathbf{k}}^{-2s_\ell - r} (\Delta^{\alpha\beta} k_\alpha k_\beta)^\ell \widetilde{\mathcal{H}}_{\mathbf{k}n}^{(\ell)}. \tag{36}$$

- Imposing $\widetilde{F}_{-r,n}^{(\ell)} = \delta_{rn}$ for $-s_\ell \le r, n \le N_\ell$ ensures compatibility with Eq. (20) and fully determines $\widetilde{\mathcal{H}}_{\mathbf{k}n}^{(\ell)}$.

Shakhov collision matrix

- Eq. (36) $\Rightarrow \rho_{\mathrm{S};r}^{\mu_1\cdots\mu_\ell} \neq 0$ even when $r < -s_\ell$ and $r > N_\ell$.
- $\Rightarrow \mathcal{A}_{\mathrm{S};rn}^{(\ell)}$ contains non-trivial entries when $r < -s_\ell$ and $r > N_\ell$:

$$\mathcal{A}_{rn}^{(\ell)} = \begin{pmatrix} \frac{1}{\tau_R}\delta_{rn} & \mathcal{A}_{<;rn}^{(\ell)} & 0 \\ 0 & \mathcal{A}_{\mathrm{S};rn}^{(\ell)} & 0 \\ 0 & \mathcal{A}_{>;rn}^{(\ell)} & \frac{1}{\tau_R}\delta_{rn} \end{pmatrix}, \qquad (37)$$

where $\mathcal{A}_{</>;rn}^{(\ell)}$ correspond to $r < -s_\ell$ and $r > N_\ell$, respectively.

- These entries supplement the $\tau_R^{-1}\delta_{rn}$ structure of AW with

$$\mathcal{A}_{</>;rn}^{(\ell)} = -\frac{1}{\tau_R}\widetilde{\mathcal{F}}_{-(r+s_\ell),n+s_\ell}^{(\ell)} + \sum_{j=-s_\ell}^{N_\ell} \widetilde{\mathcal{F}}_{-(r+s_\ell),j+s_\ell}^{(\ell)} \mathcal{A}_{\mathrm{S};jn}^{(\ell)}. \qquad (38)$$

Inverse collision matrix

- The inverse matrix $\tau_{rn}^{(\ell)}$ reads

$$\tau_{rn}^{(\ell)} = \begin{pmatrix} \tau_R \delta_{rn} & \tau_{<;rn}^{(\ell)} & 0 \\ 0 & \tau_{\mathrm{S};rn}^{(\ell)} & 0 \\ 0 & \tau_{>;rn}^{(\ell)} & \tau_R \delta_{rn} \end{pmatrix} , \tag{39}$$

with $\tau_{\mathrm{S};rn}^{(\ell)} = [\mathcal{A}_{\mathrm{S};rn}^{(\ell)}]^{-1}$ a finite $(N_\ell + s_\ell + 1)^2$ matrix and

$$\tau_{<,>;rn}^{(\ell)} = -\tau_R \widetilde{\mathcal{F}}_{-(r+s_\ell),n+s_\ell}^{(\ell)} + \sum_{j=-s_\ell}^{N_\ell} \widetilde{\mathcal{F}}_{-(r+s_\ell),j+s_\ell}^{(\ell)} \tau_{\mathrm{S};jn}^{(\ell)} . \tag{40}$$

- For example, the shear viscosities $\eta_r = \sum_n \tau_{rn}^{(2)} \alpha_n^{(2)}$ are

$$\eta_{-s_\ell \le r \le N_\ell} = \sum_{n=-s_2}^{N_2} \tau_{\mathrm{S};rn}^{(2)} \alpha_n^{(2)},$$

$$\eta_{r,</>} = \tau_R \alpha_r^{(2)} + \sum_{n=-s_2}^{N_2} \widetilde{\mathcal{F}}_{-r-s_2,n+s_2}^{(2)} (\eta_n - \tau_R \alpha_n^{(2)}). \tag{41}$$

Tunable coefficients in the Shakhov model

- The t.c. depend on

$$\begin{aligned}
\tau^{(0)}_{0,n\neq 1,2} &: N_0 + s_0 - 1 \text{ entries}; & \mathcal{C}^{(0)}_{n\neq 1,2} &\equiv \frac{\zeta_n}{\zeta_0} : N_0 + s_0 - 2 \text{ extra lines},\\
\tau^{(1)}_{0,n\neq 1} &: N_1 + s_1 \text{ entries}; & \mathcal{C}^{(1)}_{n\neq 1} &\equiv \frac{\kappa_n}{\kappa_0} : N_1 + s_1 - 1 \text{ extra lines},\\
\tau^{(2)}_{0n} &: N_2 + s_2 + 1 \text{ entries}; & \mathcal{C}^{(2)}_{n} &\equiv \frac{\eta_n}{\eta_0} : N_2 + s_2 \text{ extra lines},
\end{aligned} \tag{42}$$

so in total:

$$2(N_0 + s_0 + N_1 + s_1 + N_2 + s_2) - 3 \,\text{transport coefficients}, \tag{43}$$

plus a hidden degree of freedom given by τ_R.

- For an ultrarelativistic gas, the scalar sector is not important, leaving in total

$$2(N_1 + s_1 + N_2 + s_2) \text{ transport coefficients}, \tag{44}$$

plus τ_R.

Example: shear-diffusion coupling

- Consider a longitudinal wave propagating along z.
- The linearized hydro equations for $\delta\pi \equiv \pi^{zz}$ and $\delta V \equiv V^z$ read

$$\begin{aligned} \tau_V \partial_t \delta V + \delta V &= -\kappa \partial_z \delta\alpha + \ell_{V\pi} \partial_z \delta\pi = & 0, \\ \tau_\pi \partial_t \delta\pi + \delta\pi &= -\frac{4\eta}{3} \partial_z \delta v - \frac{2}{3} \ell_{\pi V} \partial_z \delta V & = 0, \end{aligned} \tag{45}$$

where the cross couplings read (for an UR classical gas):

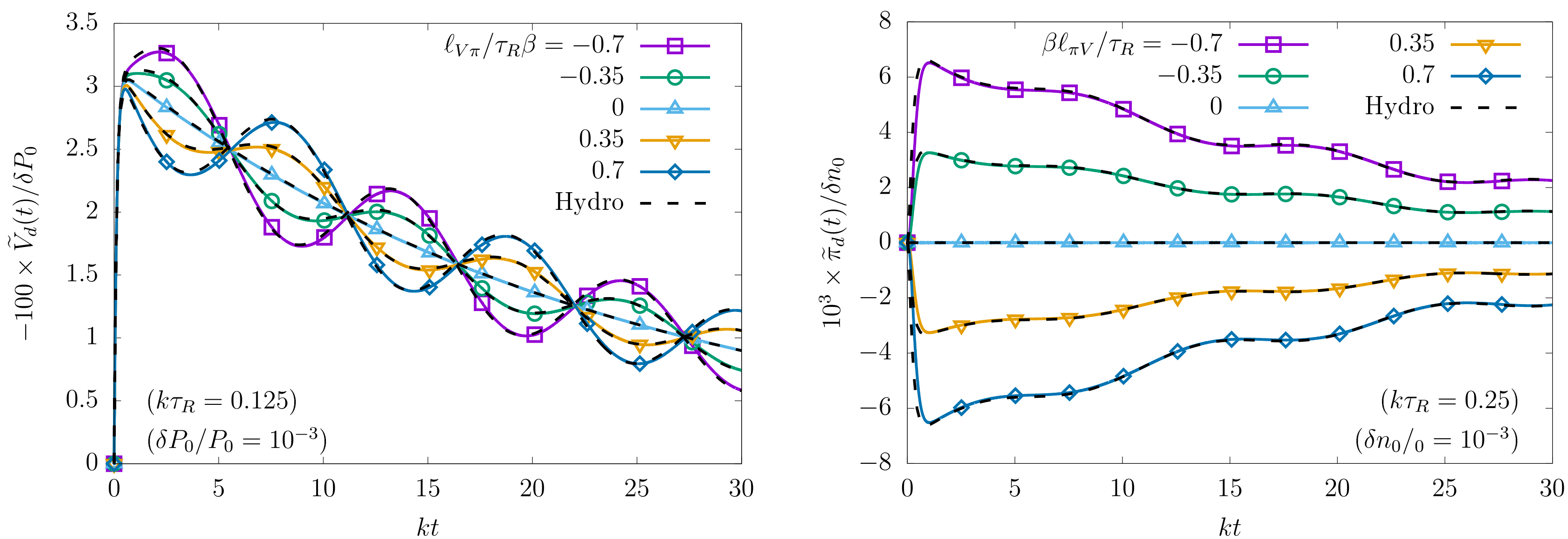
$$\ell_{V\pi} = \sum_{r\neq 1} \tau_{0r}^{(1)} \left(\frac{\beta J_{r+2,1}}{\epsilon + P} - \mathcal{C}_{r-1}^{(2)} \right), \quad \ell_{\pi V} = \frac{2}{5} \sum_r \tau_{0r}^{(2)} \mathcal{C}_{r+1}^{(1)}. \tag{46}$$

- In RTA, $\ell_{V\pi} = \tau_R \left(\frac{\beta J_{21}}{\epsilon+P} - \mathcal{C}_{-1}^{(2)} \right)$ and $\ell_{\pi V} = \tau_R \mathcal{C}_1^{(1)}$ both vanish:

$$\begin{aligned} J_{21} &= P = \frac{1}{3}\epsilon, & \mathcal{C}_{-1}^{(2)} &= \frac{\alpha_{-1}^{(2)}}{\alpha_0^{(2)}} = \frac{\beta}{4} & \Rightarrow \quad & \ell_{V\pi} = 0, \\ \kappa_1 &= \alpha_1^{(1)} = 0, & \mathcal{C}_1^{(1)} &= \frac{\alpha_1^{(1)}}{\alpha_0^{(1)}} = 0 & \Rightarrow \quad & \ell_{\pi V} = 0. \end{aligned} \tag{47}$$

- We aim to control independently 4 t.c.: κ, η, $\ell_{V\pi}$ and $\ell_{\pi V}$.

Example: shear-diffusion coupling



- We use $(N_1, N_2, s_1, s_2) = (1, 0, 0, 1)$ with $\mathcal{A}_\mathrm{S}^{(1)} = 1/\tau_R$ and

$$\mathcal{A}_\mathrm{S}^{(2)} = \frac{1}{\tau_\pi H(H + L_{V\pi}L_{\pi V})} \begin{pmatrix} H - L_{\pi V} & \frac{\beta}{4}(HL_{V\pi} + L_{\pi V}) \\ -\frac{4}{\beta}L_{\pi V} & H + L_{\pi V} \end{pmatrix}, \tag{48}$$

allowing $\ell_{V\pi}$ and $\ell_{\pi V}$ to be controlled independently via

$$L_{V\pi} = \frac{4}{\beta\tau_V}\ell_{V\pi}, \qquad L_{\pi V} = \frac{5\beta}{8\tau_\pi}\ell_{\pi V}, \qquad H = \frac{5\eta}{4\tau_\pi P}. \tag{49}$$

- $\lambda = 1$ fm; $T_0 = 1$ GeV, $\mu_0 = 0 \Rightarrow n_0 = 212.04\ \mathrm{fm}^{-3} \Rightarrow \sigma = 1.2676/\beta\eta \simeq 3.7$ mb.

Ultrarelativistic hard spheres (URHS)

- The t.c. of the URHS model are: [D. Wagner, A. Palermo, VEA, PRD **106** (2022) 016013]

[D. Wagner, VEA, E. Molnár, arXiv: 2309.09335]

$\kappa\sigma$	$\tau_V\,[\lambda_{\rm mfp}]$	$\delta_{VV}\,[\tau_V]$	$\ell_{V\pi}\,[\tau_V] = \tau_{V\pi}\,[\tau_V]$	$\lambda_{VV}\,[\tau_V]$	$\lambda_{V\pi}\,[\tau_V]$
0.15892	2.0838	1	0.028371β	0.89862	0.069273β

$\eta\sigma\beta$	$\tau_\pi\,[\lambda_{\rm mfp}]$	$\delta_{\pi\pi}\,[\tau_\pi]$	$\ell_{\pi V}\,[\tau_\pi]$	$\tau_{\pi V}\,[\tau_\pi]$	$\tau_{\pi\pi}\,[\tau_\pi]$	$\lambda_{\pi V}\,[\tau_\pi]$
1.2676	1.6557	4/3	$-0.56960/\beta$	$-2.2784/\beta$	1.6945	$0.20503/\beta$

- The t.c. of RTA with $\eta_R = \eta_{\rm HS}$ are

$\kappa\sigma$	$\tau_V\,[\lambda_{\rm mfp}]$	$\delta_{VV}\,[\tau_V]$	$\ell_{V\pi}\,[\tau_V] = \tau_{V\pi}\,[\tau_V]$	$\lambda_{VV}\,[\tau_V]$	$\lambda_{V\pi}\,[\tau_V]$
0.13204	1.5845	1	0	3/5	$\beta/16$

$\eta\sigma\beta$	$\tau_\pi\,[\lambda_{\rm mfp}]$	$\delta_{\pi\pi}\,[\tau_\pi]$	$\ell_{\pi V}\,[\tau_\pi]$	$\tau_{\pi V}\,[\tau_\pi]$	$\tau_{\pi\pi}\,[\tau_\pi]$	$\lambda_{\pi V}\,[\tau_\pi]$
1.2676	1.5845	4/3	0	0	10/7	0

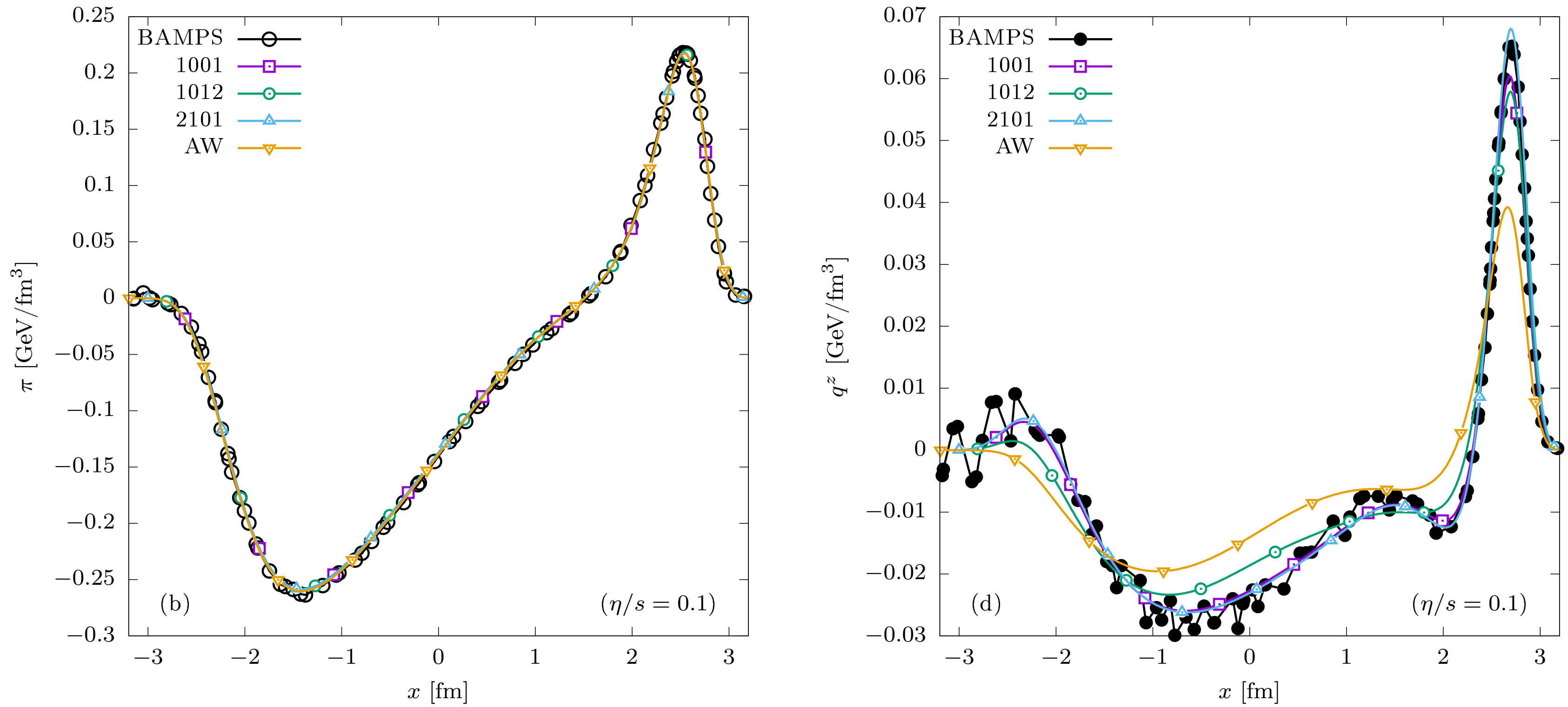
- RTA-HS mismatch for almost all coefficients, except $\delta_{VV} = \tau_V$ and $\delta_{\pi\pi} = 4\tau_\pi/3$, which are fixed for an UR gas.
- To align all transport coefficients, we need 11 parameters!

Various (N_1, N_2, s_1, s_2) models

- A Shakhov model with parameters (N_1, N_2, s_1, s_2) provides $2(N_1 + N_2 + s_1 + s_2)$.
- To test the effect of various t.c., we employed several models:
- AW: τ_R is used to fix $\eta_R = \eta_{\mathrm{HS}}$.
- (1001): discussed previously, fixes $(\kappa, \eta, \ell_{V\pi}, \ell_{\pi V})$
- (1012): has $2 \times 4 = 8$ free entries and fixes everything except λ_{VV} and $\lambda_{V\pi}$.
- (2101): has $2 \times 4 = 8$ free entries and fixes everything except λ_{VV} and $\lambda_{V\pi}$.

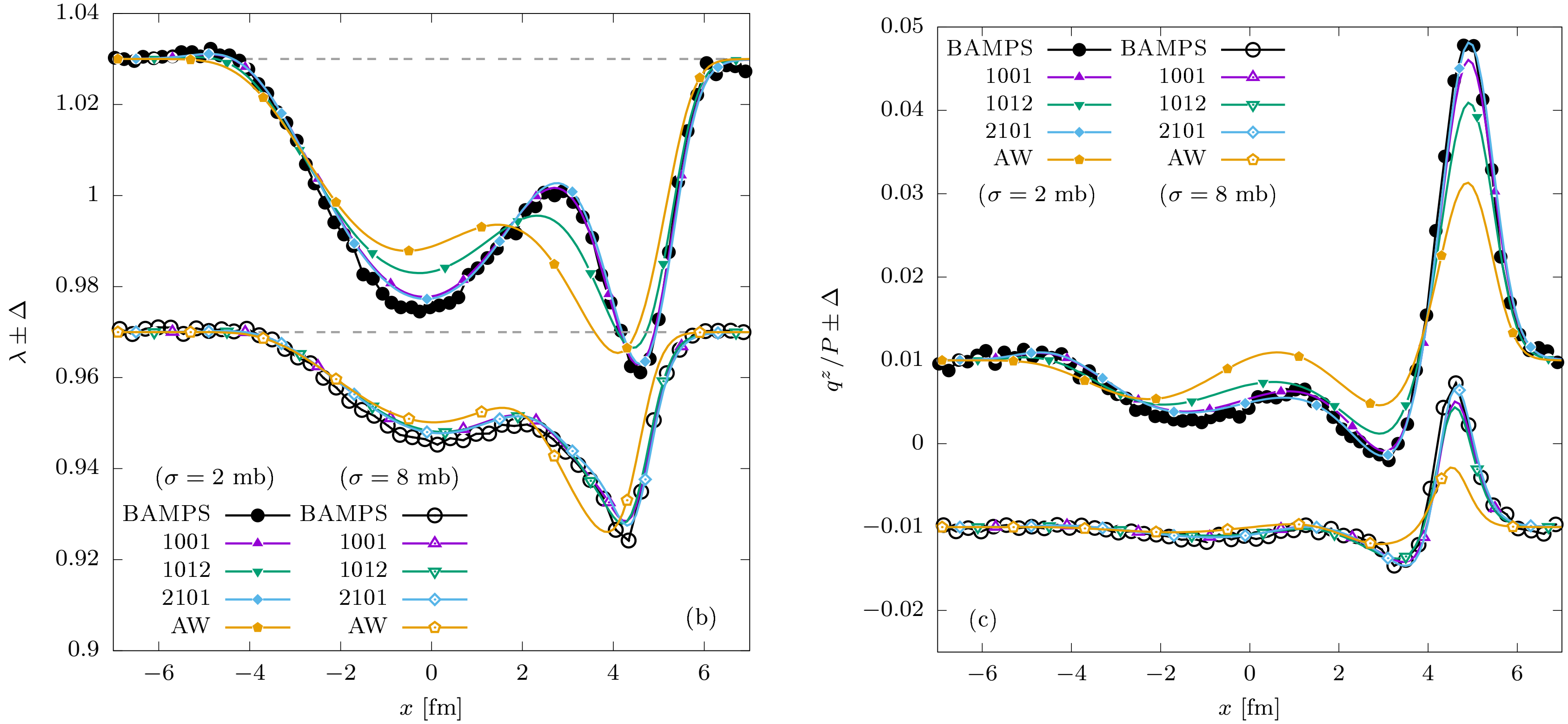
Sod shock tube: Comparison to BAMPS

[Bouras *et al*, PRC **82** (2010) 024910]



- In the frame of the Sod shock tube, we considered a comparison to BAMPS for hard-sphere interactions.
- Using τ_R to tune η, shear comes out well with AW and Shakhov.
- For diffusion: $1001 \equiv$ first-order Shakhov underestimates peak.
- Higher-order (2101) Shakhov required to tune $2^{\rm nd}$ order t. coeffs.

Sod shock tube: Comparison to BAMPS [DNBMXRG, PRD **89** (2014) 074005]



- In the heat-flow problem (const. initial λ, pressure jump), again higher-order 2101 Shahkov required.

Conclusions

- Shakhov model generalized for the relativistic Anderson-Witting RTA, allowing ζ, κ and η to be controlled independently.
- Numerical simulations of the Bjorken flow and of sound waves damping confirmed that the model is robust.
- Extending the Shakhov model allows $2^{\rm nd}$-order t. coeffs. to be controlled $\Rightarrow$ agreement with BAMPS in Sod shock tube.
- This work was supported through a grant of the Ministry of Research, Innovation and Digitization, CNCS - UEFISCDI, project number PN-III-P1-1.1-TE-2021-1707, within PNCDI III.

Appendix

First-order model

- Specifically, $\mathcal{H}^{(\ell)}_{\mathbf{k}0}$ must satisfy:

$$\int dK\, f_{0\mathbf{k}} \begin{pmatrix} 1 \\ E_{\mathbf{k}} \\ E_{\mathbf{k}}^2 \end{pmatrix} \mathcal{H}^{(0)}_{\mathbf{k}0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$
$$\frac{1}{3}\int dK\, f_{0\mathbf{k}} \begin{pmatrix} 1 \\ E_{\mathbf{k}} \end{pmatrix} (\Delta^{\alpha\beta} k_\alpha k_\beta) \mathcal{H}^{(1)}_{\mathbf{k}0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$\frac{2}{15}\int dK\, f_{0\mathbf{k}} (\Delta^{\alpha\beta} k_\alpha k_\beta)^2 \mathcal{H}^{(2)}_{\mathbf{k}0} = 1. \tag{50}$$

- The lowest-order polynomials satisfying these relations are

$$\mathcal{H}^{(0)}_{\mathbf{k}0} = \frac{G_{33} - G_{23} E_{\mathbf{k}} + G_{22} E_{\mathbf{k}}^2}{J_{00} G_{33} - J_{10} G_{23} + J_{20} G_{22}},$$
$$\mathcal{H}^{(1)}_{\mathbf{k}0} = \frac{J_{31} E_{\mathbf{k}} - J_{41}}{J_{21} J_{41} - J_{31}^2}, \quad \mathcal{H}^{(2)}_{\mathbf{k}0} = \frac{1}{2 J_{42}}, \tag{51}$$

where $G_{nm} = J_{n0} J_{m0} - J_{n-1,0} J_{m+1,0}$, while

$$J_{nq} = \frac{(-1)^q}{(2q+1)!!} \int dK\, E_{\mathbf{k}}^{n-2q} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^q f_{0\mathbf{k}}. \tag{52}$$

Sound waves: linear modes

- Inserting $A(t,x) = A_0 + \int_{-\infty}^{\infty} dk \sum_\omega e^{-i(\omega t - kz)} \delta A_\omega(k)$ gives

$$\begin{pmatrix} -3\frac{\omega}{k} & 4P_0 & 0 & 0 & 0 \\ 1 & -\frac{4\omega}{k}P_0 & 1 & 0 & 0 \\ 0 & \frac{4\eta}{3} & -\frac{i}{k} - \frac{\omega}{k}\tau_\pi & 0 & \ell_{\pi V} \\ 0 & n_0 & 0 & -\frac{\omega}{k} & 1 \\ -\frac{3\kappa}{P_0} & 0 & -\ell_{V\pi} & \frac{4\kappa}{n_0} & -\frac{i}{k} - \frac{\omega}{k}\tau_V \end{pmatrix} \begin{pmatrix} \delta P_\omega(k) \\ \delta v_\omega(k) \\ \delta \pi_\omega(k) \\ \delta n_\omega(k) \\ \delta V_\omega(k) \end{pmatrix} = 0.$$

- Thanks to $\ell_{V\pi} = \ell_{\pi V} = 0$, the shear and diffusion sectors decouple:

$$(k^2 - 3\omega^2)(1 - i\omega\tau_\pi) - \frac{ik^2\omega}{P_0}\eta = 0, \quad \omega(1 - i\omega\tau_V) + \frac{4ik^2}{n_0}\kappa = 0.$$

- The shear and diffusion modes are:

$$\begin{aligned} \omega_a^\pm &= \pm|k|c_{s;a} - i\xi_a, \qquad \omega_\eta = -i\xi_\eta; \qquad \omega_\kappa^\pm = -i\xi_\kappa^\pm, \\ c_{s;a} &\simeq \frac{1}{\sqrt{3}}, \quad \xi_a \simeq \frac{k^2\eta}{6P_0}, \quad \xi_\eta \simeq \frac{1}{\tau_\pi} - \frac{k^2\eta}{3P_0}, \\ \xi_\kappa^- &\simeq \frac{4k^2\kappa}{n_0}, \qquad \xi_\kappa^+ \simeq \frac{1}{\tau_V} - \frac{4k^2\kappa}{n_0}. \end{aligned} \tag{53}$$

Solution

- At initial time, $n(0,z) = n_0 + \delta n_0 \cos(kz)$ and $v(0,z) = \delta v_0 \sin(kz)$.
- The approximate solution is [Ambruș, PRC **97** (2018) 024914.]

$$\widetilde{\delta V} \simeq \frac{4k\kappa\delta n_0}{\tau_V n_0} \frac{e^{-\xi_\kappa^+ t} - e^{-\xi_\kappa^- t}}{\xi_\kappa^+ - \xi_\kappa^-},$$
$$\widetilde{\delta \pi} \simeq -\frac{4\eta}{3}\delta v_0 \left\{ e^{-\xi_a t}\left[\cos(kc_s t) - \frac{\xi_a}{kc_s}\sin(kc_s t)\right] - e^{-t/\tau_\pi}\right\}. \quad (54)$$

Arbitrary Shakhov matrix

- The model can be extended to control $2^{\rm nd}$-order transport coeffs..
- Systematic extensions can be obtained by writing in general

$$\mathbb{S}_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \sum_{n=-s_\ell}^{N_\ell} \rho_{\mathrm{S};n}^{\mu_1\cdots\mu_\ell} E_{\mathbf{k}}^{-s_\ell} k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle} \widetilde{\mathcal{H}}^{(\ell)}_{\mathbf{k},n+s_\ell}, \qquad (55)$$

where $N_\ell \equiv$ expansion order and $s_\ell \equiv$ basis-shift allowing to access negative-order moments.
- The Shakhov irreducible moments are taken as

$$\rho_{\mathrm{S};r}^{\mu_1\cdots\mu_\ell} = \sum_{n=-s_\ell}^{N_\ell} \left(\delta_{rn} - \tau_R \mathcal{A}^{(\ell)}_{\mathrm{S};rn}\right) \rho_n^{\mu_1\cdots\mu_\ell}. \qquad (56)$$

with arbitrary entries $\mathcal{A}^{(\ell)}_{\mathrm{S};rn}$ defined for $-s_\ell \le r, n \le N_\ell$.
- The irreducible moments $C_{\mathrm{S};r-1}^{\mu_1\cdots\mu_\ell}$ of the collision term can be written as

$$C_{\mathrm{S};r-1}^{\mu_1\cdots\mu_\ell} = -\sum_n \mathcal{A}^{(\ell)}_{rn} \rho_n^{\mu_1\cdots\mu_\ell}, \quad \mathcal{A}^{(\ell)}_{rn} = \begin{pmatrix} \frac{1}{\tau_R}\delta_{rn} & \mathcal{A}^{(\ell)}_{<;rn} & 0 \\ 0 & \mathcal{A}^{(\ell)}_{\mathrm{S};rn} & 0 \\ 0 & \mathcal{A}^{(\ell)}_{>;rn} & \frac{1}{\tau_R}\delta_{rn} \end{pmatrix}. \qquad (57)$$