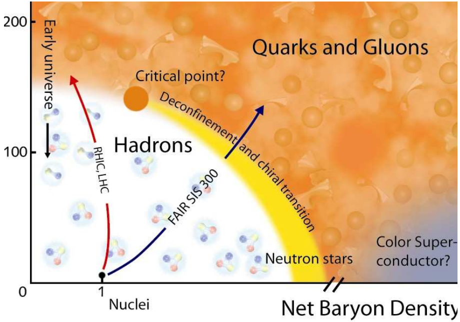
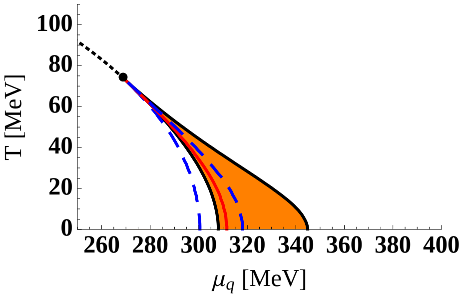
Outline

1. Introduction ✔
2. Chiral phase transition and critical endpoint ✔
3. Color superconductivity ✔
4. Inhomogeneous chiral phases



INHOMOGENEOUS CHIRAL PHASES



[Nickel (2009)]

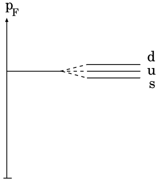
Why should we expect inhomogeneous chiral-symmetry breaking phases?

Why should we expect inhomogeneous chiral-symmetry breaking phases?

TECHNISCHE UNIVERSITÄT DARMSTADT

Analogy:

- CSC = quark-quark pairing
  - favored for equal Fermi momenta
  - stressed by unequal densities

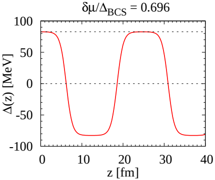


Why should we expect inhomogeneous chiral-symmetry breaking phases?

TECHNISCHE UNIVERSITÄT DARMSTADT

Analogy:

- CSC = quark-quark pairing
  - favored for equal Fermi momenta
  - stressed by unequal densities
  - tradeoff: spatially varying diquark condensate
  - excess quarks in regions of low $\langle qq \rangle$

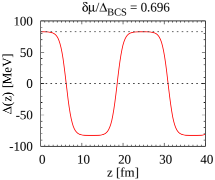


Why should we expect inhomogeneous chiral-symmetry breaking phases?

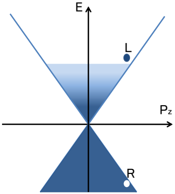
TECHNISCHE UNIVERSITÄT DARMSTADT

Analogy:

- CSC = quark-quark pairing
  - favored for equal Fermi momenta
  - stressed by unequal densities
  - tradeoff: spatially varying diquark condensate
  - excess quarks in regions of low $\langle qq \rangle$



- χSB = quark-antiquark pairing
  - favored for vanishing Fermi momenta
  - stressed by nonzero densities
  - tradeoff: spatially varying chiral condensate
  - quarks in regions of low $\langle \bar{q}q \rangle$



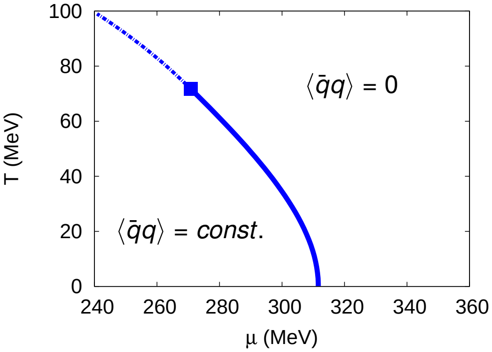
[Kojo et al. (2010)]

Highlight example

TECHNISCHE UNIVERSITÄT DARMSTADT

- chiral phase transition in the NJL model [D. Nickel, PRD (2009)]

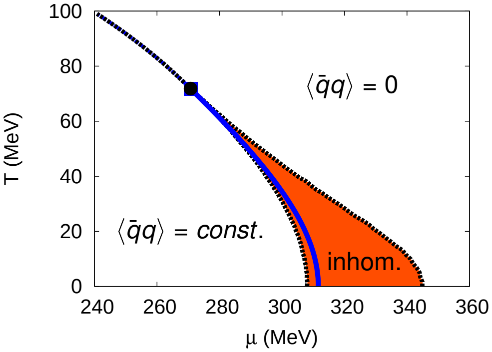
homogeneous phases only



Highlight example

- chiral phase transition in the NJL model [D. Nickel, PRD (2009)]

including inhomogeneous phase



- first-order phase boundary completely covered by the inhomogeneous phase
- all phase boundaries second order (mean-field artifact?)
- tricritical point $\to$ Lifshitz point
  [Nickel, PRL (2009)]

Inhomogeneous chiral phases: (incomplete) historical overview

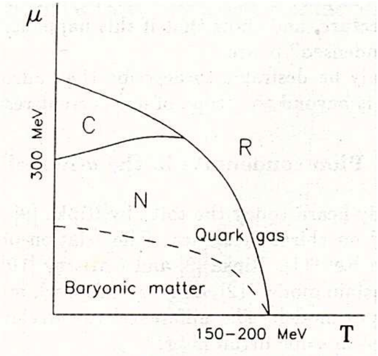
TECHNISCHE UNIVERSITÄT DARMSTADT

- 1960s:
  - spin-density waves in nuclear matter (Overhauser)
- 1970s – 1990s:
  - p-wave pion condensation (Migdal)
  - chiral density wave (Dautry, Nyman)
  - Skyrme crystals (Goldhaber, Manton)
- after 2000:
  - 1+1 D Gross-Neveu model (Thies et al.)
  - quarkyonic matter (Kojo, McLerran, Pisarski, ...)

Inhomogeneous chiral phases: (incomplete) historical overview

TECHNISCHE UNIVERSITÄT DARMSTADT

- 1960s:
  - spin-density waves in nuclear matter (Overhauser)
- 1970s – 1990s:
  - p-wave pion condensation (Migdal)
  - chiral density wave (Dautry, Nyman)
  - Skyrme crystals (Goldhaber, Manton)
- after 2000:
  - 1+1 D Gross-Neveu model (Thies et al.)
  - quarkyonic matter (Kojo, McLerran, Pisarski, ...)

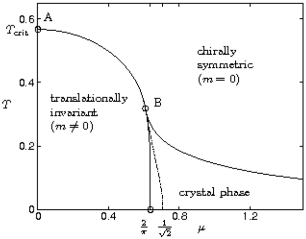


Broniowski et al. (1991)

Inhomogeneous chiral phases: (incomplete) historical overview

TECHNISCHE UNIVERSITÄT DARMSTADT

- 1960s:
  - spin-density waves in nuclear matter (Overhauser)
- 1970s – 1990s:
  - p-wave pion condensation (Migdal)
  - chiral density wave (Dautry, Nyman)
  - Skyrme crystals (Goldhaber, Manton)
- after 2000:
  - 1+1 D Gross-Neveu model (Thies et al.)
  - quarkyonic matter (Kojo, McLerran, Pisarski, ...)

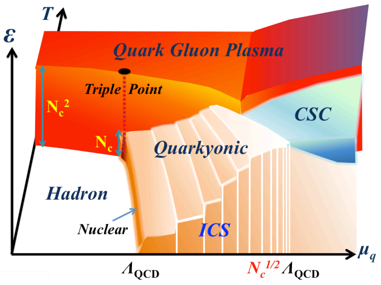


Thies, Urlichs (2003)

Inhomogeneous chiral phases: (incomplete) historical overview

TECHNISCHE UNIVERSITÄT DARMSTADT

- 1960s:
  - spin-density waves in nuclear matter (Overhauser)
- 1970s – 1990s:
  - p-wave pion condensation (Migdal)
  - chiral density wave (Dautry, Nyman)
  - Skyrme crystals (Goldhaber, Manton)
- after 2000:
  - 1+1 D Gross-Neveu model (Thies et al.)
  - quarkyonic matter (Kojo, McLerran, Pisarski, ...)



Kojo et al. (2011)

NJL Model

- Lagrangian:

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + G\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2\right]$$

NJL Model

TECHNISCHE
UNIVERSITÄT
DARMSTADT

- Lagrangian:

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + G\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2\right]$$

- bosonize: $\sigma(x) = \bar{\psi}(x)\psi(x), \quad \vec{\pi}(x) = \bar{\psi}(x)i\gamma_5\vec{\tau}\psi(x)$

$$\Rightarrow \quad \mathscr{L} = \bar{\psi}\left(i\partial\!\!\!/ - m + 2G\left(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi}\right)\right)\psi - G\left(\sigma^2 + \vec{\pi}^{\,2}\right)$$

NJL Model

TECHNISCHE UNIVERSITÄT DARMSTADT

- Lagrangian:

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + G\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2\right]$$

- bosonize: $\sigma(x) = \bar{\psi}(x)\psi(x), \quad \vec{\pi}(x) = \bar{\psi}(x)i\gamma_5\vec{\tau}\psi(x)$

$$\Rightarrow \quad \mathscr{L} = \bar{\psi}\left(i\partial\!\!\!/ - m + 2G\left(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi}\right)\right)\psi - G\left(\sigma^2 + \vec{\pi}^{\,2}\right)$$

- mean-field approximation:

$$\sigma(x) \to \langle\sigma(x)\rangle \equiv S(\vec{x}), \quad \pi_a(x) \to \langle\pi_a(x)\rangle \equiv P(\vec{x})\,\delta_{a3}$$

  - $S(\vec{x})$, $P(\vec{x})$ time independent classical fields
  - retain space dependence !

NJL Model

TECHNISCHE UNIVERSITÄT DARMSTADT

- Lagrangian:

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + G\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2\right]$$

- bosonize: $\sigma(x) = \bar{\psi}(x)\psi(x), \quad \vec{\pi}(x) = \bar{\psi}(x)i\gamma_5\vec{\tau}\psi(x)$

$$\Rightarrow \quad \mathscr{L} = \bar{\psi}\left(i\partial\!\!\!/ - m + 2G\left(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi}\right)\right)\psi - G\left(\sigma^2 + \vec{\pi}^{\,2}\right)$$

- mean-field approximation:

$$\sigma(x) \to \langle\sigma(x)\rangle \equiv S(\vec{x}), \quad \pi_a(x) \to \langle\pi_a(x)\rangle \equiv P(\vec{x})\,\delta_{a3}$$

  - $S(\vec{x})$, $P(\vec{x})$ time independent classical fields
  - retain space dependence !

- mean-field thermodynamic potential:

$$\Omega_{MF}(T,\mu) = -\frac{T}{V}\ln\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\,\exp\left(\int_{x\in[0,\frac{1}{T}]\times V}\left(\mathscr{L}_{MF} + \mu\bar{\psi}\gamma^0\psi\right)\right)$$

Mean-field model

TECHNISCHE UNIVERSITÄT DARMSTADT

- mean-field Lagrangian:

$$\mathscr{L}_{MF} = \bar{\psi}(x)\,\mathcal{S}^{-1}(x)\,\psi(x) - G\left[S^2(\vec{x}) + P^2(\vec{x})\right]$$

  - bilinear in ψ and $\bar{\psi}$ $\quad\Rightarrow\quad$ quark fields can be integrated out!

Mean-field model

- mean-field Lagrangian:

$$\mathscr{L}_{MF} = \bar{\psi}(x)\,\mathcal{S}^{-1}(x)\,\psi(x) - G\left[S^2(\vec{x}) + P^2(\vec{x})\right]$$

  - bilinear in ψ and $\bar{\psi}$ $\Rightarrow$ quark fields can be integrated out!

- inverse dressed propagator:

$$\mathcal{S}^{-1}(x) = i\partial\!\!\!/ - m + 2G\left(S(\vec{x}) + i\gamma_5\tau_3 P(\vec{x})\right) \equiv \gamma^0\,(i\partial_0 - H_{MF})$$

Mean-field model

TECHNISCHE UNIVERSITÄT DARMSTADT

- mean-field Lagrangian:

$$\mathscr{L}_{MF} = \bar{\psi}(x)\,\mathcal{S}^{-1}(x)\,\psi(x) - G\left[S^2(\vec{x}) + P^2(\vec{x})\right]$$

  - bilinear in ψ and $\bar{\psi}$ $\quad\Rightarrow\quad$ quark fields can be integrated out!

- inverse dressed propagator:

$$\mathcal{S}^{-1}(x) \;=\; i\partial\!\!\!/ - m + 2G\left(S(\vec{x}) + i\gamma_5\tau_3 P(\vec{x})\right) \;\equiv\; \gamma^0\,\left(i\partial_0 - H_{MF}\right)$$

- effective Hamiltonian (in chiral representation):

$$H_{MF} = H_{MF}[S, P] = \begin{pmatrix} -i\vec{\sigma}\cdot\vec{\partial} & M(\vec{x}) \\ M^*(\vec{x}) & i\vec{\sigma}\cdot\vec{\partial} \end{pmatrix}$$

  - constituent mass functions: $\quad M(\vec{x}) = m - 2G\,[S(\vec{x}) + iP(\vec{x})]$

Mean-field model

TECHNISCHE UNIVERSITÄT DARMSTADT

- mean-field Lagrangian:

$$\mathscr{L}_{MF} = \bar{\psi}(x)\,\mathcal{S}^{-1}(x)\,\psi(x) - G\left[S^2(\vec{x}) + P^2(\vec{x})\right]$$

  - bilinear in ψ and $\bar{\psi}$ $\Rightarrow$ quark fields can be integrated out!

- inverse dressed propagator:

$$\mathcal{S}^{-1}(x) = i\partial\!\!\!/ - m + 2G\left(S(\vec{x}) + i\gamma_5\tau_3 P(\vec{x})\right) \equiv \gamma^0\,(i\partial_0 - H_{MF})$$

- effective Hamiltonian (in chiral representation):

$$H_{MF} = H_{MF}[S,P] = \begin{pmatrix} -i\vec{\sigma}\cdot\vec{\partial} & M(\vec{x}) \\ M^*(\vec{x}) & i\vec{\sigma}\cdot\vec{\partial} \end{pmatrix}$$

  - constituent mass functions: $M(\vec{x}) = m - 2G\,[S(\vec{x}) + iP(\vec{x})]$
  - H_{MF} hermitean $\Rightarrow$ can (in principle) be diagonalized (eigenvalues E_λ)
  - H_{MF} time-independent $\Rightarrow$ Matsubara sum as usual

Mean-field thermodynamic potential

TECHNISCHE UNIVERSITÄT DARMSTADT

- thermodynamic potential:

$$\Omega_{MF}(T,\mu;S,P) = -\frac{T}{V}\mathbf{Tr}\ln\left(\frac{1}{T}(i\partial_0 - H_{MF} + \mu)\right) + \frac{G}{V}\int_V d^3x\,\left(S^2(\vec{x}) + P^2(\vec{x})\right)$$

Mean-field thermodynamic potential

TECHNISCHE UNIVERSITÄT DARMSTADT

- thermodynamic potential:

$$\Omega_{MF}(T,\mu;S,P) = -\frac{T}{V}\mathbf{Tr}\ln\left(\frac{1}{T}(i\partial_0 - H_{MF} + \mu)\right) + \frac{G}{V}\int_V d^3x\,\left(S^2(\vec{x}) + P^2(\vec{x})\right)$$

$$= -\frac{1}{V}\sum_\lambda\left[\frac{E_\lambda - \mu}{2} + T\ln\left(1 + e^{\frac{E_\lambda - \mu}{T}}\right)\right] + \frac{1}{V}\int_V d^3x\,\frac{|M(\vec{x}) - m|^2}{4G}$$

Mean-field thermodynamic potential

TECHNISCHE UNIVERSITÄT DARMSTADT

- thermodynamic potential:

$$\Omega_{MF}(T,\mu;S,P) = -\frac{T}{V}\mathbf{Tr}\ln\left(\frac{1}{T}(i\partial_0 - H_{MF} + \mu)\right) + \frac{G}{V}\int_V d^3x\,\left(S^2(\vec{x}) + P^2(\vec{x})\right)$$

$$= -\frac{1}{V}\sum_\lambda\left[\frac{E_\lambda - \mu}{2} + T\ln\left(1 + e^{\frac{E_\lambda-\mu}{T}}\right)\right] + \frac{1}{V}\int_V d^3x\,\frac{|M(\vec{x}) - m|^2}{4G}$$

- remaining tasks:
  - Calculate eigenvalue spectrum $E_\lambda[M(\vec{x})]$ of H_{MF} for given $M(\vec{x})$.
  - Minimize Ω_{MF} w.r.t. $M(\vec{x})$

Mean-field thermodynamic potential

- thermodynamic potential:

$$\Omega_{MF}(T,\mu;S,P) = -\frac{T}{V}\mathbf{Tr}\ln\left(\frac{1}{T}(i\partial_0 - H_{MF} + \mu)\right) + \frac{G}{V}\int_V d^3x\,\left(S^2(\vec{x}) + P^2(\vec{x})\right)$$

$$= -\frac{1}{V}\sum_\lambda\left[\frac{E_\lambda - \mu}{2} + T\ln\left(1 + e^{\frac{E_\lambda - \mu}{T}}\right)\right] + \frac{1}{V}\int_V d^3x\,\frac{|M(\vec{x}) - m|^2}{4G}$$

- remaining tasks:
  - Calculate eigenvalue spectrum $E_\lambda[M(\vec{x})]$ of H_{MF} for given $M(\vec{x})$.
    difficulty: H_{MF} is nondiagonal in momentum space
  - Minimize Ω_{MF} w.r.t. $M(\vec{x})$

Mean-field thermodynamic potential

TECHNISCHE UNIVERSITÄT DARMSTADT

- thermodynamic potential:

$$\Omega_{MF}(T,\mu;S,P) = -\frac{T}{V}\mathbf{Tr}\ln\left(\frac{1}{T}(i\partial_0 - H_{MF} + \mu)\right) + \frac{G}{V}\int_V d^3x\,\left(S^2(\vec{x}) + P^2(\vec{x})\right)$$

$$= -\frac{1}{V}\sum_\lambda\left[\frac{E_\lambda - \mu}{2} + T\ln\left(1 + e^{\frac{E_\lambda-\mu}{T}}\right)\right] + \frac{1}{V}\int_V d^3x\,\frac{|M(\vec{x}) - m|^2}{4G}$$

- remaining tasks:
  - Calculate eigenvalue spectrum $E_\lambda[M(\vec{x})]$ of H_{MF} for given $M(\vec{x})$.

    difficulty: H_{MF} is nondiagonal in momentum space
  - Minimize Ω_{MF} w.r.t. $M(\vec{x})$

    difficulty: functional minimization w.r.t. arbitrary shapes

Strategies

TECHNISCHE UNIVERSITÄT DARMSTADT

- Restricted ansätze for the condensate modulation

  → minimize Ω_{MF} w.r.t. a finite number of parameters

  - ansätze for which H_{MF} can be diagonalized analytically
  - brute-force numerical diagonalization of H_{MF}

- Stability and Ginzburg-Landau anlayses

  → investigate the stability of the homogeneous ground state w.r.t. small inhomogeneous fluctuations

Ansätze which can be diagonalized analytically

Chiral density wave

Chiral density wave

TECHNISCHE UNIVERSITÄT DARMSTADT

- popular ansatz: $M(\vec{x}) = \Delta e^{i\vec{q}\cdot\vec{x}}$ (dual) chiral density wave, "chiral spiral"

Chiral density wave

TECHNISCHE UNIVERSITÄT DARMSTADT

- popular ansatz: $M(\vec{x}) = \Delta e^{i\vec{q}\cdot\vec{x}}$ (dual) chiral density wave, “chiral spiral”

$$\Leftrightarrow \quad S(\vec{x}) = -\frac{\Delta}{2G}\cos(\vec{q}\cdot\vec{x})\,, \quad P(\vec{x}) = -\frac{\Delta}{2G}\sin(\vec{q}\cdot\vec{x})$$

$$\Rightarrow \quad \mathcal{L}_{MF} = \bar{\psi}\left[i\partial\!\!\!/ - m + 2G(S(\vec{x}) + i\gamma_5\tau_3 P(\vec{x}))\right]\psi - G\left(S^2 + P^2\right)$$

$$= \bar{\psi}\Big[i\partial\!\!\!/ - \Delta\Big(\cos(\vec{q}\cdot\vec{x}) + i\gamma_5\tau_3\sin(\vec{q}\cdot\vec{x})\Big)\Big]\psi - \frac{\Delta^2}{4G}$$

$$= \bar{\psi}\Big[i\partial\!\!\!/ - \Delta\exp\left(i\gamma_5\tau_3\vec{q}\cdot\vec{x}\right)\Big]\psi - \frac{\Delta^2}{4G}$$

Chiral density wave

TECHNISCHE UNIVERSITÄT DARMSTADT

- popular ansatz: $M(\vec{x}) = \Delta e^{i\vec{q}\cdot\vec{x}}$ (dual) chiral density wave, “chiral spiral”

$$\Leftrightarrow \quad S(\vec{x}) = -\frac{\Delta}{2G}\cos(\vec{q}\cdot\vec{x})\,, \quad P(\vec{x}) = -\frac{\Delta}{2G}\sin(\vec{q}\cdot\vec{x})$$

$$\Rightarrow \quad \mathcal{L}_{MF} = \bar{\psi}\left[i\partial\!\!\!/ - m + 2G(S(\vec{x}) + i\gamma_5\tau_3 P(\vec{x}))\right]\psi - G\left(S^2 + P^2\right)$$

$$= \bar{\psi}\left[i\partial\!\!\!/ - \Delta\left(\cos(\vec{q}\cdot\vec{x}) + i\gamma_5\tau_3\sin(\vec{q}\cdot\vec{x})\right)\right]\psi - \frac{\Delta^2}{4G}$$

$$= \bar{\psi}\left[i\partial\!\!\!/ - \Delta\exp\left(i\gamma_5\tau_3\vec{q}\cdot\vec{x}\right)\right]\psi - \frac{\Delta^2}{4G}$$

- unitary transformation: $\psi(x) = \exp\left(-\frac{i}{2}\gamma_5\tau_3\vec{q}\cdot\vec{x}\right)\psi'(x)$ [Dautry, Nyman (1979)]

$$\Rightarrow \quad \mathcal{L}_{MF} = \bar{\psi}'\left[i\partial\!\!\!/ + \frac{1}{2}\vec{\gamma}\gamma_5\tau_3\cdot q - \Delta\right]\psi' - \frac{\Delta^2}{4G}$$

no explicit $\vec{x}$ dependence $\longrightarrow$ can be diagonalized analytically!

Chiral density wave

TECHNISCHE
UNIVERSITÄT
DARMSTADT

- popular ansatz: $M(\vec{x}) = \Delta e^{i\vec{q}\cdot\vec{x}}$ (dual) chiral density wave, "chiral spiral"

$$\Leftrightarrow \quad S(\vec{x}) = -\frac{\Delta}{2G}\cos(\vec{q}\cdot\vec{x}) \, , \quad P(\vec{x}) = -\frac{\Delta}{2G}\sin(\vec{q}\cdot\vec{x})$$

$$\Rightarrow \quad \mathcal{L}_{MF} = \bar{\psi}\left[i\partial\!\!\!/ - m + 2G(S(\vec{x}) + i\gamma_5\tau_3 P(\vec{x}))\right]\psi - G\left(S^2 + P^2\right)$$

$$= \bar{\psi}\left[i\partial\!\!\!/ - \Delta\Big(\cos(\vec{q}\cdot\vec{x}) + i\gamma_5\tau_3\sin(\vec{q}\cdot\vec{x})\Big)\right]\psi - \frac{\Delta^2}{4G}$$

$$= \bar{\psi}\left[i\partial\!\!\!/ - \Delta\exp\left(i\gamma_5\tau_3\vec{q}\cdot\vec{x}\right)\right]\psi - \frac{\Delta^2}{4G}$$

- unitary transformation: $\psi(x) = \exp\left(-\frac{i}{2}\gamma_5\tau_3\vec{q}\cdot\vec{x}\right)\psi'(x)$ [Dautry, Nyman (1979)]

$$\Rightarrow \quad \mathcal{L}_{MF} = \bar{\psi}'\left[i\partial\!\!\!/ + \frac{1}{2}\vec{\gamma}\gamma_5\tau_3\cdot q - \Delta\right]\psi' - \frac{\Delta^2}{4G}$$

no explicit $\vec{x}$ dependence → can be diagonalized analytically!

- dispersion relations: $E^2_\pm(\vec{p}) = \vec{p}^{\,2} + \Delta^2 + \frac{\vec{q}^{\,2}}{4} \pm \sqrt{\Delta^2\vec{q}^{\,2} + (\vec{q}\cdot\vec{p})^2}$

Real kink crystal

TECHNISCHE UNIVERSITÄT DARMSTADT

- important observation: [D. Nickel, PRD (2009)]

  general problem with 1D modulations in 3+1D
  can be mapped to the $1 + 1$ dimensional case

Real kink crystal

TECHNISCHE UNIVERSITÄT DARMSTADT

- important observation: [D. Nickel, PRD (2009)]

  general problem with 1D modulations in 3+1D
  can be mapped to the $1+1$ dimensional case

- $1+1$D solutions known analytically: [M. Thies, J. Phys. A (2006)]

  $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu)$ (chiral limit)

  - $\mathrm{sn}(\xi|\nu)$: Jacobi elliptic functions
  - $M(z)$ real $\Rightarrow$ purely scalar "real kink cystal" (RKC)

Real kink crystal

- important observation: [D. Nickel, PRD (2009)]

  general problem with 1D modulations in 3+1D
  can be mapped to the $1+1$ dimensional case

- $1+1$D solutions known analytically: [M. Thies, J. Phys. A (2006)]

  $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu)$ (chiral limit)

  - $\mathrm{sn}(\xi|\nu)$: Jacobi elliptic functions
  - $M(z)$ real $\Rightarrow$ purely scalar "real kink cystal" (RKC)

- remaining task:

  - minimize w.r.t. 2 parameters: Δ, ν
  - (almost) as simple as CDW, but more powerful
  - $m \neq 0$: 3 parameters

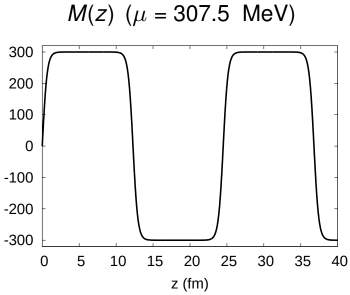
Mass functions and density profiles ($T = 0$)

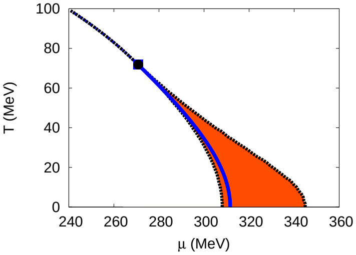
TECHNISCHE UNIVERSITÄT DARMSTADT

- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \rightarrow \quad \begin{cases} \Delta\tanh(\Delta z) & \text{for} \quad \nu \rightarrow 1 \\ \sqrt{\nu}\Delta\sin(\Delta z) & \text{for} \quad \nu \rightarrow 0 \end{cases}$

Mass functions and density profiles ($T = 0$)

- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \rightarrow \quad \begin{cases} \Delta \tanh(\Delta z) & \text{for} \quad \nu \rightarrow 1 \\ \sqrt{\nu}\Delta \sin(\Delta z) & \text{for} \quad \nu \rightarrow 0 \end{cases}$

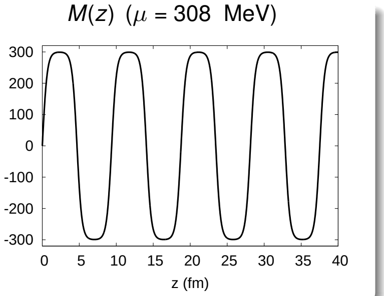


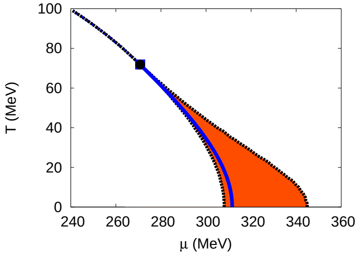


Mass functions and density profiles ($T = 0$)

TECHNISCHE UNIVERSITÄT DARMSTADT

- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \rightarrow \quad \begin{cases} \Delta \tanh(\Delta z) & \text{for} \quad \nu \rightarrow 1 \\ \sqrt{\nu}\Delta \sin(\Delta z) & \text{for} \quad \nu \rightarrow 0 \end{cases}$



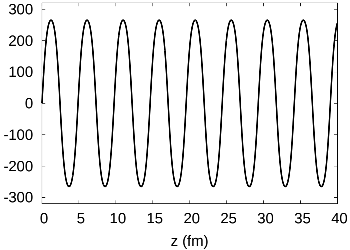


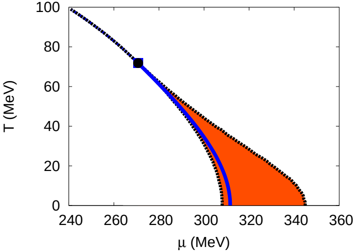
Mass functions and density profiles ($T = 0$)

TECHNISCHE UNIVERSITÄT DARMSTADT

- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \to \quad \begin{cases} \Delta\tanh(\Delta z) & \text{for} \quad \nu \to 1 \\ \sqrt{\nu}\Delta\sin(\Delta z) & \text{for} \quad \nu \to 0 \end{cases}$

$M(z)$ (μ = 309 MeV)

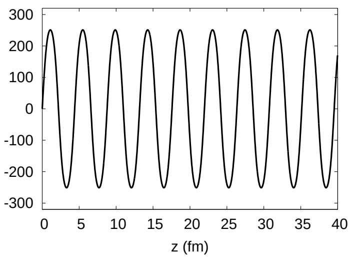


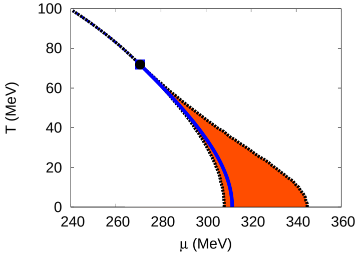


Mass functions and density profiles ($T = 0$)

- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \rightarrow \quad \begin{cases} \Delta \tanh(\Delta z) & \text{for} \quad \nu \to 1 \\ \sqrt{\nu}\Delta \sin(\Delta z) & \text{for} \quad \nu \to 0 \end{cases}$

$M(z)$ ($\mu = 310$ MeV)

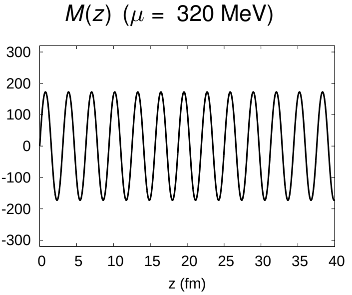


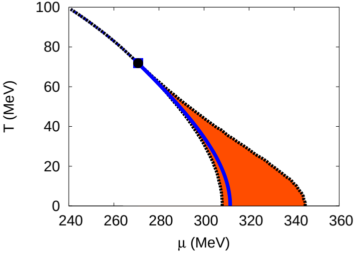


Mass functions and density profiles ($T = 0$)

TECHNISCHE UNIVERSITÄT DARMSTADT

- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \to \quad \begin{cases} \Delta\tanh(\Delta z) & \text{for} \quad \nu \to 1 \\ \sqrt{\nu}\Delta\sin(\Delta z) & \text{for} \quad \nu \to 0 \end{cases}$



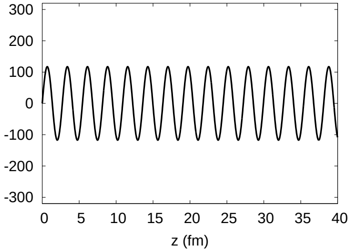


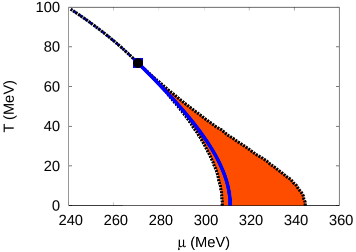
Mass functions and density profiles ($T = 0$)

TECHNISCHE UNIVERSITÄT DARMSTADT

- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \to \quad \begin{cases} \Delta\tanh(\Delta z) & \text{for} \quad \nu \to 1 \\ \sqrt{\nu}\Delta\sin(\Delta z) & \text{for} \quad \nu \to 0 \end{cases}$

$M(z)$ (μ = 330 MeV)



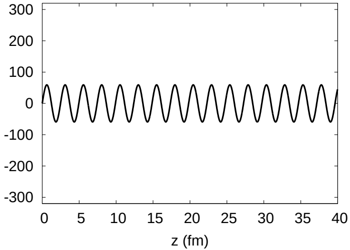


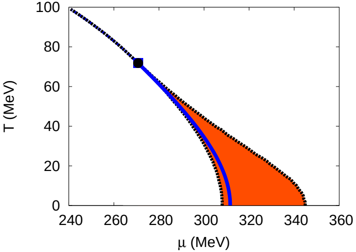
Mass functions and density profiles ($T = 0$)

TECHNISCHE UNIVERSITÄT DARMSTADT

- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \rightarrow \quad \begin{cases} \Delta \tanh(\Delta z) & \text{for} \quad \nu \rightarrow 1 \\ \sqrt{\nu}\Delta \sin(\Delta z) & \text{for} \quad \nu \rightarrow 0 \end{cases}$

$M(z)$ (μ = 340 MeV)

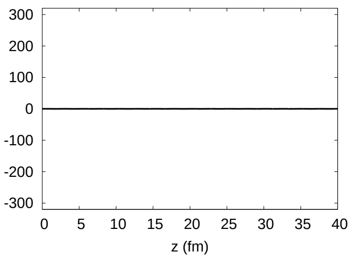


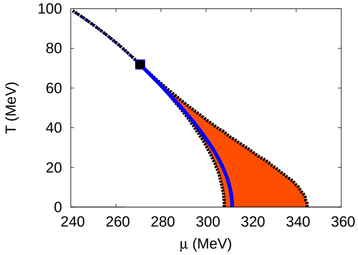


Mass functions and density profiles (T = 0)

- $M(z) = \sqrt{\nu}\Delta\, \mathrm{sn}(\Delta z|\nu) \quad \rightarrow \quad \begin{cases} \Delta \tanh(\Delta z) & \text{for} \quad \nu \to 1 \\ \sqrt{\nu}\Delta \sin(\Delta z) & \text{for} \quad \nu \to 0 \end{cases}$

$M(z)$ (μ = 345 MeV)

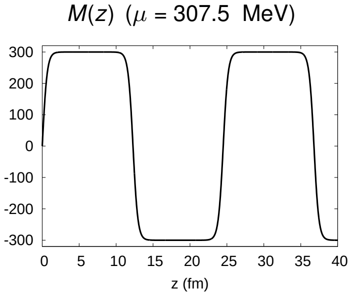


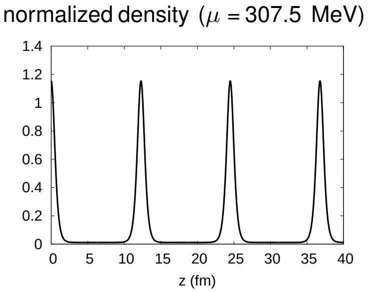


Mass functions and density profiles ($T = 0$)

TECHNISCHE UNIVERSITÄT DARMSTADT

- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \rightarrow \quad \begin{cases} \Delta \tanh(\Delta z) & \text{for} \quad \nu \to 1 \\ \sqrt{\nu}\Delta \sin(\Delta z) & \text{for} \quad \nu \to 0 \end{cases}$

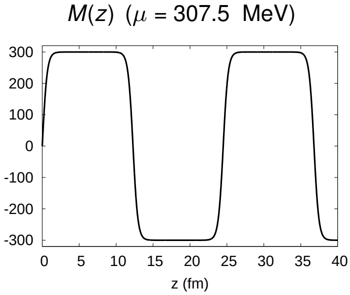


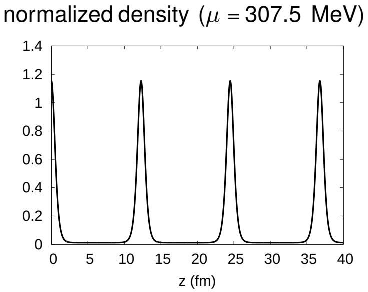


Mass functions and density profiles ($T = 0$)

TECHNISCHE UNIVERSITÄT DARMSTADT

- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \to \quad \begin{cases} \Delta \tanh(\Delta z) & \text{for} \quad \nu \to 1 \\ \sqrt{\nu}\Delta \sin(\Delta z) & \text{for} \quad \nu \to 0 \end{cases}$

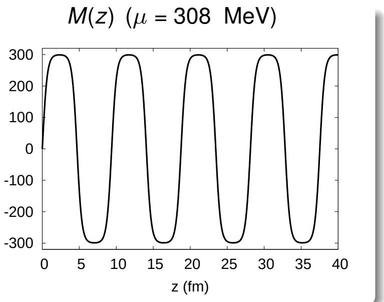


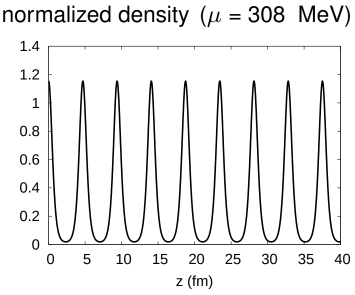


- Quarks reside in the chirally restored regions.

Mass functions and density profiles ($T = 0$)

- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \rightarrow \quad \begin{cases} \Delta \tanh(\Delta z) & \text{for} \quad \nu \rightarrow 1 \\ \sqrt{\nu}\Delta \sin(\Delta z) & \text{for} \quad \nu \rightarrow 0 \end{cases}$

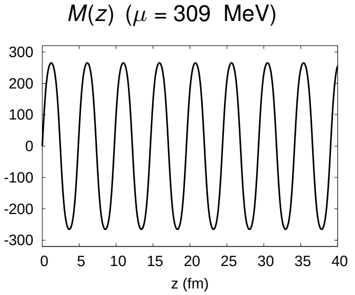


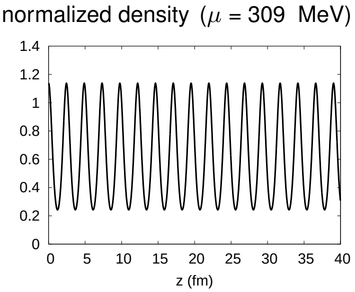


- Quarks reside in the chirally restored regions.

Mass functions and density profiles ($T = 0$)

- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \rightarrow \quad \begin{cases} \Delta\tanh(\Delta z) & \text{for} \quad \nu \to 1 \\ \sqrt{\nu}\Delta\sin(\Delta z) & \text{for} \quad \nu \to 0 \end{cases}$

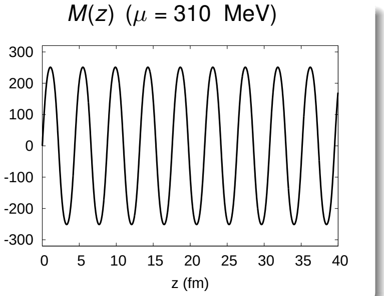


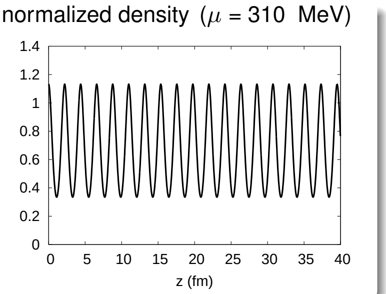


- Quarks reside in the chirally restored regions.

Mass functions and density profiles ($T = 0$)

- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \rightarrow \quad \begin{cases} \Delta\tanh(\Delta z) & \text{for} \quad \nu \rightarrow 1 \\ \sqrt{\nu}\Delta\sin(\Delta z) & \text{for} \quad \nu \rightarrow 0 \end{cases}$

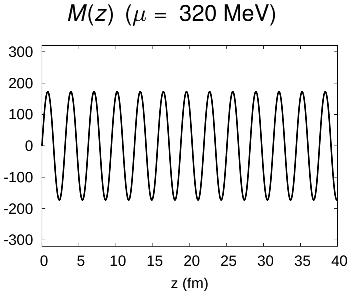


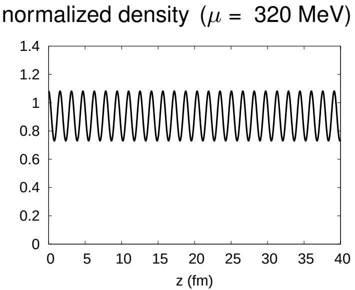


- Quarks reside in the chirally restored regions.

Mass functions and density profiles ($T = 0$)

- $M(z) = \sqrt{\nu}\Delta\, \mathrm{sn}(\Delta z|\nu) \quad \rightarrow \quad \begin{cases} \Delta \tanh(\Delta z) & \text{for} \quad \nu \rightarrow 1 \\ \sqrt{\nu}\Delta \sin(\Delta z) & \text{for} \quad \nu \rightarrow 0 \end{cases}$



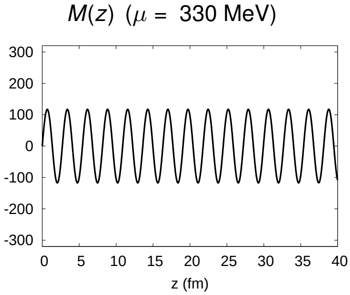


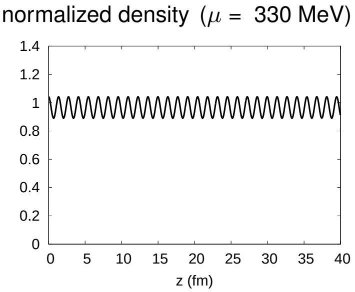
- Quarks reside in the chirally restored regions.
- Density gets smoothened with increasing μ and T.

Mass functions and density profiles ($T = 0$)

TECHNISCHE UNIVERSITÄT DARMSTADT

- $M(z) = \sqrt{\nu}\Delta\, \mathrm{sn}(\Delta z|\nu) \quad \to \quad \begin{cases} \Delta \tanh(\Delta z) & \text{for} \quad \nu \to 1 \\ \sqrt{\nu}\Delta \sin(\Delta z) & \text{for} \quad \nu \to 0 \end{cases}$



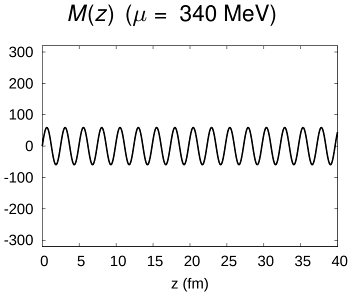


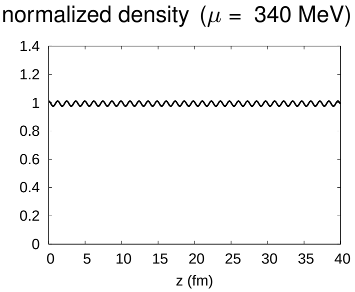
- Quarks reside in the chirally restored regions.
- Density gets smoothened with increasing μ and T.

Mass functions and density profiles ($T = 0$)

TECHNISCHE UNIVERSITÄT DARMSTADT

- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \rightarrow \quad \begin{cases} \Delta\tanh(\Delta z) & \text{for} \quad \nu \to 1 \\ \sqrt{\nu}\Delta\sin(\Delta z) & \text{for} \quad \nu \to 0 \end{cases}$

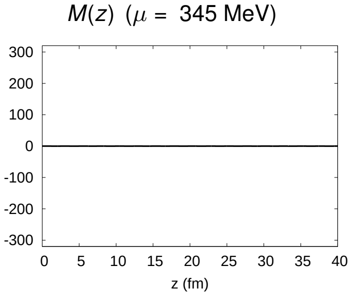


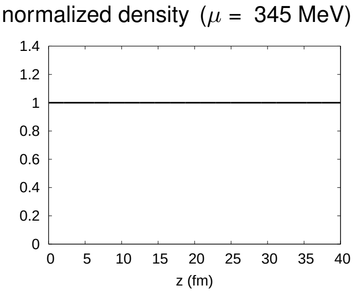


- Quarks reside in the chirally restored regions.
- Density gets smoothened with increasing μ and T.

Mass functions and density profiles ($T = 0$)

- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \rightarrow \quad \begin{cases} \Delta\tanh(\Delta z) & \text{for} \quad \nu \rightarrow 1 \\ \sqrt{\nu}\Delta\sin(\Delta z) & \text{for} \quad \nu \rightarrow 0 \end{cases}$

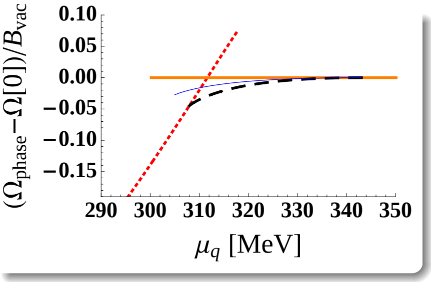




- Quarks reside in the chirally restored regions.
- Density gets smoothened with increasing μ and T.

Free energy difference

[D. Nickel, PRD (2009)]



- homogeneous chirally broken
- Jacobi elliptic functions
- chiral density wave:
  $M_{CDW}(z) = M_1\, e^{iqz}$

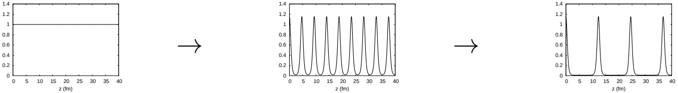
- soliton lattice favored, when it exists
- $\delta\Omega_{Jacobi} \approx 2\delta\Omega_{CDW} \quad \Rightarrow$ CDW never favored

Self-bound quark matter

[M.B., S. Carignano, PRD (2013)]

- 1D inhomogeneous solutions:

  homogeneous matter decays into domain-wall solitons



- If it was 3D: Hadronization!
- single-soliton properties:
  - $\frac{E}{N} = \mu_{c,inh} \sim 325$ MeV $\Rightarrow$ "baryon" mass: $M_B = 3\frac{E}{N} \sim 975$ MeV
  - central density: $\rho_B = \frac{1}{4\pi} M_{vac}\mu_{c,inh}^2 \sim 2.1\rho_0$
  - longitudinal size: $\sqrt{\langle z^2 \rangle} = \frac{\pi}{\sqrt{12}} \frac{1}{M_{vac}} \sim .5$ fm
- but it's only 1D modulations ...

  → revisit chiral solitons !? [Alkofer, Reinhardt, Weigel; Goeke et al.; Ripka; ...]

Two-dimensional modulations

[S. Carignano, M.B., PRD (2012)]

Two-dimensional modulations

[S. Carignano, M.B., PRD (2012)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- no known analytical solutions

  $\rightarrow$ brute-force numerical diagonalization of H for a given ansatz

Two-dimensional modulations

[S. Carignano, M.B., PRD (2012)]

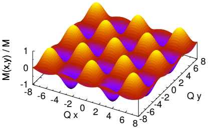
TECHNISCHE UNIVERSITÄT DARMSTADT

- no known analytical solutions

  $\rightarrow$ brute-force numerical diagonalization of H for a given ansatz

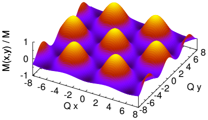
- consider two shapes:
  - square lattice ("egg carton")

    $M(x, y) = M \cos(Qx) \cos(Qy)$



  - hexagonal lattice

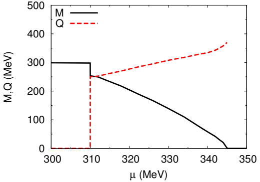
    $M(x, y) = \frac{M}{3} \left[2 \cos(Qx) \cos\left(\frac{1}{\sqrt{3}} Qy\right) + \cos(\frac{2}{\sqrt{3}} Qy) \right]$



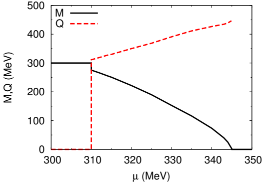
- minimize both cases numerically w.r.t. M and Q

Two-dimensional modulations: results

- amplitudes and wave numbers:
  - egg carton:

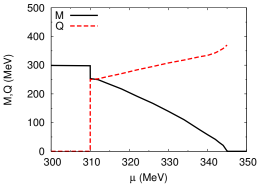


  - hexagon:

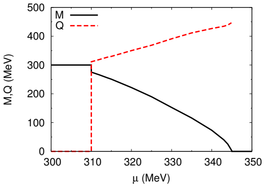


Two-dimensional modulations: results

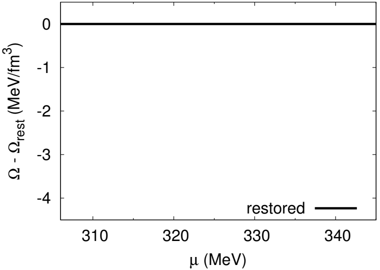
- amplitudes and wave numbers:
  - egg carton:



  - hexagon:



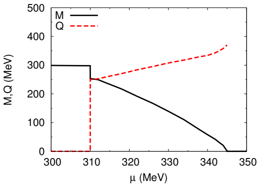
free-energy gain at $T = 0$:



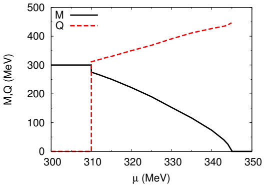
Two-dimensional modulations: results

TECHNISCHE UNIVERSITÄT DARMSTADT

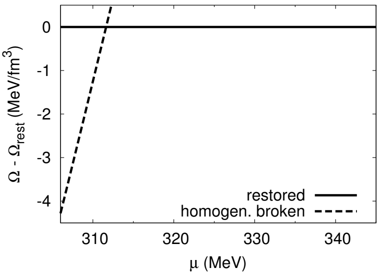
- amplitudes and wave numbers:
  - egg carton:



  - hexagon:

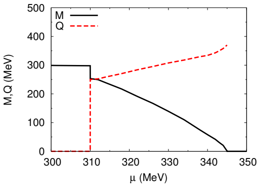


free-energy gain at $T = 0$:

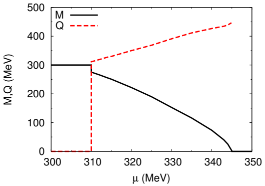


Two-dimensional modulations: results

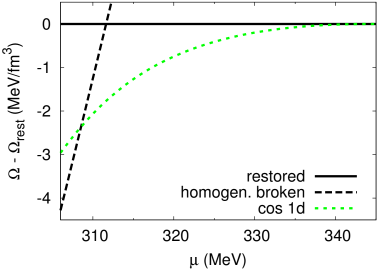
- amplitudes and wave numbers:
  - egg carton:



  - hexagon:

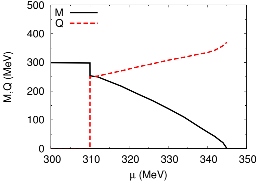


free-energy gain at $T = 0$:

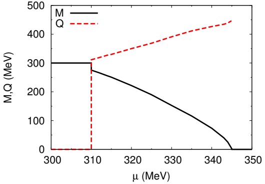


Two-dimensional modulations: results

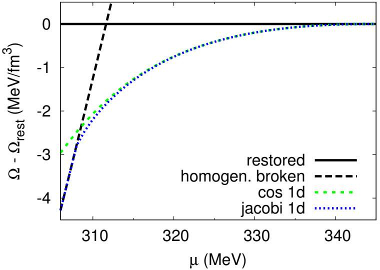
- amplitudes and wave numbers:
  - egg carton:



  - hexagon:

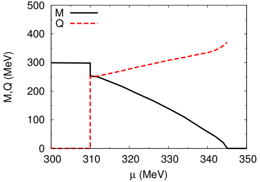


free-energy gain at $T = 0$:

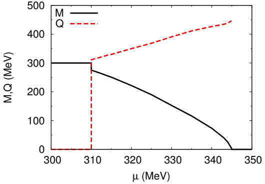


Two-dimensional modulations: results

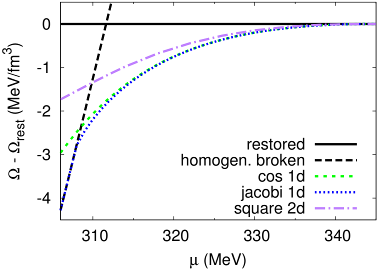
- amplitudes and wave numbers:
  - egg carton:



  - hexagon:

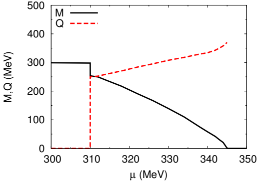


free-energy gain at $T = 0$:

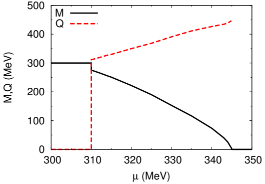


Two-dimensional modulations: results

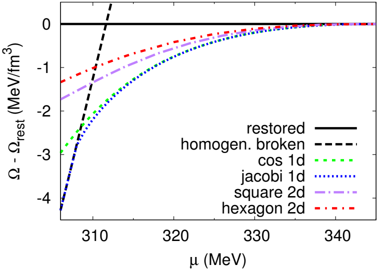
- amplitudes and wave numbers:
  - egg carton:



  - hexagon:

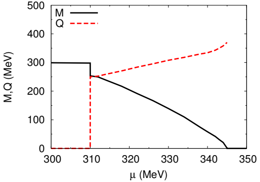


free-energy gain at $T = 0$:

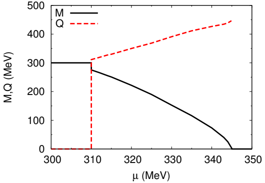


Two-dimensional modulations: results

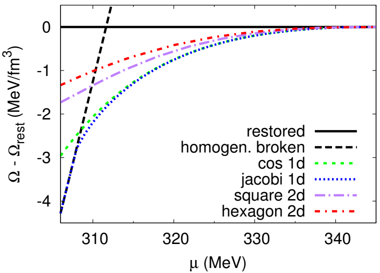
- amplitudes and wave numbers:
  - egg carton:



  - hexagon:



free-energy gain at $T = 0$:



- 2d not favored over 1d in this regime

Stability and Ginzburg-Landau analyses

Stability and Ginzburg-Landau analyses

TECHNISCHE UNIVERSITÄT DARMSTADT

General idea:

- Stability analysis:
  - Minimize Ω_{MF} w.r.t. homogeneous mean fields $\rightarrow$ $S = \bar{S} = const.$, $P = 0$
  - Study effect of small inhomogeneous fluctuations $\delta S(\vec{x})$, $\delta P(\vec{x})$

Stability and Ginzburg-Landau analyses

TECHNISCHE UNIVERSITÄT DARMSTADT

General idea:

- Stability analysis:
  - Minimize Ω_{MF} w.r.t. homogeneous mean fields $\rightarrow$ $S = \bar{S} = const.$, $P = 0$
  - Study effect of small inhomogeneous fluctuations $\delta S(\vec{x})$, $\delta P(\vec{x})$

→ sufficient but not necessary criterion for an inhomogeneous phase

☹ instabilities w.r.t large inhomogeneous fluctuations not excluded

☺ no ansatz functions for $S(\vec{x})$ and $P(\vec{x})$ needed

Stability and Ginzburg-Landau analyses

TECHNISCHE UNIVERSITÄT DARMSTADT

General idea:

- Stability analysis:
  - Minimize Ω_{MF} w.r.t. homogeneous mean fields $\rightarrow$ $S = \bar{S} = const.$, $P = 0$
  - Study effect of small inhomogeneous fluctuations $\delta S(\vec{x})$, $\delta P(\vec{x})$

  → sufficient but not necessary criterion for an inhomogeneous phase

  ☹ instabilities w.r.t large inhomogeneous fluctuations not excluded

  ☺ no ansatz functions for $S(\vec{x})$ and $P(\vec{x})$ needed

  → well suited to identify 2nd-order phase transitions

Stability and Ginzburg-Landau analyses

TECHNISCHE UNIVERSITÄT DARMSTADT

General idea:

- Stability analysis:
  - Minimize Ω_{MF} w.r.t. homogeneous mean fields $\rightarrow$ $S = \bar{S} = const.$, $P = 0$
  - Study effect of small inhomogeneous fluctuations $\delta S(\vec{x})$, $\delta P(\vec{x})$

  → sufficient but not necessary criterion for an inhomogeneous phase

  ☹ instabilities w.r.t large inhomogeneous fluctuations not excluded

  ☺ no ansatz functions for $S(\vec{x})$ and $P(\vec{x})$ needed

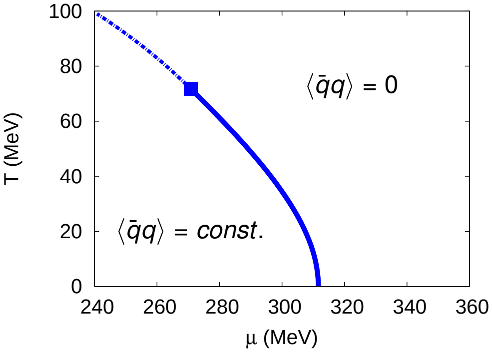
  → well suited to identify 2nd-order phase transitions

- Ginzburg-Landau analysis:
  - additional expansion in small gradients $\vec{\nabla} S(\vec{x})$, $\vec{\nabla} P(\vec{x})$
  - best suited to identify critical and Lifshitz points

Reminder

- chiral phase transition in the NJL model (chiral limit) [D. Nickel, PRD (2009)]

homogeneous phases only

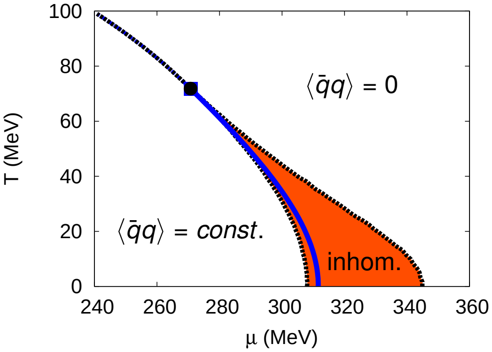


- tricritical point

Reminder

- chiral phase transition in the NJL model (chiral limit) [D. Nickel, PRD (2009)]

including inhomogeneous phase

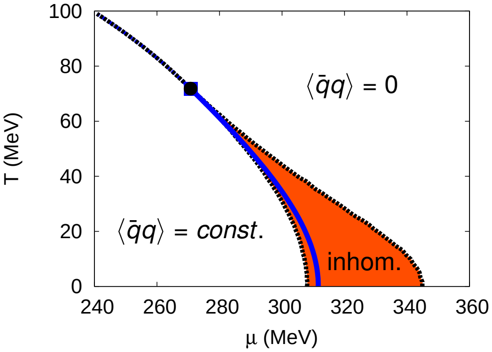


- tricritical point
  $\rightarrow$ Lifshitz point

Reminder

- chiral phase transition in the NJL model (chiral limit) [D. Nickel, PRD (2009)]

including inhomogeneous phase

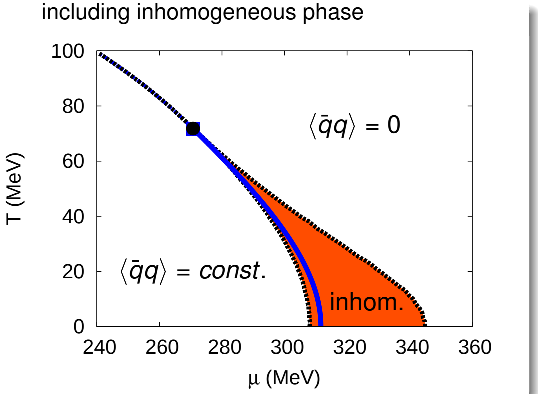


- tricritical point
  $\rightarrow$ Lifshitz point
- How was this shown?
  [Nickel, PRL (2009)]

Reminder

TECHNISCHE UNIVERSITÄT DARMSTADT

- chiral phase transition in the NJL model (chiral limit) [D. Nickel, PRD (2009)]



- tricritical point
  $\rightarrow$ Lifshitz point
- How was this shown?
  [Nickel, PRL (2009)]
- How is it away from the chiral limit?
  [MB, Carignano, PRB (2018)]

Ginzburg-Landau analysis

- Simplifications:
  - chiral limit $m = 0$ (will be relaxed later)
  - $P = 0$ (to simplify the notation, can be included straightforwardly)

$\rightarrow$ order parameter $M(\vec{x}) = -2G\,S(\vec{x})$ ("constituent quark mass")

$\rightarrow$ $\Omega_{MF} = \Omega_{MF}[M]$

Ginzburg-Landau analysis

TECHNISCHE UNIVERSITÄT DARMSTADT

- Simplifications:
  - chiral limit $m = 0$ (will be relaxed later)
  - $P = 0$ (to simplify the notation, can be included straightforwardly)

  $\rightarrow$ order parameter $M(\vec{x}) = -2G\,S(\vec{x})$ ("constituent quark mass")

  $\rightarrow$ $\Omega_{MF} = \Omega_{MF}[M]$

- Assumptions: M, $|\nabla M|$ small (holds near the LP)

  $\rightarrow$ expansion of the thermodynamic potential.

$$\Omega[M] = \Omega[0] + \frac{1}{V}\int_V d^3x\,\left\{\alpha_2 M^2(\vec{x}) + \alpha_{4,a} M^4(\vec{x}) + \alpha_{4,b}|\vec{\nabla} M(\vec{x})|^2 + ...\right\}$$

  - $\alpha_n = \alpha_n(T,\mu)$: GL coefficients
  - chiral symmetry: only even powers allowed
  - stability: higher-order coeffs. positive

Tricritical and Lifshitz point

TECHNISCHE UNIVERSITÄT DARMSTADT

- GL expansion: $\Omega[M] = \Omega[0] + \frac{1}{V} \int\limits_V d^3x \left\{ \alpha_2 M^2 + \alpha_{4,a} M^4 + \alpha_{4,b} |\vec{\nabla} M|^2 + ... \right\}$

Tricritical and Lifshitz point

- GL expansion: $\Omega[M] = \Omega[0] + \frac{1}{V} \int_V d^3x \left\{ \alpha_2 M^2 + \alpha_{4,a} M^4 + \alpha_{4,b} |\vec{\nabla} M|^2 + \ldots \right\}$
- <u>case 1:</u> $\alpha_{4,b} > 0 \;\Rightarrow$ gradients disfavored $\Rightarrow$ homogeneous

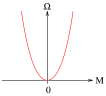
Tricritical and Lifshitz point

- GL expansion: $\Omega[M] = \Omega[0] + \frac{1}{V} \int\limits_V d^3x \left\{ \alpha_2 M^2 + \alpha_{4,a} M^4 + \alpha_{4,b} |\vec{\nabla} M|^2 + \dots \right\}$

- <u>case 1:</u> $\alpha_{4,b} > 0 \Rightarrow$ gradients disfavored $\Rightarrow$ homogeneous

  <u>case 1.1:</u> $\alpha_{4,a} > 0$

  - $\alpha_2 > 0 \Rightarrow$ restored phase

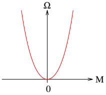


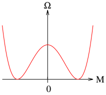
Tricritical and Lifshitz point

- GL expansion: $\Omega[M] = \Omega[0] + \frac{1}{V}\int_V d^3x \left\{ \alpha_2 M^2 + \alpha_{4,a} M^4 + \alpha_{4,b} |\vec{\nabla} M|^2 + ... \right\}$
- case 1: $\alpha_{4,b} > 0 \Rightarrow$ gradients disfavored $\Rightarrow$ homogeneous

case 1.1: $\alpha_{4,a} > 0$

- $\alpha_2 < 0 \Rightarrow$ hom. broken phase





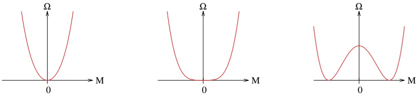
Tricritical and Lifshitz point

- GL expansion: $\Omega[M] = \Omega[0] + \frac{1}{V}\int_V d^3x \left\{ \alpha_2 M^2 + \alpha_{4,a} M^4 + \alpha_{4,b} |\vec{\nabla} M|^2 + ... \right\}$

- <u>case 1:</u> $\alpha_{4,b} > 0 \Rightarrow$ gradients disfavored $\Rightarrow$ homogeneous

  <u>case 1.1:</u> $\alpha_{4,a} > 0$

  - 2nd-order p.t. at $\alpha_2 = 0$



Tricritical and Lifshitz point

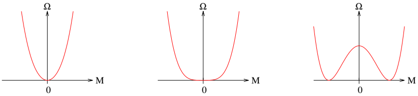
TECHNISCHE
UNIVERSITÄT
DARMSTADT

- GL expansion: $\Omega[M] = \Omega[0] + \frac{1}{V}\int_V d^3x \left\{ \alpha_2 M^2 + \alpha_{4,a} M^4 + \alpha_{4,b} |\vec{\nabla} M|^2 + \ldots \right\}$

- case 1: $\alpha_{4,b} > 0 \;\Rightarrow$ gradients disfavored $\Rightarrow$ homogeneous

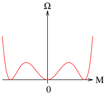
  case 1.1: $\alpha_{4,a} > 0$

  - 2nd-order p.t. at $\alpha_2 = 0$



  case 1.2: $\alpha_{4,a} < 0$

  - 1st-order phase trans. at $\alpha_2 > 0$



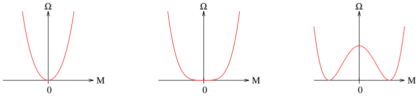
Tricritical and Lifshitz point

- GL expansion: $\Omega[M] = \Omega[0] + \frac{1}{V} \int_V d^3x \left\{ \alpha_2 M^2 + \alpha_{4,a} M^4 + \alpha_{4,b} |\vec{\nabla} M|^2 + \dots \right\}$

- <u>case 1:</u> $\alpha_{4,b} > 0 \;\Rightarrow$ gradients disfavored $\Rightarrow$ homogeneous

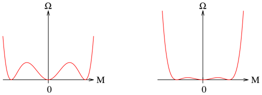
  <u>case 1.1:</u> $\alpha_{4,a} > 0$

  - 2nd-order p.t. at $\alpha_2 = 0$



  <u>case 1.2:</u> $\alpha_{4,a} < 0$

  - 1st-order phase trans. at $\alpha_2 > 0$



Tricritical and Lifshitz point

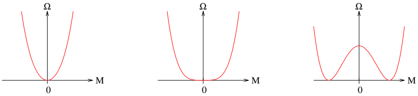
TECHNISCHE UNIVERSITÄT DARMSTADT

- GL expansion: $\Omega[M] = \Omega[0] + \frac{1}{V}\int_V d^3x \left\{\alpha_2 M^2 + \alpha_{4,a} M^4 + \alpha_{4,b}|\vec{\nabla} M|^2 + \ldots\right\}$

- case 1: $\alpha_{4,b} > 0 \;\Rightarrow$ gradients disfavored $\Rightarrow$ homogeneous

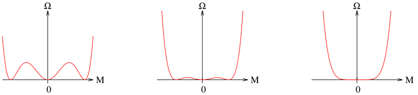
case 1.1: $\alpha_{4,a} > 0$

- 2nd-order p.t. at $\alpha_2 = 0$



case 1.2: $\alpha_{4,a} < 0$

- 1st-order phase trans. at $\alpha_2 > 0$



$\Rightarrow$ tricritical point (TCP): $\alpha_2 = \alpha_{4,a} = 0$

Tricritical and Lifshitz point

TECHNISCHE UNIVERSITÄT DARMSTADT

- GL expansion: $\Omega[M] = \Omega[0] + \frac{1}{V}\int_V d^3x \left\{\alpha_2 M^2 + \alpha_{4,a} M^4 + \alpha_{4,b} |\vec{\nabla} M|^2 + \ldots\right\}$

- case 1: $\alpha_{4,b} > 0 \Rightarrow$ gradients disfavored $\Rightarrow$ homogeneous

  case 1.1: $\alpha_{4,a} > 0$

  - 2nd-order p.t. at $\alpha_2 = 0$

  case 1.2: $\alpha_{4,a} < 0$

  - 1st-order phase trans. at $\alpha_2 > 0$

  $\Rightarrow$ tricritical point (TCP): $\alpha_2 = \alpha_{4,a} = 0$

- case 2: $\alpha_{4,b} < 0$

  - inhomogeneous phase possible

Tricritical and Lifshitz point

- GL expansion: $\Omega[M] = \Omega[0] + \frac{1}{V}\int_V d^3x \left\{\alpha_2 M^2 + \alpha_{4,a} M^4 + \alpha_{4,b} |\vec{\nabla} M|^2 + \dots\right\}$
- case 1: $\alpha_{4,b} > 0 \Rightarrow$ gradients disfavored $\Rightarrow$ homogeneous

  case 1.1: $\alpha_{4,a} > 0$

  - 2nd-order p.t. at $\alpha_2 = 0$

  case 1.2: $\alpha_{4,a} < 0$

  - 1st-order phase trans. at $\alpha_2 > 0$

  $\Rightarrow$ tricritical point (TCP): $\alpha_2 = \alpha_{4,a} = 0$

- case 2: $\alpha_{4,b} < 0$
  - inhomogeneous phase possible
  - 2nd-order phase boundary inhom. - restored: $\alpha_{4,b} < 0, \ \alpha_2 > 0$
    finite wavelength, amplitude $\to 0$

Tricritical and Lifshitz point

TECHNISCHE UNIVERSITÄT DARMSTADT

- GL expansion: $\Omega[M] = \Omega[0] + \frac{1}{V}\int_V d^3x \left\{ \alpha_2 M^2 + \alpha_{4,a} M^4 + \alpha_{4,b} |\vec{\nabla} M|^2 + ... \right\}$

- case 1: $\alpha_{4,b} > 0 \Rightarrow$ gradients disfavored $\Rightarrow$ homogeneous

  case 1.1: $\alpha_{4,a} > 0$

  - 2nd-order p.t. at $\alpha_2 = 0$

  case 1.2: $\alpha_{4,a} < 0$

  - 1st-order phase trans. at $\alpha_2 > 0$

  $\Rightarrow$ tricritical point (TCP): $\alpha_2 = \alpha_{4,a} = 0$

- case 2: $\alpha_{4,b} < 0$

  - inhomogeneous phase possible
  - 2nd-order phase boundary inhom. - restored: $\alpha_{4,b} < 0,\ \alpha_2 > 0$
    finite wavelength, amplitude $\to 0$

  Lifshitz point (LP): $\alpha_2 = \alpha_{4,b} = 0$

Away from the chiral limit

TECHNISCHE UNIVERSITÄT DARMSTADT

- $m \neq 0$: no chirally restored solution $M = 0$

  $\rightarrow$ expand about a priory unknown constant mass M_0:

$$\Omega[M] = \Omega[M_0] + \frac{1}{V}\int d^3x \left(\alpha_1 \delta M + \alpha_2 \delta M^2 + \alpha_3 \delta M^3 + \alpha_{4,a}\delta M^4 + \alpha_{4,b}(\nabla \delta M)^2 + \ldots\right)$$

  - small parameters: $\delta M(\vec{x}) \equiv M(\vec{x}) - M_0, \quad |\nabla \delta M(\vec{x})|$
  - GL coefficients: $\alpha_j = \alpha_j(T, \mu, M_0)$
  - odd powers allowed
  - require M_0 = extremum of Ω at given T and μ

    $\Rightarrow \quad \alpha_1(T, \mu, M_0) = 0 \quad \rightarrow \quad M_0 = M_0(T, \mu)$ (= homogeneous gap equation)

CEP and pseudo Lifshitz point

TECHNISCHE UNIVERSITÄT DARMSTADT

- GL expansion:

$$\Omega[M] = \Omega[M_0] + \frac{1}{V}\int d^3x \left(\alpha_2 \delta M^2 + \alpha_3 \delta M^3 + \alpha_{4,a}\delta M^4 + \alpha_{4,b}(\nabla \delta M)^2 + \ldots\right)$$

CEP and pseudo Lifshitz point

- GL expansion:

$$\Omega[M] = \Omega[M_0] + \frac{1}{V}\int d^3x \left(\alpha_2 \delta M^2 + \alpha_3 \delta M^3 + \alpha_{4,a} \delta M^4 + \alpha_{4,b} (\nabla \delta M)^2 + ...\right)$$

- case 1: $\alpha_{4,b} > 0 \Rightarrow$ homogeneous

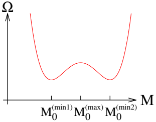
CEP and pseudo Lifshitz point

TECHNISCHE UNIVERSITÄT DARMSTADT

- GL expansion:

$$\Omega[M] = \Omega[M_0] + \frac{1}{V}\int d^3x \left(\alpha_2 \delta M^2 + \alpha_3 \delta M^3 + \alpha_{4,a}\delta M^4 + \alpha_{4,b}(\nabla \delta M)^2 + \dots\right)$$

- case 1: $\alpha_{4,b} > 0 \Rightarrow$ homogeneous



  - no restored phase, but 1st-order ph. trans. between different minima possible

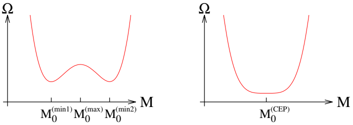
CEP and pseudo Lifshitz point

TECHNISCHE UNIVERSITÄT DARMSTADT

- GL expansion:

$$\Omega[M] = \Omega[M_0] + \frac{1}{V}\int d^3x \left(\alpha_2 \delta M^2 + \alpha_3 \delta M^3 + \alpha_{4,a}\delta M^4 + \alpha_{4,b}(\nabla \delta M)^2 + \ldots\right)$$

- <u>case 1:</u> $\alpha_{4,b} > 0 \;\Rightarrow$ homogeneous



  - no restored phase, but 1st-order ph. trans. between different minima possible
  - 2 minima + 1 maximum $\rightarrow$ 1 minimum

$\Rightarrow$ critical endpoint (CEP): $\alpha_2 = \alpha_3 = 0$

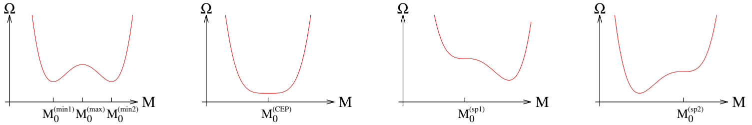
CEP and pseudo Lifshitz point

TECHNISCHE UNIVERSITÄT DARMSTADT

- GL expansion:

$$\Omega[M] = \Omega[M_0] + \frac{1}{V}\int d^3x \left(\alpha_2 \delta M^2 + \alpha_3 \delta M^3 + \alpha_{4,a}\delta M^4 + \alpha_{4,b}(\nabla \delta M)^2 + \ldots\right)$$

- case 1: $\alpha_{4,b} > 0 \;\Rightarrow$ homogeneous



  - no restored phase, but 1st-order ph. trans. between different minima possible
  - 2 minima + 1 maximum $\to$ 1 minimum

    $\Rightarrow$ critical endpoint (CEP): $\alpha_2 = \alpha_3 = 0$
  - spinodals: left: $\alpha_2 = 0,\ \alpha_3 < 0$, right: $\alpha_2 = 0,\ \alpha_3 > 0$,

CEP and pseudo Lifshitz point

TECHNISCHE UNIVERSITÄT DARMSTADT

- GL expansion:

$$\Omega[M] = \Omega[M_0] + \frac{1}{V}\int d^3x \left(\alpha_2 \delta M^2 + \alpha_3 \delta M^3 + \alpha_{4,a}\delta M^4 + \alpha_{4,b}(\nabla \delta M)^2 + ...\right)$$

- <u>case 1:</u> $\alpha_{4,b} > 0 \Rightarrow$ homogeneous CEP: $\alpha_2 = \alpha_3 = 0$
- <u>case 2:</u> $\alpha_{4,b} < 0 \Rightarrow$ inhomogeneous phases possible

CEP and pseudo Lifshitz point

- GL expansion:

  $$\Omega[M] = \Omega[M_0] + \frac{1}{V}\int d^3x \left(\alpha_2 \delta M^2 + \alpha_3 \delta M^3 + \alpha_{4,a}\delta M^4 + \alpha_{4,b}(\nabla \delta M)^2 + \ldots\right)$$

- <u>case 1:</u> $\alpha_{4,b} > 0 \Rightarrow$ homogeneous   CEP: $\alpha_2 = \alpha_3 = 0$
- <u>case 2:</u> $\alpha_{4,b} < 0 \Rightarrow$ inhomogeneous phases possible
  - strictly: only two phases – homogeneous and inhomogeneous $\Rightarrow$ no LP

CEP and pseudo Lifshitz point

TECHNISCHE UNIVERSITÄT DARMSTADT

- GL expansion:

$$\Omega[M] = \Omega[M_0] + \frac{1}{V}\int d^3x\,\left(\alpha_2\delta M^2 + \alpha_3\delta M^3 + \alpha_{4,a}\delta M^4 + \alpha_{4,b}(\nabla\delta M)^2 + \ldots\right)$$

- <u>case 1:</u> $\alpha_{4,b} > 0 \;\Rightarrow$ homogeneous CEP: $\alpha_2 = \alpha_3 = 0$
- <u>case 2:</u> $\alpha_{4,b} < 0 \;\Rightarrow$ inhomogeneous phases possible
  - strictly: only two phases – homogeneous and inhomogeneous $\Rightarrow$ no LP
  - There can be a 2nd-order transition between inhom. and hom. phase where the amplitude of the *inhomogeneous* part of $M(\vec{x})$ goes to zero

CEP and pseudo Lifshitz point

TECHNISCHE UNIVERSITÄT DARMSTADT

- GL expansion:

$$\Omega[M] = \Omega[M_0] + \frac{1}{V}\int d^3x\,\left(\alpha_2\delta M^2 + \alpha_3\delta M^3 + \alpha_{4,a}\delta M^4 + \alpha_{4,b}(\nabla\delta M)^2 + \ldots\right)$$

- <u>case 1:</u> $\alpha_{4,b} > 0 \Rightarrow$ homogeneous   CEP: $\alpha_2 = \alpha_3 = 0$
- <u>case 2:</u> $\alpha_{4,b} < 0 \Rightarrow$ inhomogeneous phases possible
  - strictly: only two phases – homogeneous and inhomogeneous $\Rightarrow$ no LP
  - There can be a 2nd-order transition between inhom. and hom. phase where the amplitude of the *inhomogeneous* part of $M(\vec{x})$ goes to zero
  - M_0 homogeneous ground state $\Rightarrow \delta M(\vec{x}) \to 0$ along this phase boundary

CEP and pseudo Lifshitz point

- GL expansion:

  $$\Omega[M] = \Omega[M_0] + \frac{1}{V}\int d^3x \left(\alpha_2 \delta M^2 + \alpha_3 \delta M^3 + \alpha_{4,a}\delta M^4 + \alpha_{4,b}(\nabla \delta M)^2 + \dots\right)$$

- case 1: $\alpha_{4,b} > 0 \Rightarrow$ homogeneous   CEP: $\alpha_2 = \alpha_3 = 0$
- case 2: $\alpha_{4,b} < 0 \Rightarrow$ inhomogeneous phases possible
  - strictly: only two phases – homogeneous and inhomogeneous $\Rightarrow$ no LP
  - There can be a 2nd-order transition between inhom. and hom. phase where the amplitude of the *inhomogeneous* part of $M(\vec{x})$ goes to zero
  - M_0 homogeneous ground state $\Rightarrow \delta M(\vec{x}) \to 0$ along this phase boundary
  - in general: $\nabla \delta M(\vec{x}) \neq 0$ along this phase boundary

    $\Rightarrow$ as in the chiral limit: $\alpha_{4,b} < 0$, $\alpha_2 > 0$

CEP and pseudo Lifshitz point

- GL expansion:

  $$\Omega[M] = \Omega[M_0] + \frac{1}{V}\int d^3x \left(\alpha_2 \delta M^2 + \alpha_3 \delta M^3 + \alpha_{4,a}\delta M^4 + \alpha_{4,b}(\nabla \delta M)^2 + \ldots\right)$$

- <u>case 1:</u> $\alpha_{4,b} > 0 \;\Rightarrow$ homogeneous   CEP: $\alpha_2 = \alpha_3 = 0$

- <u>case 2:</u> $\alpha_{4,b} < 0 \;\Rightarrow$ inhomogeneous phases possible
  - strictly: only two phases – homogeneous and inhomogeneous $\Rightarrow$ no LP
  - There can be a 2nd-order transition between inhom. and hom. phase where the amplitude of the *inhomogeneous* part of $M(\vec{x})$ goes to zero
  - M_0 homogeneous ground state $\Rightarrow \delta M(\vec{x}) \to 0$ along this phase boundary
  - in general: $\nabla \delta M(\vec{x}) \neq 0$ along this phase boundary

    $\Rightarrow$ as in the chiral limit: $\alpha_{4,b} < 0$, $\alpha_2 > 0$

    $\to$ pseudo Lifshitz point (PLP): $\alpha_2 = \alpha_{4,b} = 0$

Summarizing:
GL analysis of critical and Lifshitz points

- chiral limit ($m = 0$):
  - expansion about $M = 0$
  - TCP: $\alpha_2 = \alpha_{4,a} = 0$
  - LP: $\alpha_2 = \alpha_{4,b} = 0$
- away from the chiral limit ($m \neq 0$):
  - expansion about $M_0(T, \mu)$ solving $\alpha_1(T, \mu, M_0) = 0$
  - CEP: $\alpha_2 = \alpha_3 = 0$
  - PLP: $\alpha_2 = \alpha_{4,b} = 0$

Determination of the GL coefficients

TECHNISCHE UNIVERSITÄT DARMSTADT

Determination of the GL coefficients

- NJL mean-field thermodynamic potential:

$$\Omega_{MF}(T,\mu) = -\frac{T}{V}\mathbf{Tr}\log\left(\frac{S^{-1}}{T}\right) + G\,\frac{1}{V}\int d^3x\;\left(S^2(\vec{x}) + P^2(\vec{x})\right)$$

Determination of the GL coefficients

- NJL mean-field thermodynamic potential:

$$\Omega_{MF}(T,\mu) = -\frac{T}{V}\mathbf{Tr}\log\left(\frac{S^{-1}}{T}\right) + G\,\frac{1}{V}\int d^3x\,\left(S^2(\vec{x}) + P^2(\vec{x})\right)$$

- again assume $P = 0 \quad \rightarrow \quad M(\vec{x}) = m - 2G\,S(\vec{x}) \equiv M_0 + \delta M(\vec{x})$

Determination of the GL coefficients

TECHNISCHE UNIVERSITÄT DARMSTADT

- NJL mean-field thermodynamic potential:

$$\Omega_{MF}(T,\mu) = -\frac{T}{V}\mathbf{Tr}\log\left(\frac{S^{-1}}{T}\right) + G\,\frac{1}{V}\int d^3x\;\left(S^2(\vec{x}) + P^2(\vec{x})\right)$$

- again assume $P = 0 \quad \rightarrow \quad M(\vec{x}) = m - 2G\,S(\vec{x}) \equiv M_0 + \delta M(\vec{x})$

$$\Rightarrow \quad \Omega_{MF} = -\frac{T}{V}\,\mathbf{Tr}\,\log(S_0^{-1} - \delta M) + \frac{1}{V}\int_V d^3x\,\frac{(M_0 - m + \delta M(\vec{x}))^2}{4G}$$

  - $S_0^{-1}(x) = i\partial\!\!\!/ + \mu\gamma^0 - M_0$ inverse propagator of a free fermion with mass M_0

Determination of the GL coefficients

TECHNISCHE UNIVERSITÄT DARMSTADT

- NJL mean-field thermodynamic potential:

$$\Omega_{MF}(T,\mu) = -\frac{T}{V}\mathbf{Tr}\log\left(\frac{S^{-1}}{T}\right) + G\,\frac{1}{V}\int d^3x\;\left(S^2(\vec{x}) + P^2(\vec{x})\right)$$

- again assume $P = 0 \quad \to \quad M(\vec{x}) = m - 2G\,S(\vec{x}) \equiv M_0 + \delta M(\vec{x})$

$$\Rightarrow \quad \Omega_{MF} = -\frac{T}{V}\,\mathbf{Tr}\,\log(S_0^{-1} - \delta M) + \frac{1}{V}\int_V d^3x\,\frac{(M_0 - m + \delta M(\vec{x}))^2}{4G}$$

  - $S_0^{-1}(x) = i\partial\!\!\!/ + \mu\gamma^0 - M_0$ inverse propagator of a free fermion with mass M_0

- expand logarithm:

$$\log(S_0^{-1} - \delta M) = \log(S_0^{-1}) + \log(1 - S_0\delta M) = \log(S_0^{-1}) - \sum_{n=1}^{\infty}\frac{1}{n}(S_0\delta M)^n$$

Determination of the GL coefficients

TECHNISCHE UNIVERSITÄT DARMSTADT

- Thermodynamic potential: $\Omega_{MF} = \sum_{n=0}^{\infty} \Omega^{(n)}$

$\Omega^{(n)}$: contribution of order $(\delta M)^n$:

$$\Omega^{(0)} = -\frac{T}{V}\,\mathbf{Tr}\,\log S_0^{-1} + \frac{1}{V}\int_V d^3x\,\frac{(M_0-m)^2}{4G}$$

$$\Omega^{(1)} = \frac{T}{V}\,\mathbf{Tr}\,(S_0\delta M) + \frac{M_0-m}{2G}\,\frac{1}{V}\int_V d^3x\,\delta M(\vec{x})\,,$$

$$\Omega^{(2)} = \frac{1}{2}\frac{T}{V}\,\mathbf{Tr}\,(S_0\delta M)^2 + \frac{1}{4G}\,\frac{1}{V}\int_V d^3x\,\delta M^2(\vec{x})\,,$$

$$\Omega^{(n)} = \frac{1}{n}\frac{T}{V}\,\mathbf{Tr}\,(S_0\delta M)^n \quad \text{for } n \geq 3.$$

Determination of the GL coefficients

- functional trace:

$$\mathbf{Tr}\,(S_0\delta M)^n = 2N_c \int \prod_{i=1}^{n} d^4x_i\, \mathrm{tr}_D \left[S_0(x_n, x_1)\delta M(\vec{x}_1) S_0(x_1, x_2)\delta M(\vec{x}_2) \ldots S_0(x_{n-1}, x_n)\delta M(\vec{x}_n) \right]$$

Determination of the GL coefficients

TECHNISCHE UNIVERSITÄT DARMSTADT

- functional trace:

$$\mathbf{Tr}\,(S_0\delta M)^n = 2N_c \int \prod_{i=1}^{n} d^4x_i\, \mathrm{tr}_D\left[S_0(x_n,x_1)\delta M(\vec{x}_1)S_0(x_1,x_2)\delta M(\vec{x}_2)\ldots S_0(x_{n-1},x_n)\delta M(\vec{x}_n)\right]$$

- gradient expansion: $\delta M(\vec{x}_i) = \delta M(\vec{x}_1) + \nabla M(\vec{x}_1)\cdot(\vec{x}_i - \vec{x}_1) + \ldots$

$\Rightarrow \quad \Omega^{(n)} = \sum_{j=0}^{\infty} \Omega^{(n,j)}\ , \quad j$ = number of gradients

Determination of the GL coefficients

TECHNISCHE UNIVERSITÄT DARMSTADT

- functional trace:

$$\mathbf{Tr}\,(S_0\delta M)^n = 2N_c \int \prod_{i=1}^{n} d^4x_i\, \mathrm{tr_D}\left[S_0(x_n,x_1)\delta M(\vec{x}_1)S_0(x_1,x_2)\delta M(\vec{x}_2)\,...\,S_0(x_{n-1},x_n)\delta M(\vec{x}_n)\right]$$

- gradient expansion: $\delta M(\vec{x}_i) = \delta M(\vec{x}_1) + \nabla M(\vec{x}_1)\cdot(\vec{x}_i - \vec{x}_1) + ...$

$$\Rightarrow \quad \Omega^{(n)} = \sum_{j=0}^{\infty} \Omega^{(n,j)}\ , \qquad j = \text{number of gradients}$$

- final steps:
  - Insert momentum-space rep. of the free propagators S_0 and turn out all but one d^4x_i integrals.
  - Compare results with GL expansion of Ω_{MF} to read off the GL coefficients.

GL coefficients: results

TECHNISCHE UNIVERSITÄT DARMSTADT

- Resulting coefficients:

$$\alpha_1 = \frac{M_0 - m}{2G} + M_0 F_1 \,, \qquad \alpha_2 = \frac{1}{4G} + \frac{1}{2} F_1 + M_0^2 F_2 \,, \qquad \alpha_3 = M_0 \left(F_2 + \frac{4}{3} M_0^2 F_3 \right) \,,$$

$$\alpha_{4,a} = \frac{1}{4} F_2 + 2 M_0^2 F_3 + 2 M_0^4 F_4 \,, \qquad \alpha_{4,b} = \frac{1}{4} F_2 + \frac{1}{3} M_0^2 F_3$$

- $F_n = 8 N_c \int \frac{d^3p}{(2\pi)^3} \, T \sum_j \frac{1}{[(i\omega_j + \mu)^2 - \vec{p}^{\,2} - M_0^2]^n} \,, \qquad \omega_j = (2j+1)\pi T$

GL coefficients: results

TECHNISCHE UNIVERSITÄT DARMSTADT

- Resulting coefficients:

$$\alpha_1 = \frac{M_0 - m}{2G} + M_0 F_1 \,, \qquad \alpha_2 = \frac{1}{4G} + \frac{1}{2} F_1 + M_0^2 F_2 \,, \qquad \alpha_3 = M_0 \left(F_2 + \frac{4}{3} M_0^2 F_3 \right) \,,$$

$$\alpha_{4,a} = \frac{1}{4} F_2 + 2 M_0^2 F_3 + 2 M_0^4 F_4 \,, \qquad \alpha_{4,b} = \frac{1}{4} F_2 + \frac{1}{3} M_0^2 F_3$$

  - $F_n = 8 N_c \int \frac{d^3p}{(2\pi)^3} \, T \sum_j \frac{1}{[(i\omega_j + \mu)^2 - \vec{p}^{\,2} - M_0^2]^n}$, $\quad \omega_j = (2j+1)\pi T$

- chiral limit:
  - $m = 0 \;\Rightarrow\; M_0 = 0$ solves gap equation $\alpha_1 = 0$
  - $M_0 = 0 \;\Rightarrow\; \alpha_3 = 0$ (no odd powers)
  - $M_0 = 0 \;\Rightarrow\; \alpha_{4,a} = \alpha_{4,b} \;\Rightarrow\;$ TCP = LP [Nickel, PRL (2009)]

GL coefficients: results

TECHNISCHE UNIVERSITÄT DARMSTADT

- Resulting coefficients:

$$\alpha_1 = \frac{M_0 - m}{2G} + M_0 F_1 \,, \qquad \alpha_2 = \frac{1}{4G} + \frac{1}{2} F_1 + M_0^2 F_2 \,, \qquad \alpha_3 = M_0 \left(F_2 + \frac{4}{3} M_0^2 F_3 \right) \,,$$

$$\alpha_{4,a} = \frac{1}{4} F_2 + 2 M_0^2 F_3 + 2 M_0^4 F_4 \,, \qquad \alpha_{4,b} = \frac{1}{4} F_2 + \frac{1}{3} M_0^2 F_3$$

  - $F_n = 8 N_c \int \frac{d^3p}{(2\pi)^3} \, T \sum_j \frac{1}{[(i\omega_j + \mu)^2 - \vec{p}^{\,2} - M_0^2]^n} \,, \quad \omega_j = (2j+1)\pi T$

- towards the chiral limit:

  - $M_0 \to 0 \;\Rightarrow\; \alpha_3, \alpha_{4ba}, \alpha_{4,b} \propto F_2 \;\Rightarrow\;$ CEP $\to$ TCP = LP

GL coefficients: results

- Resulting coefficients:

$$\alpha_1 = \frac{M_0 - m}{2G} + M_0 F_1 \,, \qquad \alpha_2 = \frac{1}{4G} + \frac{1}{2} F_1 + M_0^2 F_2 \,, \qquad \alpha_3 = M_0 \left(F_2 + \frac{4}{3} M_0^2 F_3 \right) \,,$$

$$\alpha_{4,a} = \frac{1}{4} F_2 + 2 M_0^2 F_3 + 2 M_0^4 F_4 \,, \qquad \alpha_{4,b} = \frac{1}{4} F_2 + \frac{1}{3} M_0^2 F_3$$

  - $F_n = 8 N_c \int \frac{d^3p}{(2\pi)^3} \, T \sum_j \frac{1}{[(i\omega_j + \mu)^2 - \vec{p}^{\,2} - M_0^2]^n} \,, \quad \omega_j = (2j+1)\pi T$

- away from the chiral limit:

  - $M_0 \neq 0 \;\Rightarrow\; \alpha_3 = 4 M_0 \alpha_{4,b} \;\Rightarrow\;$ CEP = PLP

GL coefficients: results

TECHNISCHE UNIVERSITÄT DARMSTADT

- Resulting coefficients:

$$\alpha_1 = \frac{M_0 - m}{2G} + M_0 F_1\,, \qquad \alpha_2 = \frac{1}{4G} + \frac{1}{2}F_1 + M_0^2 F_2\,, \qquad \alpha_3 = M_0\left(F_2 + \frac{4}{3}M_0^2 F_3\right)\,,$$

$$\alpha_{4,a} = \frac{1}{4}F_2 + 2M_0^2 F_3 + 2M_0^4 F_4\,, \qquad \alpha_{4,b} = \frac{1}{4}F_2 + \frac{1}{3}M_0^2 F_3$$

  - $F_n = 8N_c \int \frac{d^3p}{(2\pi)^3}\, T \sum_j \frac{1}{[(i\omega_j+\mu)^2 - \vec{p}^{\,2} - M_0^2]^n}\,, \qquad \omega_j = (2j+1)\pi T$

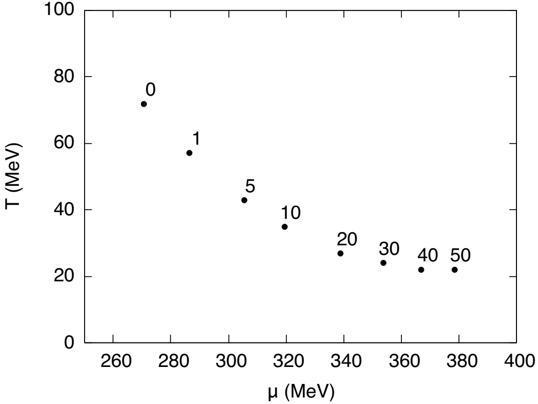
- away from the chiral limit:

  - $M_0 \neq 0 \;\Rightarrow\; \alpha_3 = 4M_0\alpha_{4,b} \;\Rightarrow\;$ CEP = PLP

  The CEP coincides with the PLP!

Results:

- position of the CEP=PLP for different m:



m/MeV	m_π/MeV
0.	0.
1.	43.
5.	96.
10.	135.
20.	191.
30.	235.
40.	271.
50.	303.

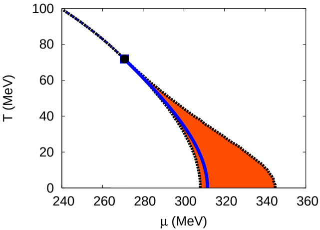
GL results for critical points and Lifshitz points

TECHNISCHE UNIVERSITÄT DARMSTADT

	chiral limit	explicitly broken
NJL model		
QM model		

GL results for critical points and Lifshitz points

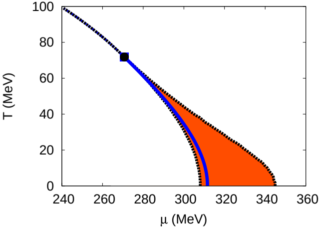
	chiral limit	explicitly broken
NJL model	LP = TCP [Nickel, PRL (2009)]	
QM model		



GL results for critical points and Lifshitz points

TECHNISCHE UNIVERSITÄT DARMSTADT

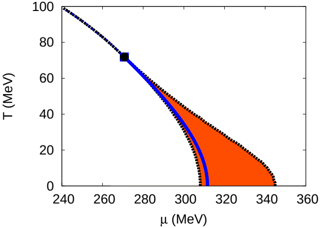
	chiral limit	explicitly broken
NJL model	LP = TCP [Nickel, PRL (2009)]	
QM model	LP = TCP if $m_\sigma = 2\bar{M}$ [MB, Carignano, Schaefer, PRD (2014)]	

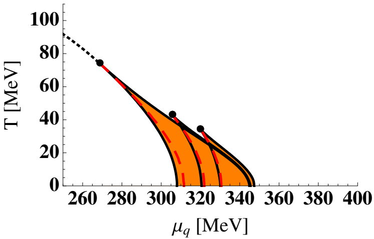


GL results for critical points and Lifshitz points

TECHNISCHE UNIVERSITÄT DARMSTADT

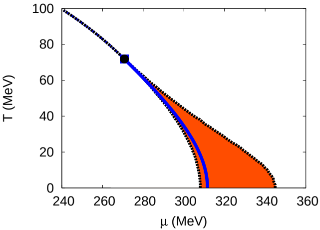
	chiral limit	explicitly broken
NJL model	LP = TCP [Nickel, PRL (2009)]	PLP = CEP [MB, Carignano, PLB (2019)]
QM model	LP = TCP if $m_\sigma = 2\bar{M}$ [MB, Carignano, Schaefer, PRD (2014)]	

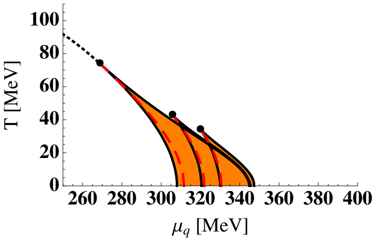




GL results for critical points and Lifshitz points

	chiral limit	explicitly broken
NJL model	LP = TCP [Nickel, PRL (2009)]	PLP = CEP [MB, Carignano, PLB (2019)]
QM model	LP = TCP if $m_\sigma = 2\bar{M}$ [MB, Carignano, Schaefer, PRD (2014)]	PLP = CEP if $m_\sigma = 2\bar{M}$ in the chiral limit [MB, Carignano, Kurth EPJST (2020)]





GL results for critical points and Lifshitz points

	chiral limit	explicitly broken
NJL model	LP = TCP [Nickel, PRL (2009)]	PLP = CEP [MB, Carignano, PLB (2019)]
QM model	LP = TCP if $m_\sigma = 2\bar{M}$ [MB, Carignano, Schaefer, PRD (2014)]	PLP = CEP if $m_\sigma = 2\bar{M}$ in the chiral limit [MB, Carignano, Kurth EPJST (2020)]

- Model results, but independent of model parameters

GL results for critical points and Lifshitz points

TECHNISCHE UNIVERSITÄT DARMSTADT

	chiral limit	explicitly broken
NJL model	LP = TCP [Nickel, PRL (2009)]	PLP = CEP [MB, Carignano, PLB (2019)]
QM model	LP = TCP if $m_\sigma = 2\bar{M}$ [MB, Carignano, Schaefer, PRD (2014)]	PLP = CEP if $m_\sigma = 2\bar{M}$ in the chiral limit [MB, Carignano, Kurth EPJST (2020)]

- Model results, but independent of model parameters

→ Model predictions of an inhomogeneous phase should be taken as seriously as those of a CEP!

Stability analysis

TECHNISCHE UNIVERSITÄT DARMSTADT

Stability analysis

TECHNISCHE UNIVERSITÄT DARMSTADT

- as before:
  Expand the thermodynamic potential in powers of small fluctuations δM around the most stable homogeneous solution M_0
- Contributions of order $(\delta M)^n$:

$$\Omega^{(0)} = -\frac{T}{V}\,\mathbf{Tr}\,\log S_0^{-1} + \frac{1}{V}\int_V d^3x\,\frac{(M_0 - m)^2}{4G}$$

$$\Omega^{(1)} = \frac{T}{V}\,\mathbf{Tr}\,(S_0\delta M) + \frac{M_0 - m}{2G}\,\frac{1}{V}\int_V d^3x\;\delta M(\vec{x})$$

$$\Omega^{(2)} = \frac{1}{2}\,\frac{T}{V}\,\mathbf{Tr}\,(S_0\delta M)^2 + \frac{1}{4G}\,\frac{1}{V}\int_V d^3x\;\delta M^2(\vec{x})$$

$$\Omega^{(n>3)} = \frac{1}{n}\,\frac{T}{V}\,\mathbf{Tr}\,(S_0\delta M)^n$$

Stability analysis

TECHNISCHE UNIVERSITÄT DARMSTADT

- as before:
  Expand the thermodynamic potential in powers of small fluctuations δM around the most stable homogeneous solution M_0
- Contributions of order $(\delta M)^n$:

$$\Omega^{(0)} \quad \text{not relevant in the following}$$

$$\Omega^{(1)} = 0 \quad \text{by the gap equation}$$

$$\Omega^{(2)} = \frac{1}{2}\frac{T}{V}\,\mathbf{Tr}\,(S_0\delta M)^2 + \frac{1}{4G}\frac{1}{V}\int_V d^3x\,\delta M^2(\vec{x})$$

$$\Omega^{(n>3)} \quad \text{not relevant in the following}$$

Quadratic contribution

TECHNISCHE UNIVERSITÄT DARMSTADT

- $\Omega^{(2)} = \frac{1}{2}\frac{T}{V}\,\mathbf{Tr}\,(S_0\delta M)^2 + \frac{1}{4G}\frac{1}{V}\int\limits_V d^3x\;\delta M^2(\vec{x})$
- functional trace:

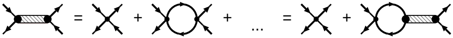
$$\mathbf{Tr}\,(S_0\delta M)^2 = 2N_c\int d^4x\,d^4x'\,\mathrm{tr_D}\left[S_0(x,x')\,\delta M(\vec{x})\,S_0(x',x)\,\delta M(\vec{x})\right]$$

Quadratic contribution

- $\Omega^{(2)} = \frac{1}{2}\frac{T}{V}\,\mathbf{Tr}\,(S_0\delta M)^2 + \frac{1}{4G}\frac{1}{V}\int\limits_V d^3x\,\delta M^2(\vec{x})$
- functional trace:

  $\mathbf{Tr}\,(S_0\delta M)^2 = 2N_c\int d^4x\,d^4x'\,\mathrm{tr}_D\left[S_0(x,x')\,\delta M(\vec{x})\,S_0(x',x)\,\delta M(\vec{x})\right]$
- Evaluate in momentum space without gradient expansion:

  $\Omega^{(2)} = \frac{1}{2V}\int\frac{d^3q}{(2\pi)^3}\,|\delta M(\vec{q})|^2\,\Gamma_S^{-1}(q)$
  - $\Gamma_S^{-1}(q) \propto$ inverse sigma propagator at $q = \begin{pmatrix} 0 \\ \vec{q} \end{pmatrix}$

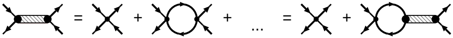
    
  - unstable region: $\Gamma_S^{-1}(q) < 0$

Quadratic contribution

- $\Omega^{(2)} = \frac{1}{2}\frac{T}{V}\,\mathbf{Tr}\,(S_0 \delta M)^2 + \frac{1}{4G}\frac{1}{V}\int_V d^3x\,\delta M^2(\vec{x})$
- functional trace:

  $\mathbf{Tr}\,(S_0\delta M)^2 = 2N_c \int d^4x\, d^4x'\, \mathrm{tr}_D\left[S_0(x,x')\,\delta M(\vec{x})\,S_0(x',x)\,\delta M(\vec{x})\right]$
- Evaluate in momentum space without gradient expansion:

  $\Omega^{(2)} = \frac{1}{2V}\int \frac{d^3q}{(2\pi)^3}\,|\delta M(\vec{q})|^2\,\Gamma_S^{-1}(q)$
  - $\Gamma_S^{-1}(q) \propto$ inverse sigma propagator at $q = \begin{pmatrix} 0 \\ \vec{q} \end{pmatrix}$

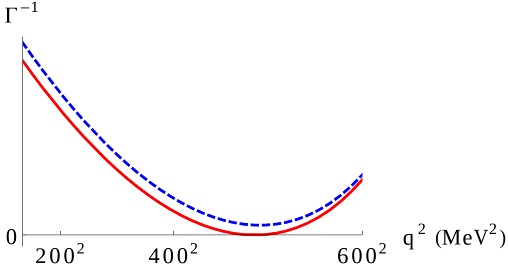
    

  - unstable region: $\Gamma_S^{-1}(q) < 0$
- including pseudoscalar fluctuations δP:

  analogous expressions involving $\Gamma_P^{-1}(q) \propto$ inverse pion propagator

Example

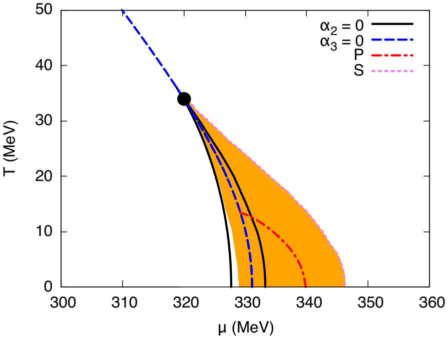
- inverse meson propagators for m = 10 MeV, T = 10 MeV, μ = 344 MeV:
  [MB, S. Carignano, PLB (2018)]



- red: Γ_S^{-1} ⟶ marginally unstable (phase boundary) w.r.t. δS at $|\vec{q}| \sim$ 500 MeV
- blue: Γ_P^{-1} ⟶ stable w.r.t. δP

Phasediagram

[MB, S. Carignano, PLB (2018)]

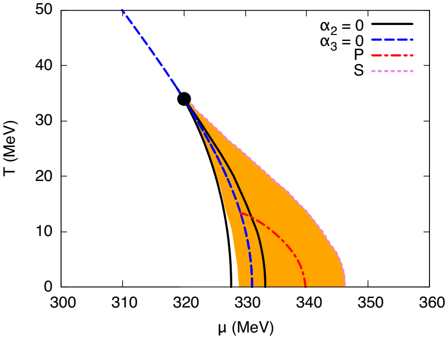


- dominant instability in the scalar channel

Phasediagram

[MB, S. Carignano, PLB (2018)]

TECHNISCHE UNIVERSITÄT DARMSTADT

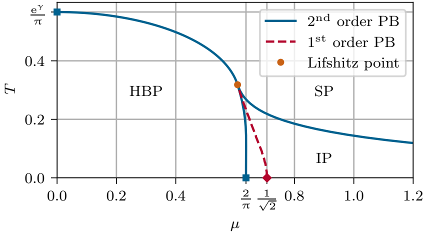


- orange region: RKC favored
- instability region $<$ RKC region (not shown)
  - “right phase” boundaries agree
  - stability analysis misses instabilites in the homogeneous broken regime w.r.t. large fluctuations

- dominant instability in the scalar channel

Are the inhomogeneous phases regularization artifacts?

Are the inhomogeneous phases regularization artifacts?

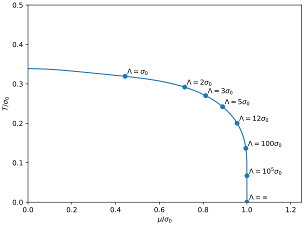


from [Koenigstein et al. (2022)]

- 1 + 1 dim Gross-Neveu model:
  - inhomogeneous phase in the renormalized limit [Thies et al.]

Are the inhomogeneous phases regularization artifacts?

TECHNISCHE UNIVERSITÄT DARMSTADT

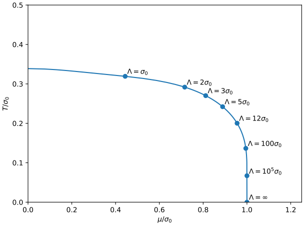


[MB, Kurth, Wagner Winstel; PRD (2021)]

- $1+1$ dim Gross-Neveu model:
  - inhomogeneous phase in the renormalized limit [Thies et al.]
- $2+1$ dim Gross-Neveu model:
  - IP for finite Λ
  - disappears for $\Lambda \to \infty$

Are the inhomogeneous phases regularization artifacts?

TECHNISCHE UNIVERSITÄT DARMSTADT



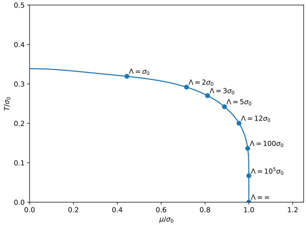
[MB, Kurth, Wagner Winstel; PRD (2021)]

- $1+1$ dim Gross-Neveu model:
  - inhomogeneous phase in the renormalized limit [Thies et al.]
- $2+1$ dim Gross-Neveu model:
  - IP for finite Λ
  - disappears for $\Lambda \to \infty$

- Then how about $3+1$ dim GN /NJL ?
  - non-renormalizable ➞ cutoff must be kept finite
  - strong regulator dependecies [Pannullo, Wagner, Winstel PoS LATTICE2022]
  - No IP in GN with $2 \leq d < 3 - \varepsilon$ spatial dimensions [Pannullo, PRD (2023)]

Are the inhomogeneous phases regularization artifacts?

TECHNISCHE UNIVERSITÄT DARMSTADT



[MB, Kurth, Wagner Winstel; PRD (2021)]

- $1+1$ dim Gross-Neveu model:
  - inhomogeneous phase in the renormalized limit [Thies et al.]
- $2+1$ dim Gross-Neveu model:
  - IP for finite Λ
  - disappears for $\Lambda \to \infty$
- Then how about $3+1$ dim GN /NJL ?
  - non-renormalizable → cutoff must be kept finite
  - strong regulator dependecies [Pannullo, Wagner, Winstel PoS LATTICE2022]
  - No IP in GN with $2 \leq d < 3 - \varepsilon$ spatial dimensions [Pannullo, PRD (2023)]
- $3+1$ dim QM model:
  IP survives $\Lambda \to \infty$, but potential not bounded from below

Are the inhomogeneous phases regularization artifacts?

TECHNISCHE UNIVERSITÄT DARMSTADT

- But maybe the cutoff contains some physics ...

Are the inhomogeneous phases regularization artifacts?

TECHNISCHE UNIVERSITÄT DARMSTADT

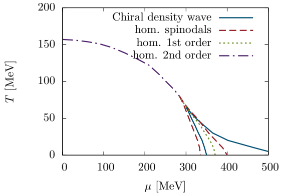
- But maybe the cutoff contains some physics ...

  Does the cutoff mimic asymptotic freedom?

Are the inhomogeneous phases regularization artifacts?

- But maybe the cutoff contains some physics ...
  Does the cutoff mimic asymptotic freedom?

- Indications of an inhomogeneous chiral phase in QCD from DSEs (CDW-like ansatz)
  [D. Müller et al., PLB (2013)]



Are the inhomogeneous phases regularization artifacts?

TECHNISCHE UNIVERSITÄT DARMSTADT

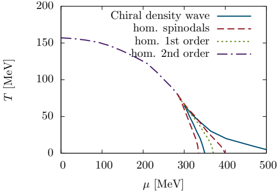
- But maybe the cutoff contains some physics ...

  Does the cutoff mimic asymptotic freedom?

- Indications of an inhomogeneous chiral phase in QCD from DSEs (CDW-like ansatz) [D. Müller et al., PLB (2013)]

- Ongoing work towards a QCD stability analysis [Motta et al., arXiv:2306.09749]

  $\rightarrow$ Theo Motta's talk on Tuesday



Conclusion

TECHNISCHE UNIVERSITÄT DARMSTADT

- Chiral models can give us hints about interesting features of the QCD phase diagram:
  - the critical endpoint
  - color-superconducting phases
  - inhomogeneous phases
  - ...
- They are not suited for quantitative predictions of them, but they have inspired more sophisticated (QCD based) investigations and are useful benchmarks for them.