Outline

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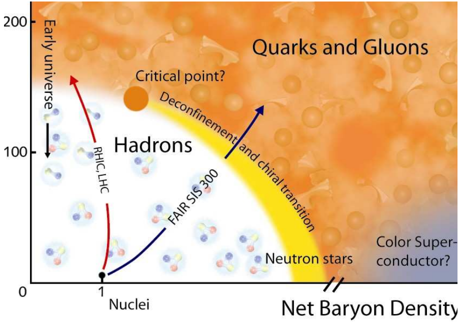
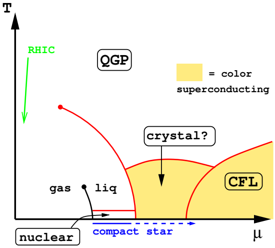
1. Introduction ✔
2. Chiral phase transition and critical endpoint ✔
3. Color superconductivity
4. Inhomogeneous chiral phases



COLOR SUPERCONDUCTIVITY



[Alford (2003)]

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- Noninteracting fermions at $T = 0$:
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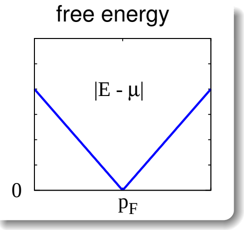

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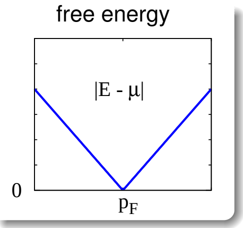

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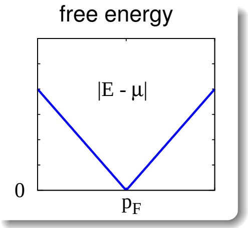

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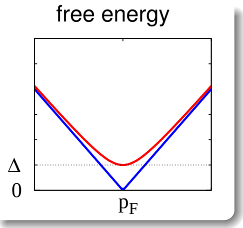

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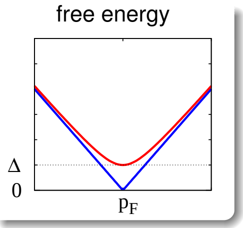

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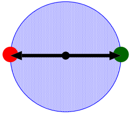
- BCS pairing:
  - pairs with vanishing total momentum: $\vec{p}^{(1)} = -\vec{p}^{(2)}$
  - each partner close to the Fermi surface

    → works only if $p_F^{(1)} \approx p_F^{(2)}$



Diquark condensates

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- QCD: attractive quark-quark interaction

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  - color $\bar{3}$ (= antisymmetric)

  → antisymmetric in flavor

  → pairing between different flavors

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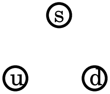
  $\rightarrow$ pairing between different flavors
- example: $(\uparrow\downarrow - \downarrow\uparrow) \otimes (r\,g - g\,r) \otimes (ud - du)$

Three-flavor systems

- Pairing patterns in flavor space:

  no pairing: "normal quark matter" (NQ)



Three-flavor systems

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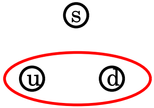
- Pairing patterns in flavor space:

  two-flavor superconducting (2SC) phase

  (+ two analogous phases with *us* or *ds* pairing)



Three-flavor systems

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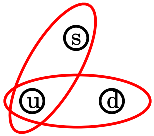
- Pairing patterns in flavor space:

  uSC phase

  (similar: dSC phase, sSC)



Three-flavor systems

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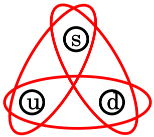
- Pairing patterns in flavor space:

  color-flavor locked (CFL) phase



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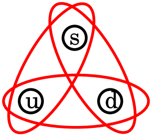
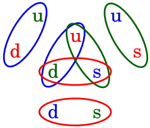


- CFL pairing (more explicitly):

$$(\uparrow\downarrow - \downarrow\uparrow) \otimes \Big((ud - du) \otimes (r\,g - g\,r) + (ds - sd) \otimes (g\,b - b\,g) + (su - us) \otimes (b\,r - r\,b) \Big)$$



(More) formal definition of the phases

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- Diquark condensates:

$$(\uparrow\downarrow - \downarrow\uparrow) \otimes (ud - du) \otimes (r\,g - g\,r) \leftrightarrow \langle q^T\, C\gamma_5\, \tau_2\, \lambda_2\, q \rangle \sim: \Delta_2$$

$$(\uparrow\downarrow - \downarrow\uparrow) \otimes (ds - sd) \otimes (g\,b - b\,g) \leftrightarrow \langle q^T\, C\gamma_5\, \tau_5\, \lambda_5\, q \rangle \sim: \Delta_5$$

$$(\uparrow\downarrow - \downarrow\uparrow) \otimes (su - us) \otimes (b\,r - r\,b) \leftrightarrow \langle q^T\, C\gamma_5\, \tau_7\, \lambda_7\, q \rangle \sim: \Delta_7$$

$C = i\gamma^2\gamma^0$ charge conjugation matrix, $C\gamma_5 \rightarrow J^P = 0^+$
τ_A: antisymmetric Gell-Mann matrices in flavor space
λ_A: antisymmetric Gell-Mann matrices in color space

- Phases:
  - NQ: $\Delta_2 = \Delta_5 = \Delta_7 = 0$
  - 2SC: $\Delta_2 \neq 0$, $\Delta_5 = \Delta_7 = 0$
  - CFL: $\Delta_2 = \Delta_5 = \Delta_7 \neq 0$ (ideal case; realistic: $\Delta_2 \approx \Delta_5 \approx \Delta_7 \neq 0$)
  - ...

Symmetries of the 2SC phase

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$$\Delta_2 \sim \langle q^T \, C\gamma_5 \, \tau_2 \, \lambda_2 \, q \rangle$$

- gauge symmetries:
  - **color:** $q \rightarrow e^{i\theta_a \frac{\lambda^a}{2}} q$ blue quarks unpaired $\Rightarrow$ $SU(3)_c \rightarrow SU(2)_c$

    → 5 of the 8 gluons get a nonzero Meissner mass.
  - **electromagnetism:** $q \rightarrow e^{i\alpha Q} q$, $Q = \text{diag}_f(\frac{2}{3}, -\frac{1}{3})$ broken

    But there is an unbroken $U(1)$ gauge symmetry with charge $\tilde{Q} = Q - \frac{1}{2\sqrt{3}}\lambda_8$.

→ color superconductor but not electromagnetic superconductor

- global symmetries:
  - **baryon number:** $q \rightarrow e^{i\alpha} q$ $\Rightarrow$ $\Delta_2 \rightarrow e^{2i\alpha} \Delta_2$ broken

    But there is an unbroken "modified baryon number" $q \rightarrow e^{i\alpha(1-\sqrt{3}\lambda_8)} q$
  - $SU(2)_L \times SU(2)_R$ **chiral symmetry:** conserved

→ same global symmetries as 2-flavor restored phase, no Goldstone bosons

Symmetries of the (ideal) CFL phase

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$$\langle q^T C\gamma_5 \tau_2 \lambda_2 q\rangle = \langle q^T C\gamma_5 \tau_2 \lambda_2 q\rangle = \langle q^T C\gamma_5 \tau_2 \lambda_2 q\rangle = \Delta$$

- **color:** $SU(3)_c$ broken completely
- **chiral symmetry:** $SU(3)_L \times SU(3)_R$ broken completely

  but:

  residual $SU(3)$ under **combined color-flavor** rotations: $q \rightarrow e^{i\theta_a(\tau_a - \lambda_a^T)} q$

  ➞ "color-flavor locking": $SU(3)_c \times SU(3)_L \times SU(3)_R \rightarrow SU(3)_{V+c}$

  ➞ 8 massive gluons + 8 pseudoscalar Goldstone bosons (chiral limit)
- **baryon number:** $U(1)$ broken ➞ 1 scalar Goldstone boson
- **electromagnetism:**

  unbroken $U(1)$ gauge symmetry with charge $\tilde{Q} = Q - \frac{1}{2}\lambda_3 - \frac{1}{2\sqrt{3}}\lambda_8$

➞ color but not electromagnetic superconductor, baryon number superfluid

Which phase is favored?
- Realistic systems

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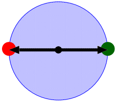

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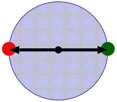

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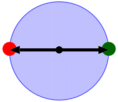

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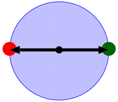

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- Expected phase structure:
  - $\mu \gg M_s \Rightarrow p_F^{(s)} \approx p_F^{(u,d)}$ → CFL
  - $\mu \lesssim M_s \Rightarrow p_F^{(s)} \ll p_F^{(u,d)}$ → 2SC

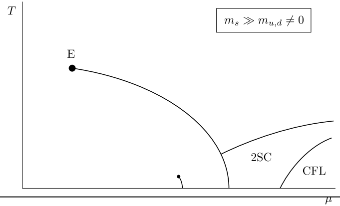

Figure: educated guess [Rajagopal (1999)]

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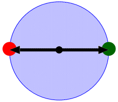

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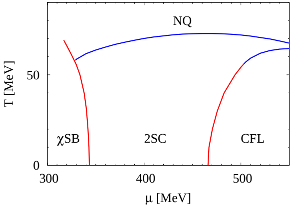

Figure: NJL [M. Oertel, MB (2002); MB (2005)]

NJL-model treatment of color superconductivity

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- NJL-type Lagrangian: $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\bar{q}q} + \mathscr{L}_{qq}$
  - free part:

    $\mathscr{L}_0 = \bar{q}(i\partial\!\!\!/ - \hat{m})q, \quad \hat{m} = \mathrm{diag}_f(m_u, m_d, m_s), \quad m_u = m_d$
  - quark-antiquark interaction:

    $\mathscr{L}_{\bar{q}q} = G\left[(\bar{q}\tau^a q)^2 + (\bar{q}i\gamma_5\tau^a q)^2\right] - K\left[\mathrm{det}_f\left(\bar{q}(1+\gamma_5)q\right) + \mathrm{det}_f\left(\bar{q}(1-\gamma_5)q\right)\right]$
  - quark-quark interaction:

    $\mathscr{L}_{qq} = H\,(\bar{q}\,i\gamma_5\tau_A\lambda_{A'}\,C\bar{q}^T)(q^T C\,i\gamma_5\tau_A\lambda_{A'}\,q) + \text{(pseudoscalar)}$

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- considered condensates:
  - $\phi_u = \langle\bar{u}u\rangle, \quad \phi_d = \langle\bar{d}d\rangle, \quad \phi_a = \langle\bar{s}s\rangle$

    → constituent masses: $M_u = m_u - 4G\phi_u + 2K\phi_d\phi_s, \quad M_d = \dots, \quad M_s = \dots$
  - gap parameters: $\Delta_A = -2H\langle q^T C\,\gamma_5\tau_A\lambda_A\,q\rangle$

Nambu-Gor’kov formalism

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- Mean-field approximation:

$$\mathscr{L}_{\text{MF}} = \bar{q}(i\partial\!\!\!/ - \hat{M})q + \frac{1}{2}\Delta_A(\bar{q}\,\gamma_5\tau_A\lambda_A\,C\bar{q}^T) + \frac{1}{2}\Delta_A^*(q^T C\,\gamma_5\tau_A\lambda_A\,q)$$
$$- 2G(\phi_u^2 + \phi_d^2 + \phi_s^2) + 4K\phi_u\phi_d\phi_s - \frac{1}{4H}\sum_A |\Delta_A|^2$$

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- Artificially double number of degrees of freedom:

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} q \\ C\bar{q}^T \end{pmatrix}$$ "Nambu-Gor'kov spinors"

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$\Rightarrow$ $\mathscr{L}_{\text{MF}} + \mu\gamma^0 = \bar{\Psi}S^{-1}\Psi - \mathcal{V}$

- inverse NG propagator: $S^{-1} = \begin{pmatrix} i\partial\!\!\!/ - \hat{M} + \mu\gamma^0 & \sum_A \Delta_A\gamma_5\tau_A\lambda_A \\ -\sum_A \Delta_A^*\gamma_5\tau_A\lambda_A & i\partial\!\!\!/ - \hat{M} - \mu\gamma^0 \end{pmatrix}$
- "potential": $\mathcal{V} = 2G(\phi_u^2 + \phi_d^2 + \phi_s^2) - 4K\phi_u\phi_d\phi_s + \frac{1}{4H}\sum_A|\Delta_A|^2$

Thermodynamic potential

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- Mean-field Lagrangian: $\mathscr{L}_{MF} + \mu\gamma^0 = \bar{\Psi} S^{-1} \Psi - \mathcal{V}$

$$\Rightarrow \; \Omega(T, \mu) = -\frac{1}{2} \frac{T}{V} \, \mathbf{Tr} \, \ln \frac{S^{-1}}{T} + \mathcal{V}$$

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- Result after turning out the Matsubara sum:

$$\Omega(T,\mu) = -\frac{1}{2}\sum_\lambda\int\frac{d^3k}{(2\pi)^3}\left\{\frac{|E_\lambda|}{2} + T\ln\left(1 + e^{-|E_\lambda|/T}\right)\right\} + \mathcal{V}$$

  - effective Dirac Hamiltonian: $S^{-1} = \gamma^0(i\omega_n - H)$
  - dispersion relations: $E_\lambda(\vec{k})$ = eigenvalues of H

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$$\Rightarrow \Omega(T,\mu) = -\frac{1}{2}\frac{T}{V}\,\mathbf{Tr}\,\ln\frac{S^{-1}}{T} + \mathcal{V} = -\frac{1}{2}T\sum_n \int \frac{d^3k}{(2\pi)^3}\ln\det\left(\frac{1}{T}S^{-1}(i\omega_n,\vec{k})\right) + \mathcal{V}$$

- Result after turning out the Matsubara sum:

$$\Omega(T,\mu) = -\frac{1}{2}\sum_\lambda \int \frac{d^3k}{(2\pi)^3}\left\{\frac{|E_\lambda|}{2} + T\ln\left(1 + e^{-|E_\lambda|/T}\right)\right\} + \mathcal{V}$$

  - effective Dirac Hamiltonian: $S^{-1} = \gamma^0(i\omega_n - H)$
  - dispersion relations: $E_\lambda(\vec{k})$ = eigenvalues of H

- Dimension: 2 (NG) $\times$ 4 (Dirac) $\times\ N_f\ \times\ N_c$
  - $N_f = 2$: 48 eigenvalues
  - $N_f = 3$: 72 eigenvalues

(always in pairs $(E_\lambda, -E_\lambda)$)

Example 1: 2SC phase in the chiral limit

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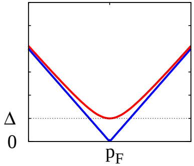
$4 \times 2 \times 3 = 24$ positive eigenvalues

- paired (red and green) quarks:
  - $\omega_-(\vec{p}) = \sqrt{(|\vec{p}| - \mu)^2 + |\Delta|^2}$ (8 fold)
  - $\omega_+(\vec{p}) = \sqrt{(|\vec{p}| + \mu)^2 + |\Delta|^2}$ (8 fold)
- unpaired (blue) quarks:
  - $\epsilon_-(\vec{p}) = \big|\, |\vec{p}| - \mu \,\big|$ (4 fold)
  - $\epsilon_+(\vec{p}) = \big|\, |\vec{p}| + \mu \,\big|$ (4 fold)



Example 2: ideal CFL phase in the chiral limit

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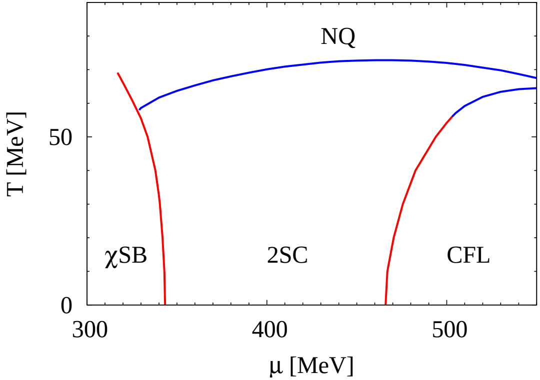
$4 \times 3 \times 3 = 36$ positive eigenvalues

- octet:
  - $\omega_{8,-}(\vec{p}) = \sqrt{(|\vec{p}| - \mu)^2 + |\Delta|^2}$ (16 fold)
  - $\omega_{8,+}(\vec{p}) = \sqrt{(|\vec{p}| + \mu)^2 + |\Delta|^2}$ (16 fold)
- singlet:
  - $\omega_{1,-}(\vec{p}) = \sqrt{(|\vec{p}| - \mu)^2 + |2\Delta|^2}$ (2 fold)
  - $\omega_{1,+}(\vec{p}) = \sqrt{(|\vec{p}| + \mu)^2 + |2\Delta|^2}$ (2 fold)

Phase diagram with realistic quark masses

[M. Oertel, MB (2002)]



- Most eigenvalues have to be found numerically.
- Cutoff artifacts at high μ (not only, see Hosein's talk)

Role of the strange quark mass

- NJL model: treatment of (dynamical) masses and gaps on an equal footing
  → T and μ dependent quantities

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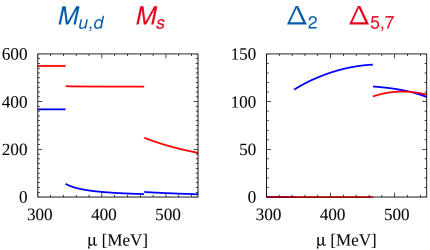
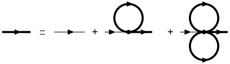


- Masses:



$$M_s = m_s - 4G\langle\bar{s}s\rangle + 2K\langle\bar{u}u\rangle\langle\bar{d}d\rangle$$

  - → M_s large in the 2SC phase
  - → stabilizes the 2SC phase

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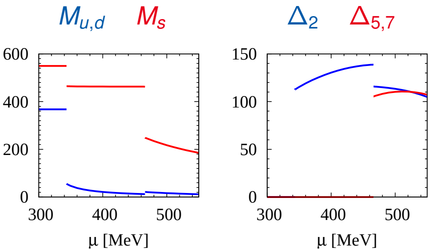

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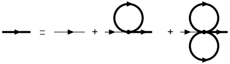

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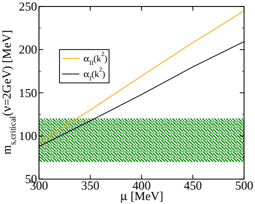
- Dyson-Schwinger QCD studies

[Nickel, Alkofer, Wambach (2006)]



  - → gluons screened by light quarks
  - → M_s smaller in the 2SC phase
  - → CFL phase favored much earlier

Compact star conditions

- constraints in compact stars:
  - color neutrality: $n_r = n_g = n_b$
  - electric neutrality: $n_Q = \frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s - n_e = 0$
  - β equilibrium: $\mu_e = \mu_d - \mu_u \quad \Rightarrow \quad n_e \ll n_{u,d}$

Compact star conditions

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$$\left.\begin{array}{l}\text{electric neutrality}\\ \beta\text{ equilibrium}\end{array}\right\} \quad \frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s \approx 0$$

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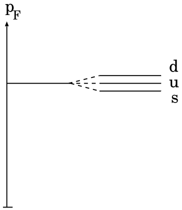
  → equidistant splitting

  → no 2SC phase in compact stars



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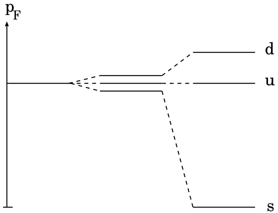

- Large M_s
  - → $n_s \approx 0, \; n_d \approx 2n_u \;\; \Rightarrow \;\; p_F^{(d)} \approx 2^{1/3} p_F^{(u)} \approx 1.26\, p_F^{(u)}$
  - → 2SC pairing possible for strong couplings

Neutral matter: further aspects

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1. role of electrons in unpaired quark matter

- two massles flavors:

  neutral matter: $\frac{2}{3} n_u - \frac{1}{3} n_d - n_e = 0$

  densities at $T = 0$: $n_u = \frac{\mu_u^3}{\pi^2}, \quad n_d = \frac{\mu_d^3}{\pi^2}, \quad n_e = \frac{\mu_e^3}{3\pi^2} = \frac{(\mu_d - \mu_u)^3}{3\pi^2}$

  → expectation: n_e very small

  $\Rightarrow \; n_d \approx 2 n_u \;\Rightarrow\; \mu_d \approx 2^{1/3} \mu_u \;\Rightarrow\; n_e \approx \frac{1}{3}(2^{1/3} - 1)^3 n_u \approx 0.006\, n_u$

  → expectation confirmed ✓

- including strange quarks: n_e even lower

⇒ electrons in electrically neutral normal quark matter negligible

2. CSC phases

- General property of (color-) super conducting matter at $T = 0$:
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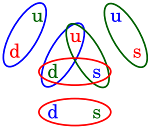

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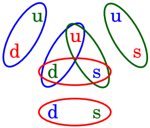

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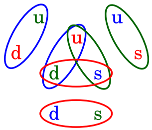

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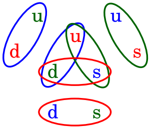

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  $\Rightarrow$ 2SC phase strongly affected by neutrality condistions

NJL model results

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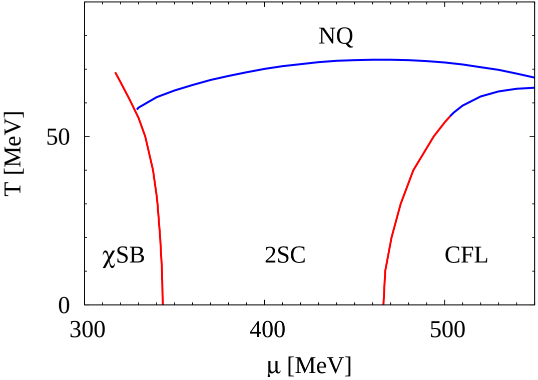
Phase diagram without neutrality constraints

[M. Oertel, MB (2002)]



NJL model results

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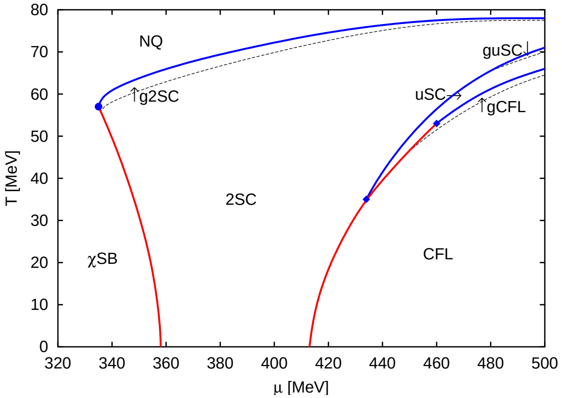
Phase diagram with neutrality constraints: "strong" qq coupling ($H = G$)

[Rüster, Werth, MB, Shovkovy, Rischke, (2005)]



NJL model results

Phase diagram with neutrality constraints: "intermediate" *qq* coupling ($H = 0.75\,G$)

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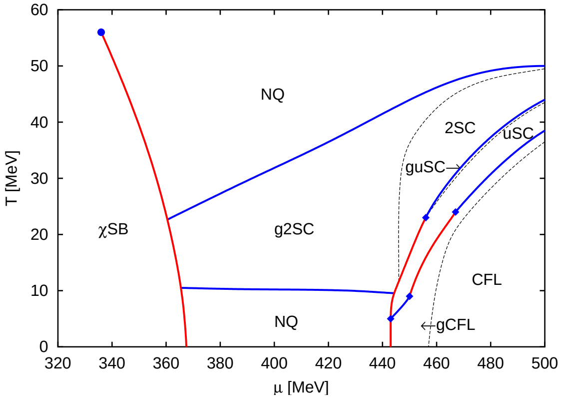

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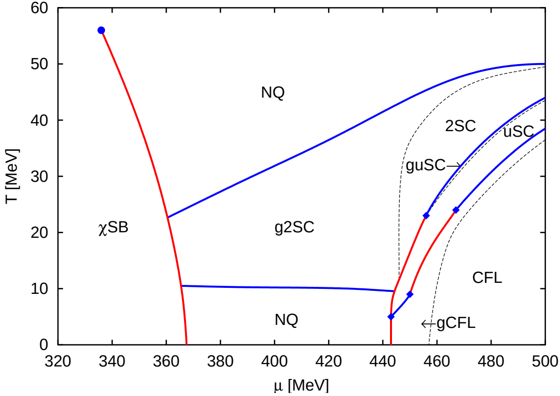

→ strong parameter dependence

Gapless color superconductors

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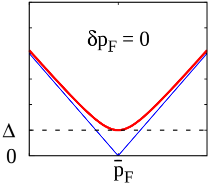
- unequal Fermi momenta: $p_F^{a,b} = \bar{p}_F \pm \delta p_F$

free energy



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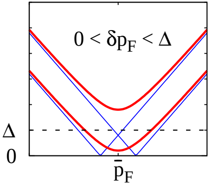

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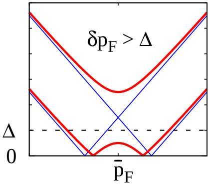

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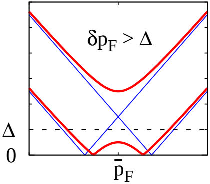

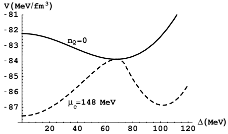

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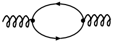
- Meissner effect:



$$m^2_{M,a} = -\frac{1}{2} \lim_{\vec{p}\to 0} \left(g_{ij} + \frac{p_i p_j}{p^2}\right) \Pi^{ij}_{aa}(0, \vec{p})$$

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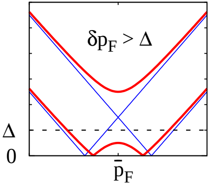

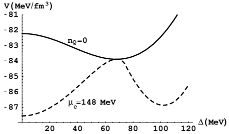

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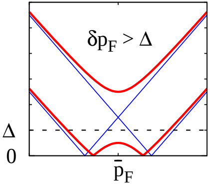

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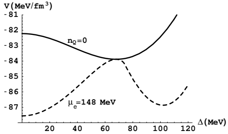

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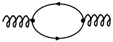

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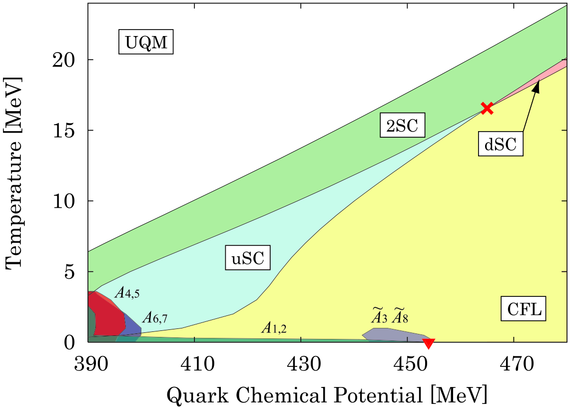
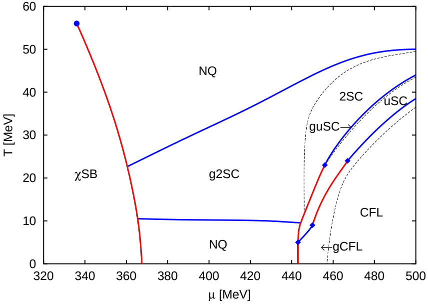
- chromomagnetic instability: $m_{M,a}^2 < 0$ for $\delta p_F > \begin{cases} \frac{\Delta}{\sqrt{2}} & a = 4, \dots, 7 \\ \Delta & a = 8 \end{cases}$

Chromomagnetic instabilities

- Phase diagram with instability regions
  [Fukushima (2005)]



Main issues



- strong parameter dependence
- unstable phases

How to reduce of the parameter dependence?

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1. Theoretical approaches: starting from QCD

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- Dyson-Schwinger equations:

  [Nickel, Alkofer, Wambach (2006, 2008),
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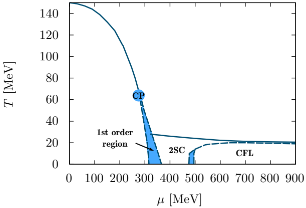
  - no cutoff artifacts at large μ
  - determination of the pressure difficult
  - still strong dependence on truncations and renormalization conditions



[Müller et al. (2013)]

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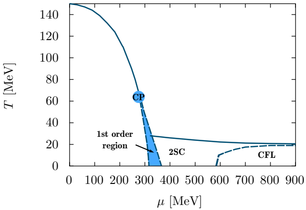

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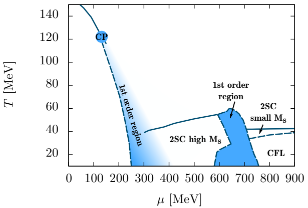

[Müller et al. (2016)]

How to reduce of the parameter dependence?

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1. Theoretical approaches: starting from QCD

- Dyson-Schwinger equations:

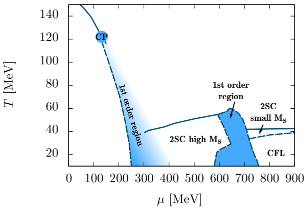
  [Nickel, Alkofer, Wambach (2006, 2008), Müller, MB, Wambach (2013, 2016)]

  - phase diagram without neutrality constraints
  - no cutoff artifacts at large μ
  - determination of the pressure difficult
  - still strong dependence on truncations and renormalization conditions

- Functional renormalization group:

  [Braun, Schallmo (2022)]

  - study 2SC pairing at $T = 0$ by solving QCD flow equations at large μ $\rightarrow$ very large gaps!



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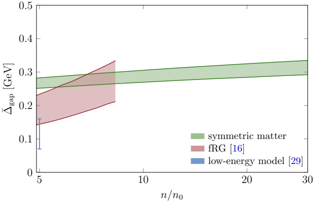

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2. Using empirical information

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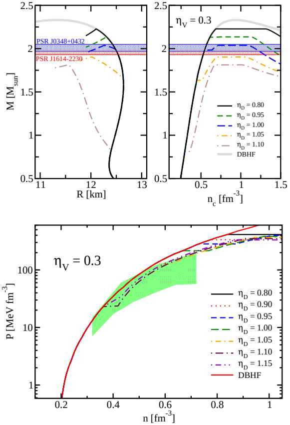
- Fitting NJL parameters to astrophysical constraints and heavy-ion data:
  [Klähn, Blaschke, ... (2006, 2007, 2013, ...)]
  - purely hadronic matter inconsistent
    (see also [Annala et al. (2020)])
  - vector repulsion to be stiff enough
  - strong *qq* interaction



[Klähn, Łastowiecki, Blaschke (2013)]

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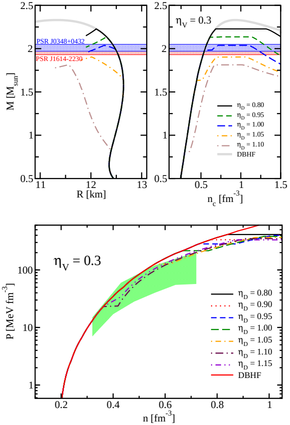
- Signals of CSC in the gravitational-wave spectrum from neutron-star mergers?



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Trapped neutrinos

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- Proto-neutron stars: neutrinos trapped during the first few seconds

  → lepton number conserved

  → more electrons:

  $$\mu_e = \mu_d - \mu_u + \mu_\nu$$

  → favors 2SC, suppresses CFL

  [Steiner, Reddy, Prakash, PRD (2002)]

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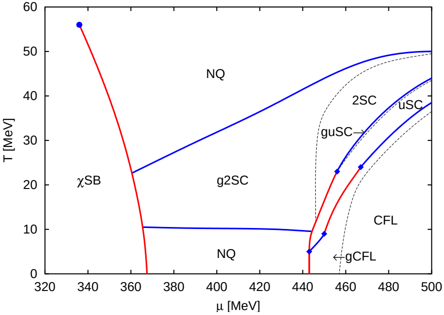

[Rüster, Werth, M.B., Shovkovy, Rischke, PRD (2006)]

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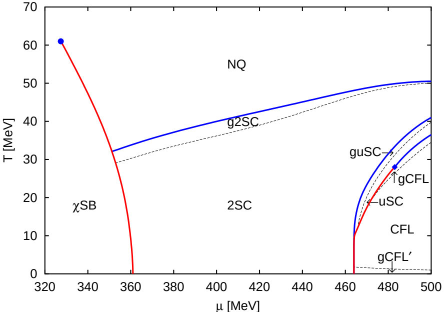
μ_ν = 200 MeV



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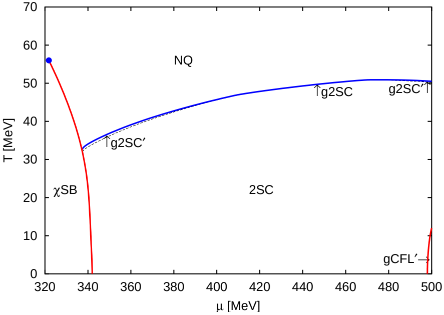

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- also relevant for neutron-star mergers!

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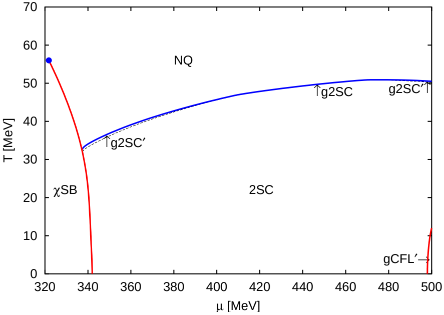

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Main issues

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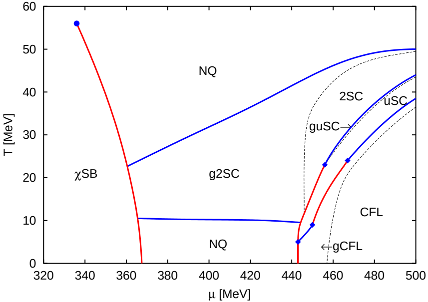

- strong parameter dependence
- unstable phases

Kaon condensation in the CFL phase

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- CFL: chiral symmetry broken → Goldstone bosons $\sim \mathcal{O}(10\text{ MeV})$

  [Son, Stephanov, PRD (2000)]

- $\mu_s^{eff} \simeq \frac{m_s^2 - m_u^2}{2\mu}$ → K^0 condensation [T. Schäfer, PRL (2000); Bedaque, Schäfer, NPA (2002)]
- NJL model: include pseudoscalar diquark conds. [M.B., PLB (2005); M.M. Forbes, PRD (2005)]

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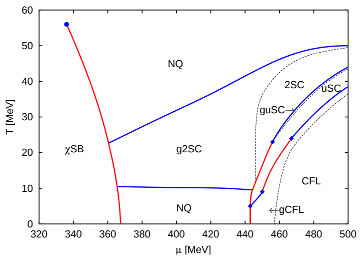

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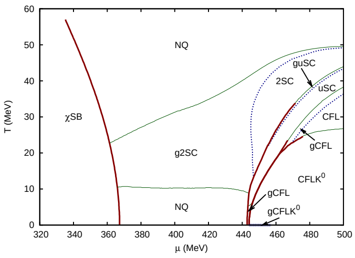
→

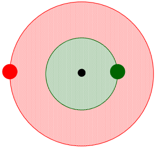


[H. Basler, M.B., PRD (2010); H. Warringa (2006)]

Inhomogeneous (crystalline) superconductors

TECHNISCHE UNIVERSITÄT DARMSTADT

- unequal Fermi momenta: $p_F^{a,b} = \bar{p}_F \pm \delta p_F$



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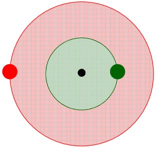

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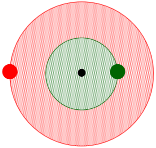

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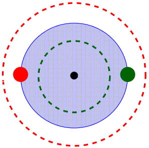

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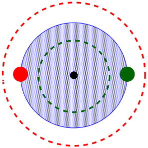

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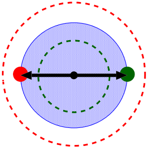

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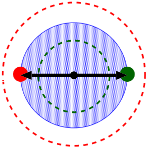

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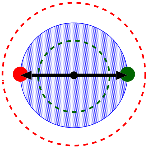

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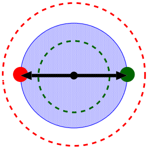

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- FF: [P. Fulde, R.A. Ferrell, Phys. Rev., 1964]
  - single plane wave, $\langle q(\vec{x})q(\vec{x})\rangle \sim \Delta\, e^{2i\vec{q}\cdot\vec{x}}$ for fixed $\vec{q}$
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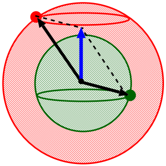

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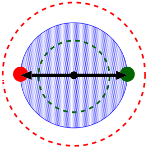

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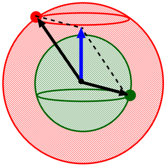

- LO: [Larkin, Ovchinnikov, Zh. Eksp. Teor. Fiz., 1964]
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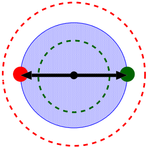

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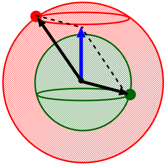

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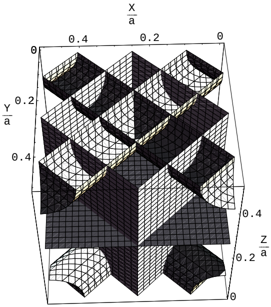
- chromomagnetic instabilities = instabilities towards LOFF phases in CSC?
  [Giannakis, Ren; Giannakis, Hou, Ren, PLB (2005)]

LOFF phases in color superconductivity

- Review: [Anglani et al., Rev. Mod. Phys. (2014)])
- Most works in literature:
  - single plane wave (FF)

    e.g., [Alford, Bowers, Rajagopal (2001), Sedrakian, Rischke (2009)]
  - superposition of several plane waves with different directions, but equal wave lengths (mostly Ginzburg-Landau analyses)

    e.g., [Bowers, Rajagopal (2002), Casalbuoni et al. (2006)]



[Rajagopal, Sharma, PRD (2006)]

- Alternative framework:

  [D. Nickel, M.B., PRD (2009)]
  - NJL model for inhomogeneous pairing
  - superimpose different wave lengths

NJL-model for LO phases

[D. Nickel, M.B., PRD (2009)]

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- space dependent mean-fields: $\Delta(\vec{x}) \;\propto\; \langle q^T(\vec{x})\, C\gamma_5\, \tau_2\, \lambda_2\, q(\vec{x})\rangle$

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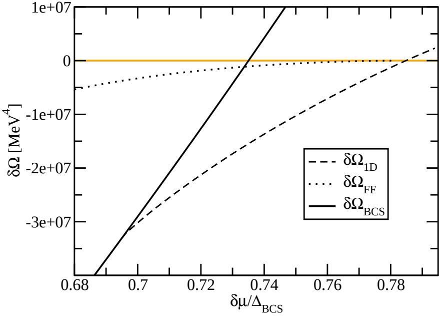

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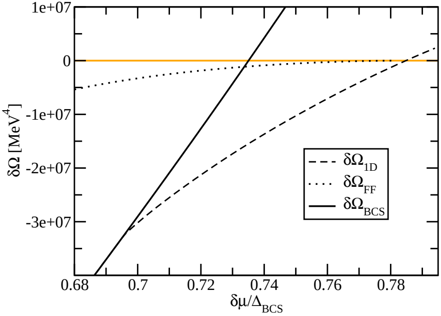

- inhomogenous solutions favored in a certain window!

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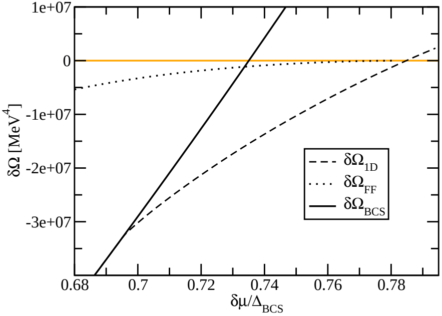
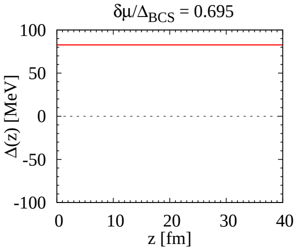


favored gap functions:



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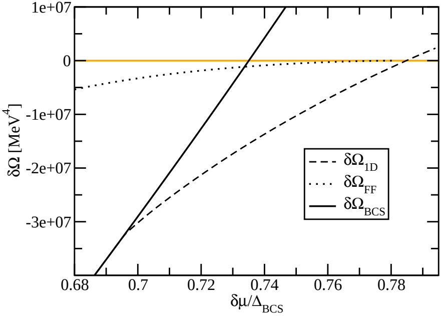

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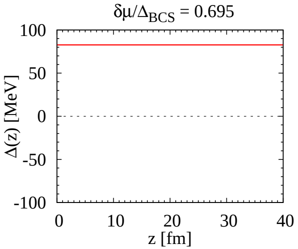

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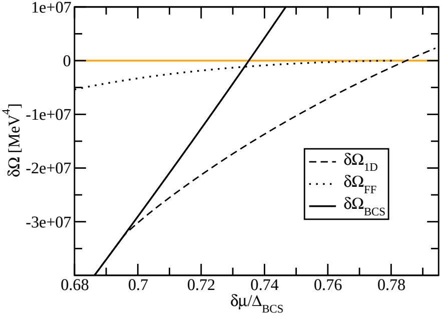

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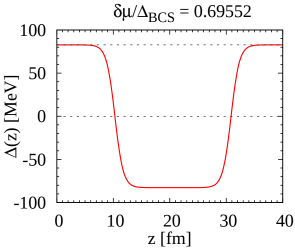

- BCS → solitonic
- inhomogenous solutions favored in a certain window!

NJL-model for LO phases

[D. Nickel, M.B., PRD (2009)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- space dependent mean-fields: $\Delta(\vec{x}) \;\propto\; \langle q^T(\vec{x})\, C\gamma_5\, \tau_2\, \lambda_2\, q(\vec{x})\rangle$
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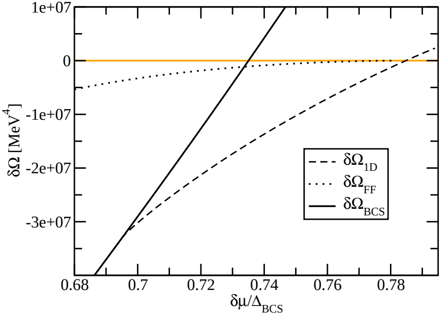

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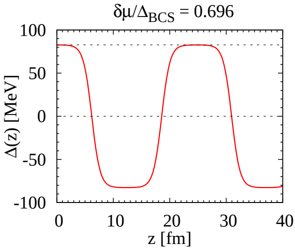

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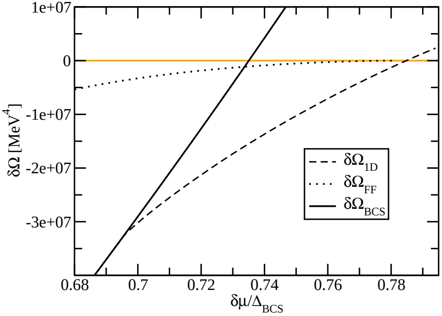

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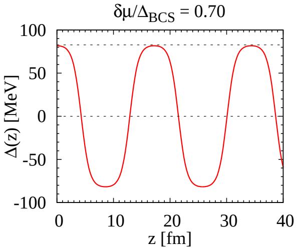

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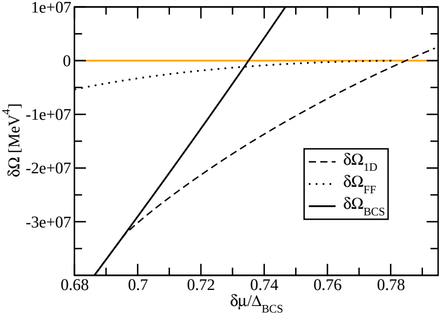

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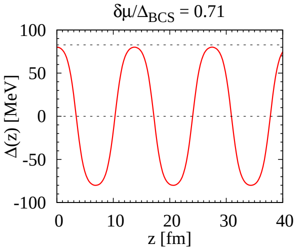

- BCS $\rightarrow$ solitonic $\rightarrow$ sinusoidal
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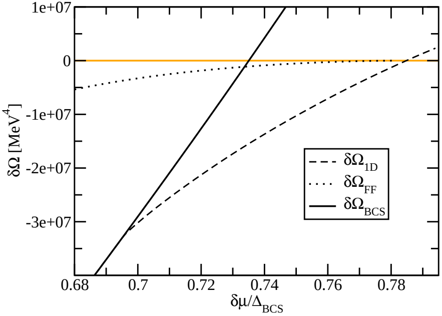

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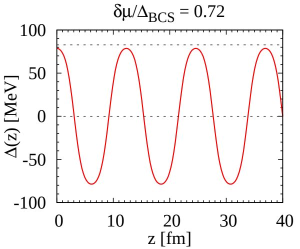

- BCS → solitonic → sinusoidal
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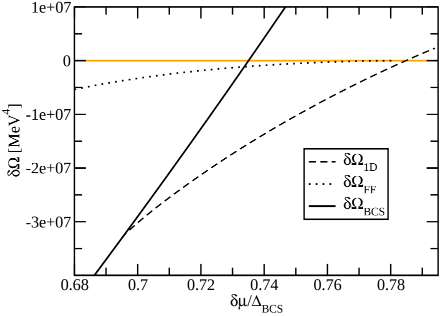

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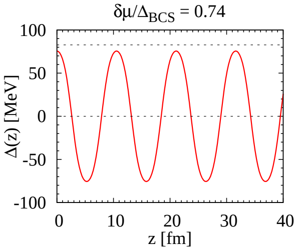

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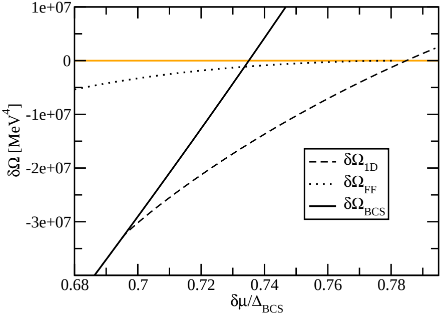

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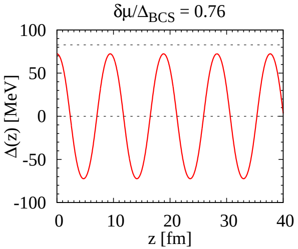

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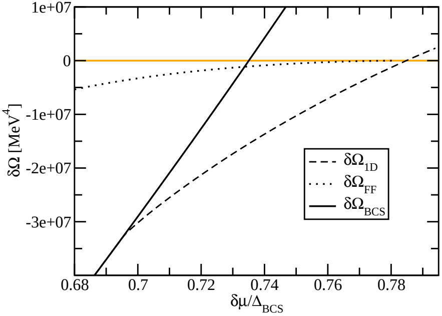

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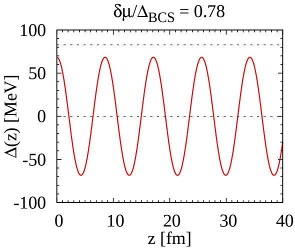

- BCS → solitonic → sinusoidal
- inhomogenous solutions favored in a certain window!

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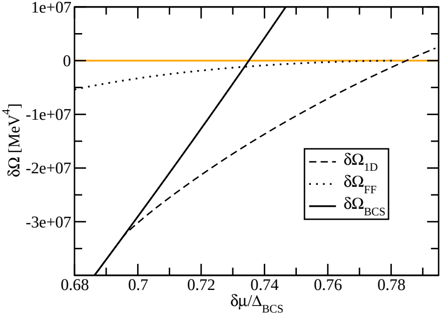

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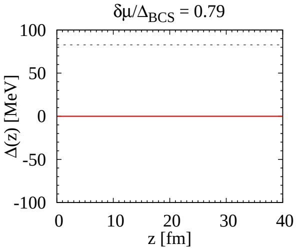

- BCS → solitonic → sinusoidal → normal
- inhomogenous solutions favored in a certain window!