Towards the cold and dense regime of QCD with effective lattice theories

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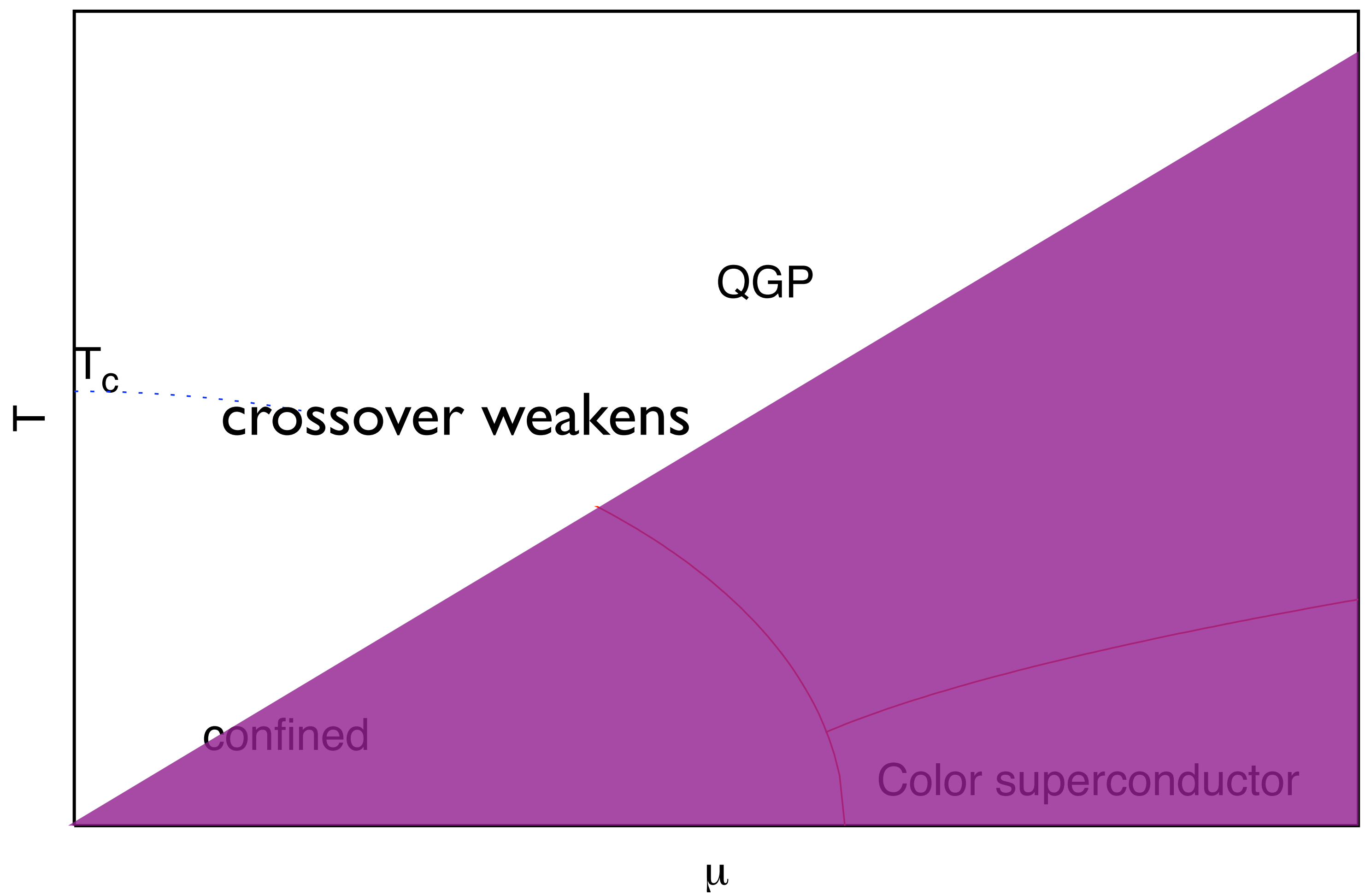
- QCD phase diagram and sign problem
- Towards cold and dense QCD: effective lattice theories

HIC for FAIR Helmholtz International Center

NIC

CRC-TR 211 Strong-interaction matter under extreme conditions

The lattice-calculable region of the phase diagram



- Sign problem prohibits direct simulation, circumvented by approximate methods: reweigthing, Taylor expansion, imaginary chem. pot., need $\mu/T \lesssim 1 \quad (\mu = \mu_B/3)$
- No critical point in the controllable region, some signals beyond

Effective lattice theories for finite density

- General idea: two-step treatment
- I. Derivation of effective theory from LQCD by expansion methods $\sim \frac{1}{g^2}, \frac{1}{m_q}$
- Part of d.o.f's integrated out, sign problem becomes milder
- II. Simulate effective theory (flux rep. + worm algorithm, complex Langevin);

  or solve analytically by "high T expansion" techniques from Stat. Mech.

Two possibilities for effective degrees of freedom:

$$Z = \int DU_0 DU_i \, (\det Q)^{N_f} \, e^{S_g[U]} = \int DU_0 \, e^{S_{eff}[U_0]} = \int DL \, e^{S_{eff}[L]}$$ Polyakov loops

$$Z = \int DU D\bar{\psi} D\psi \, e^{S_g[U]+S_f[\bar{\psi},\psi,U]} = \int D\bar{\psi} D\psi \, e^{S_{eff}[\bar{\psi},\psi]}$$ Baryons and mesons

These formulations are, in principle, exact; in practice truncations of systematic expansions

Polyakov loops: start from Wilson's lattice action

Pure gauge part: character expansion

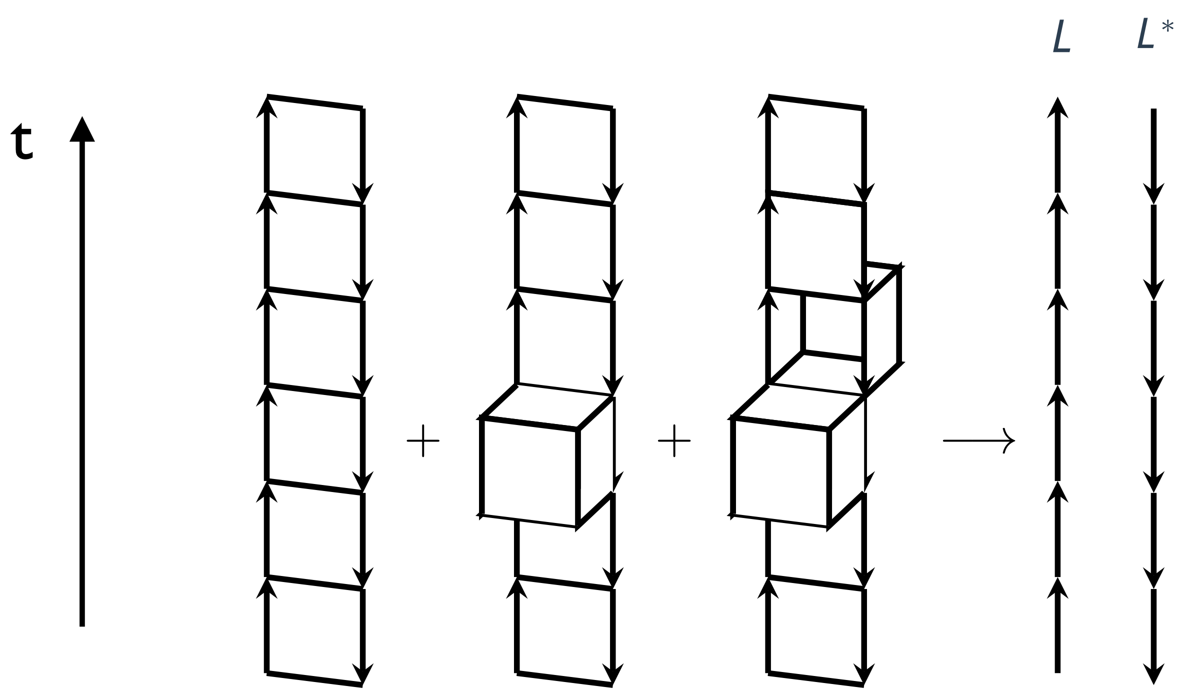
$$u(\beta) = \frac{\beta}{18} + \frac{\beta^2}{216} + \ldots < 1$$

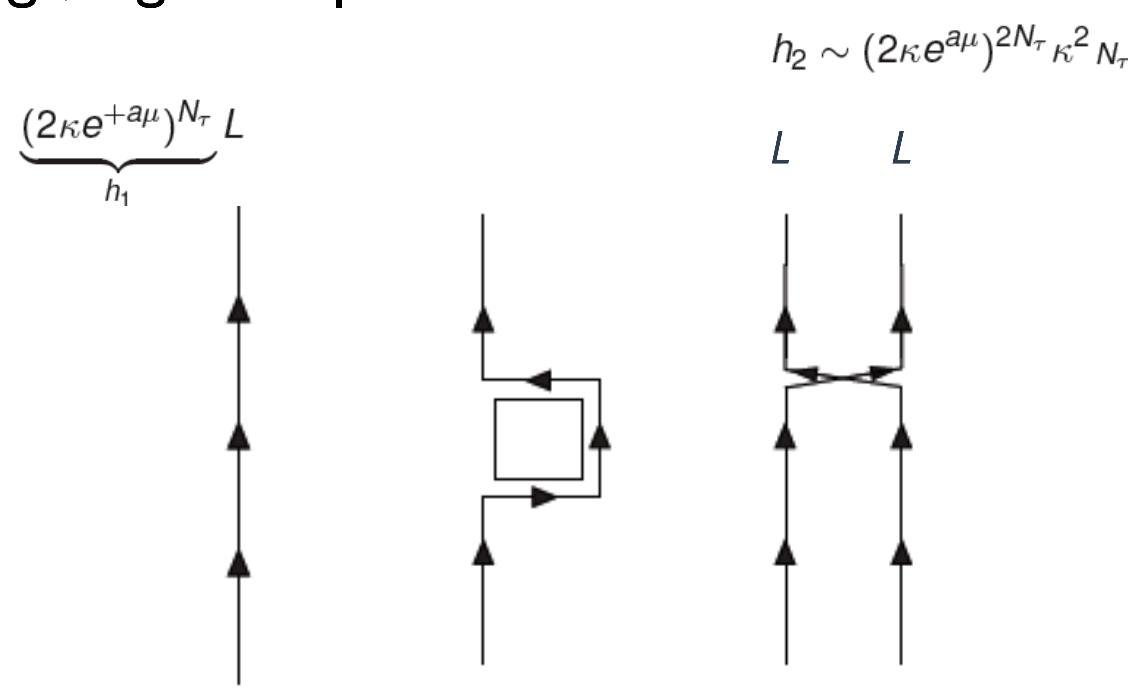
$$\beta = \frac{2N}{g^2}$$

Fermion determinant: hopping expansion

$$\kappa = \frac{1}{2am+8}$$

Generates couplings over all distances, n-pt. couplings, higher reps.... :





$$\lambda(u, N_\tau \geq 5) \;=\; u^{N_\tau} \exp\left[N_\tau\left(4u^4 + 12u^5 - 14u^6 - 36u^7 + \frac{295}{2}u^8 + \frac{1851}{10}u^9 + \frac{1055797}{5120}u^{10}\right)\right]$$

The effective 3d theory

$$-S_{\text{eff}} = \sum_i \lambda_i(u,\kappa,N_\tau) S_i^{\text{S}} - 2N_f \sum_i \left[h_i(u,\kappa,\mu,N_\tau) S_i^{\text{A}} + \bar{h}_i(u,\kappa,\mu,N_\tau) S_i^{\dagger\text{A}} \right]$$

effective couplings

$$S_i^{A,S} = S_i^{A,S}[L, L^*]$$

This is a 3d continuous spin model!

"Duality transformation": $\beta = \frac{2N}{g^2} \longleftrightarrow u = \frac{\beta}{18} + O(\beta^2) < 1$

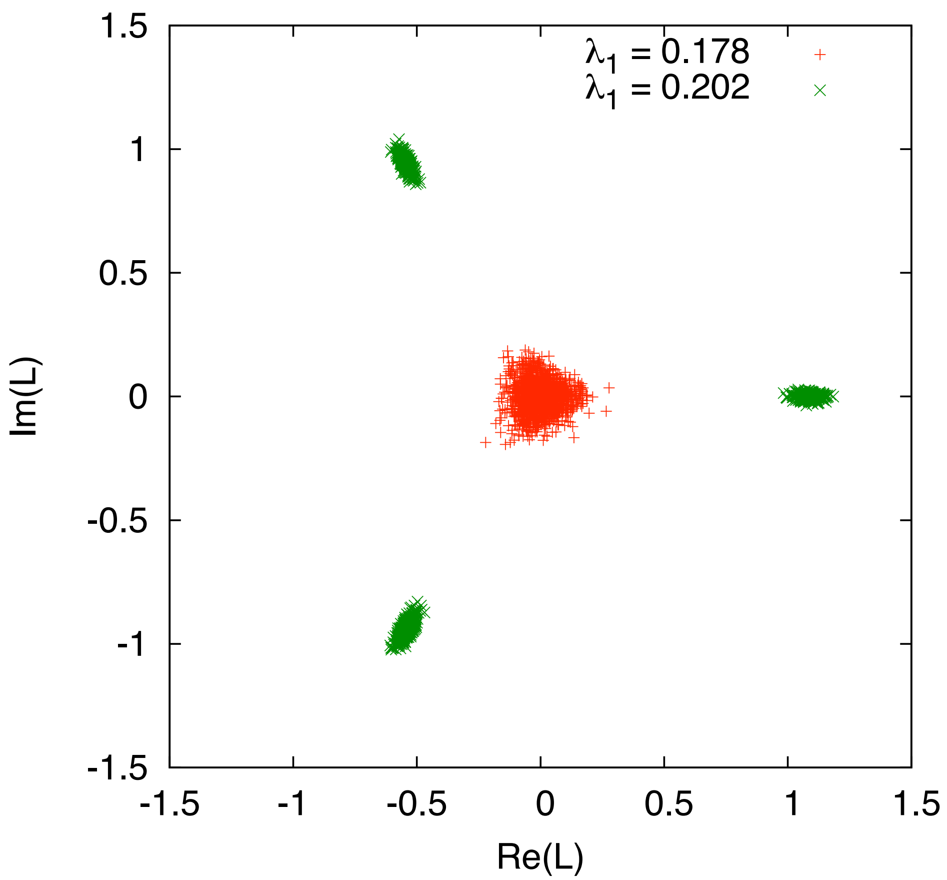
$$L = \mathrm{Tr} W$$

$$Z = \int DW \prod_{<\mathbf{x},\mathbf{y}>} \left[1 + \lambda(L_\mathbf{x} L_\mathbf{y}^* + L_\mathbf{x}^* L_\mathbf{y})\right]$$

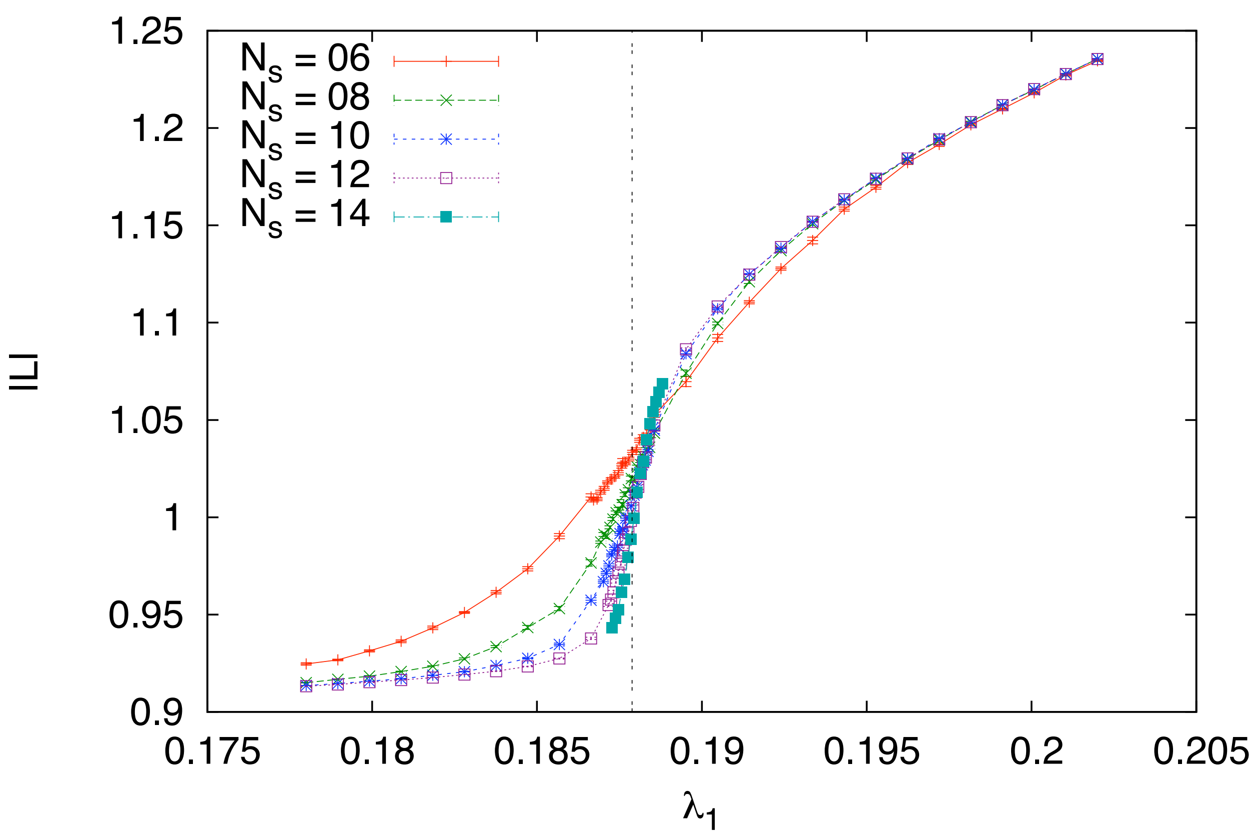
$$\times \prod_\mathbf{x} [1 + h_1 L_\mathbf{x} + h_1^2 L_\mathbf{x}^* + h_1^3]^{2N_f} [1 + \bar{h}_1 L_\mathbf{x}^* + \bar{h}_1^2 L_\mathbf{x} + \bar{h}_1^3]^{2N_f}$$

$$\times \prod_{<\mathbf{x},\mathbf{y}>} \left(1 - h_2 \mathrm{Tr}\frac{h_1 W_\mathbf{x}}{1 + h_1 W_\mathbf{x}} \mathrm{Tr}\frac{h_1 W_\mathbf{y}}{1 + h_1 W_\mathbf{y}}\right)\left(1 - h_2 \mathrm{Tr}\frac{\bar{h}_1 W_\mathbf{x}^\dagger}{1 + \bar{h}_1 W_\mathbf{x}^\dagger} \mathrm{Tr}\frac{\bar{h}_1 W_\mathbf{y}^\dagger}{1 + \bar{h}_1 W_\mathbf{y}^\dagger}\right) \quad \ldots.$$

Example SU(3) Yang-Mills

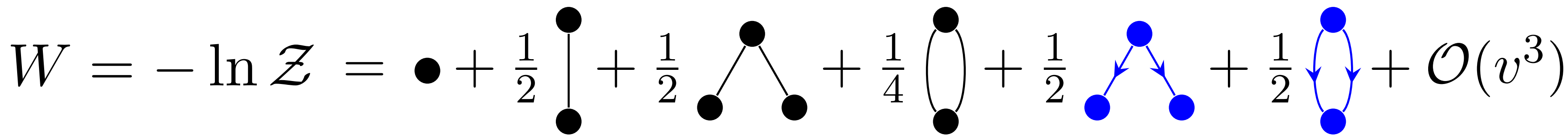


Order-disorder transition
=Z(3) breaking

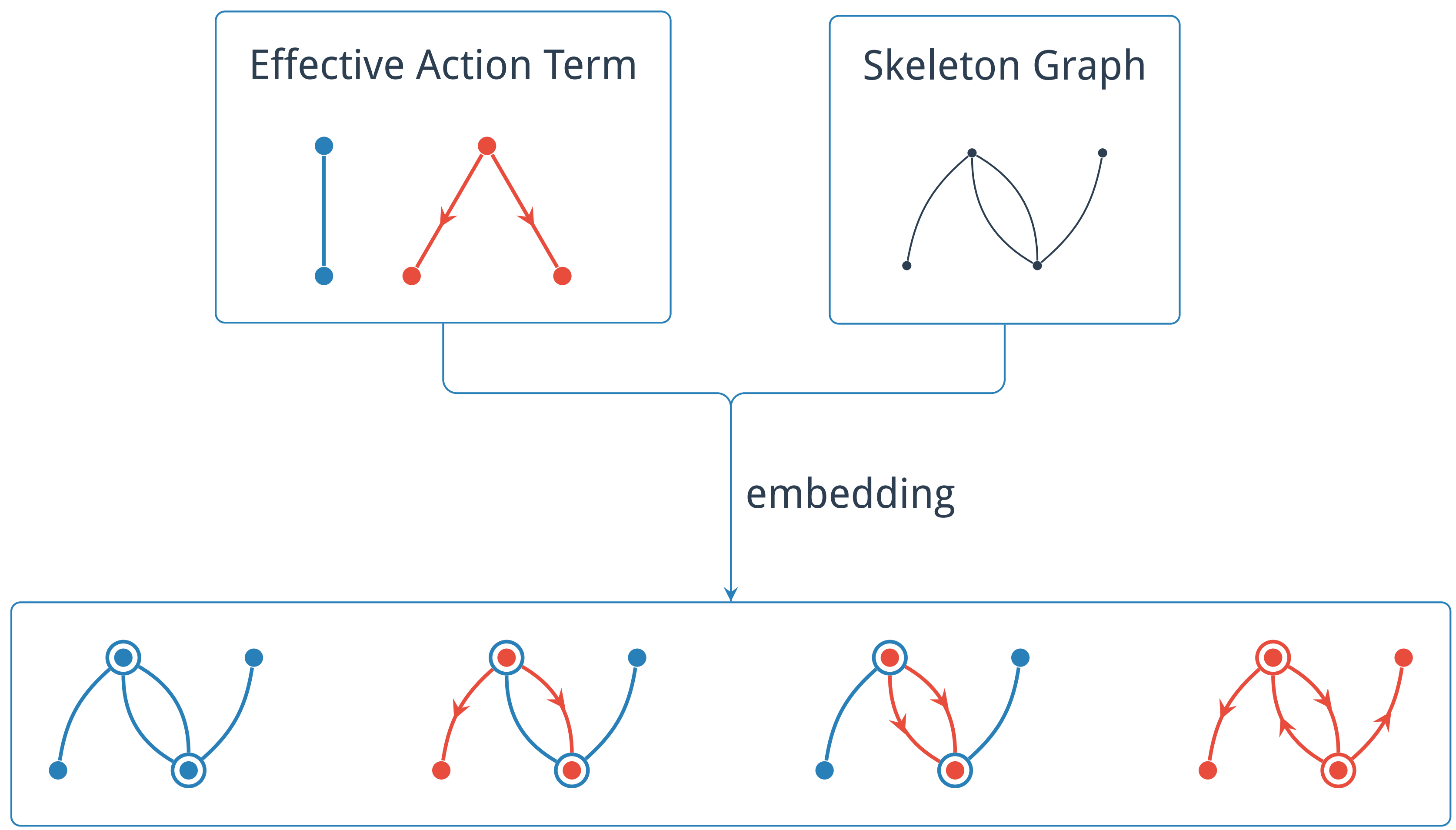


Linked cluster expansion of effective theory

$$\mathcal{Z} = \int \mathcal{D}\phi\, e^{-S_0[\phi]+\frac{1}{2}\sum v_{ij}(x,y)\phi_i(x)\phi_j(y)+\frac{1}{3!}\sum u_{ijk}(x,y,z)\phi_i(x)\phi_j(y)\phi_k(z)+\dots}$$

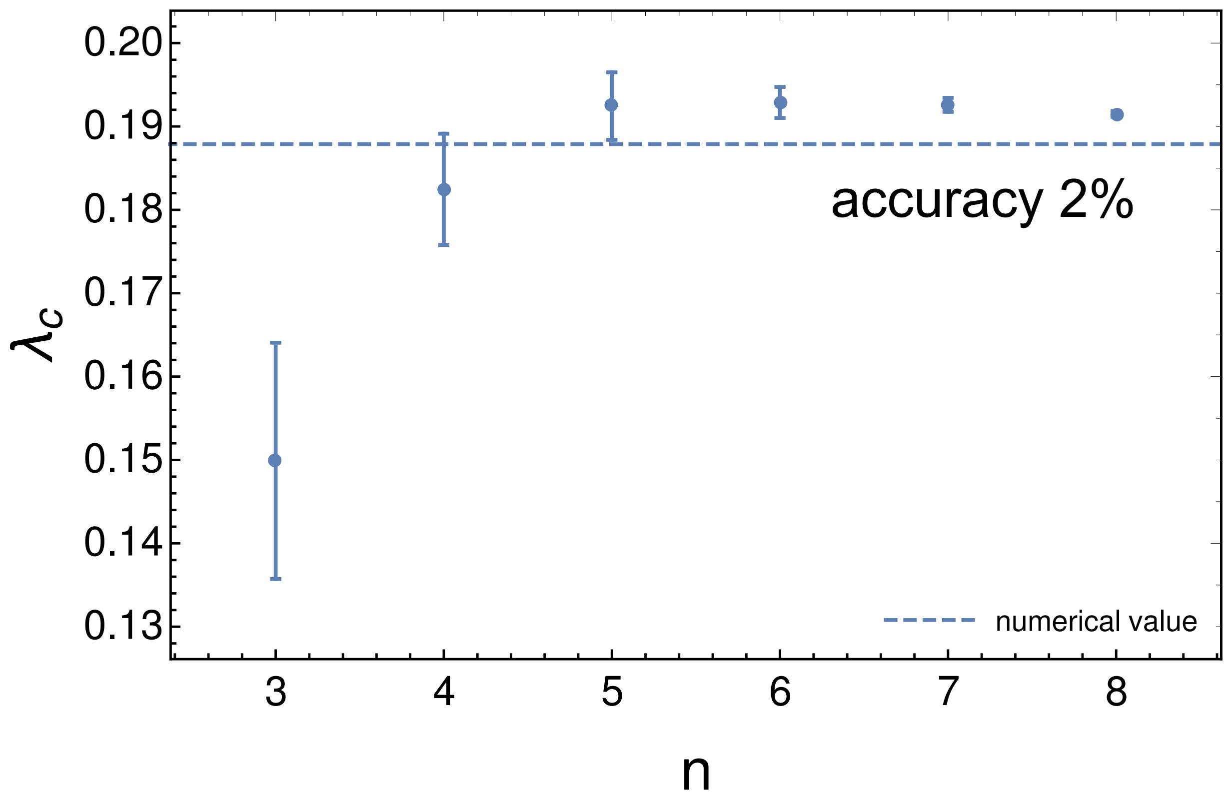


Mapping of the effective theory by embedding: Glesaaen, Neuman, O.P. 15



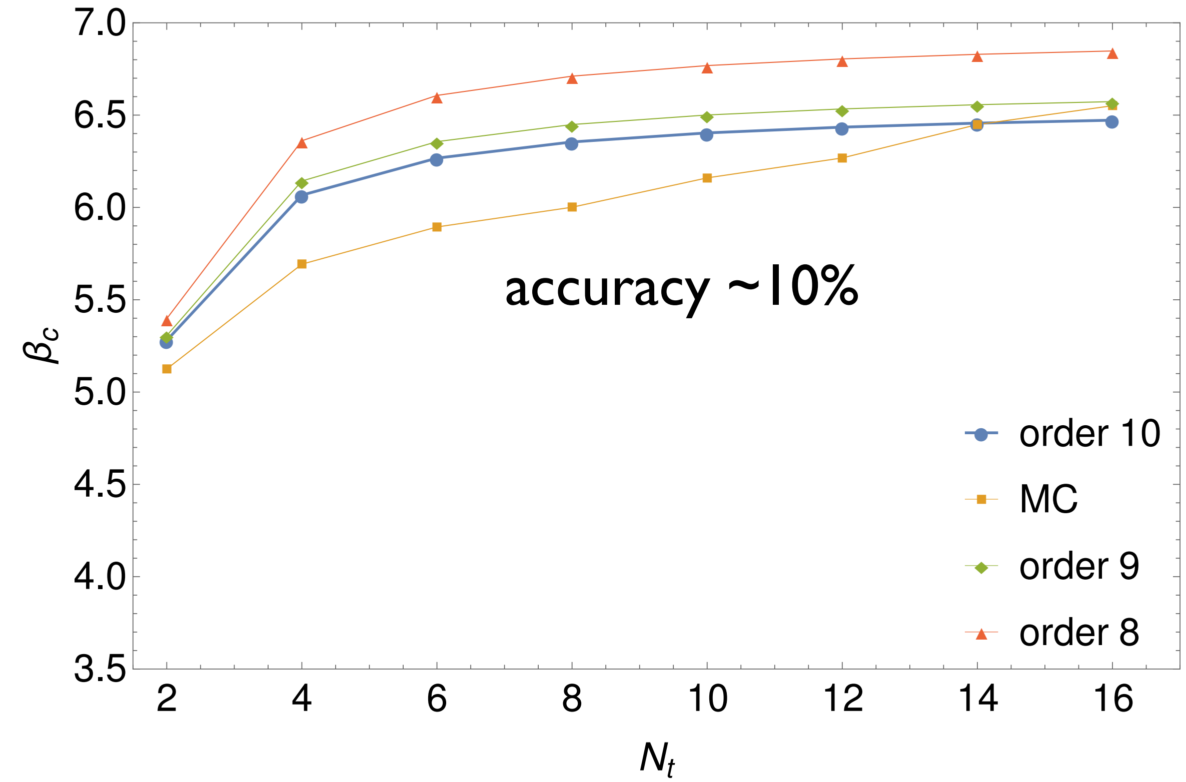
Yang-Mills fully analytic

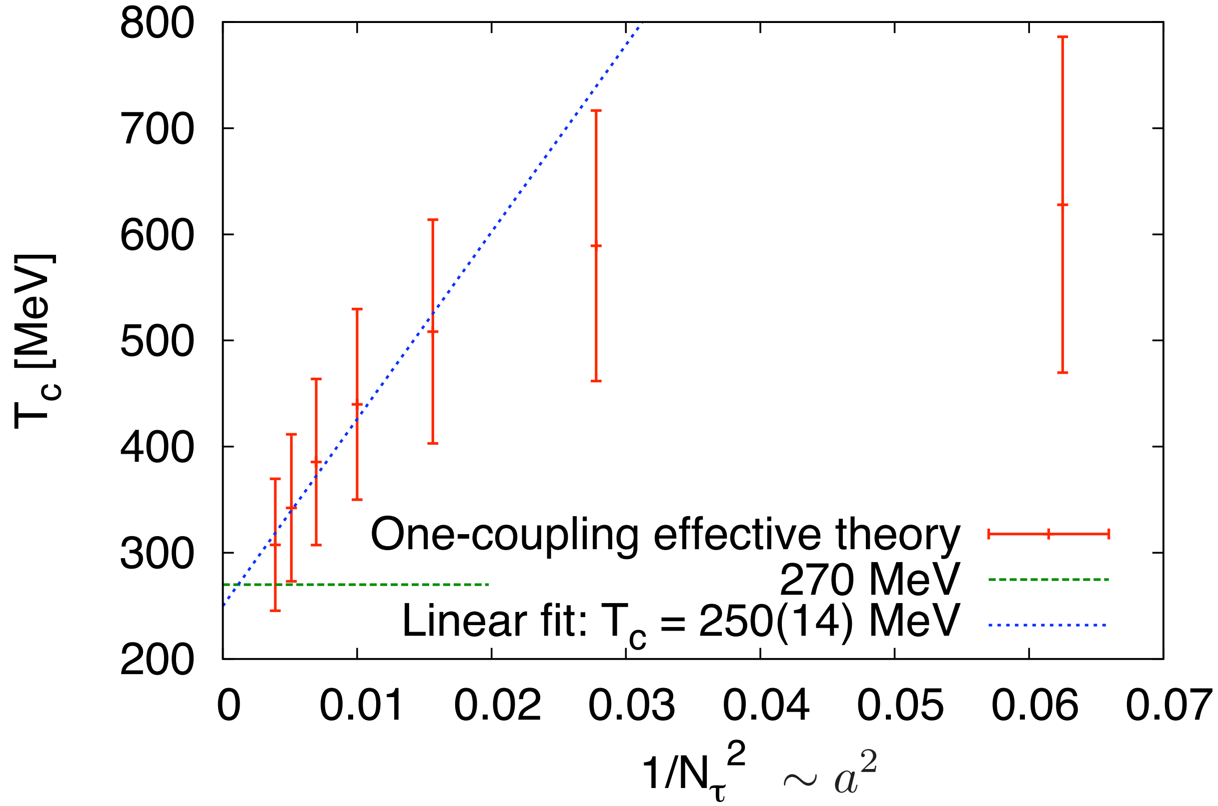
Solution of eff.th.



order of expansion

Conversion to 4d YM

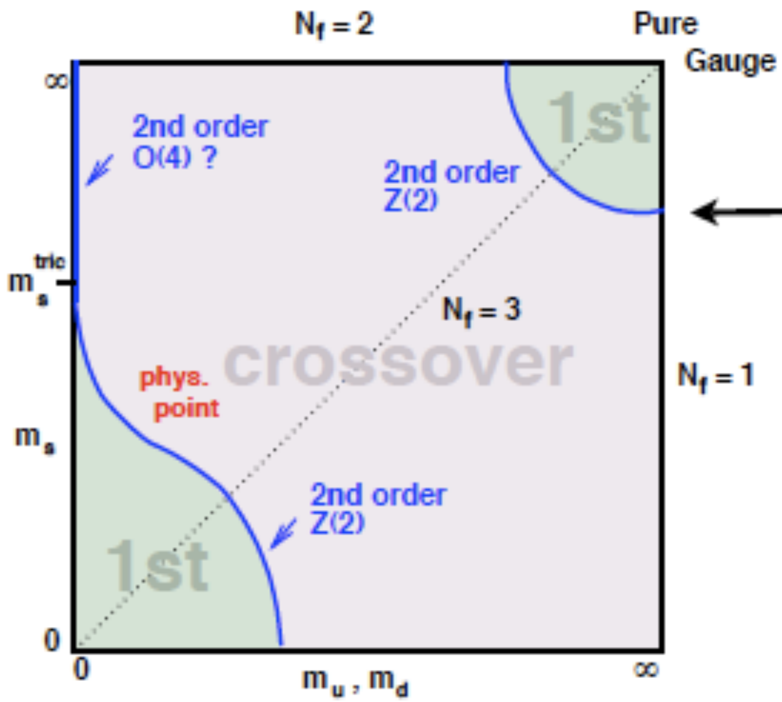




Two calculations:

1. by "hand" (Q. Pham, J. Scheunert, GU)
2. automatic graph generation (J. Kim, GU)

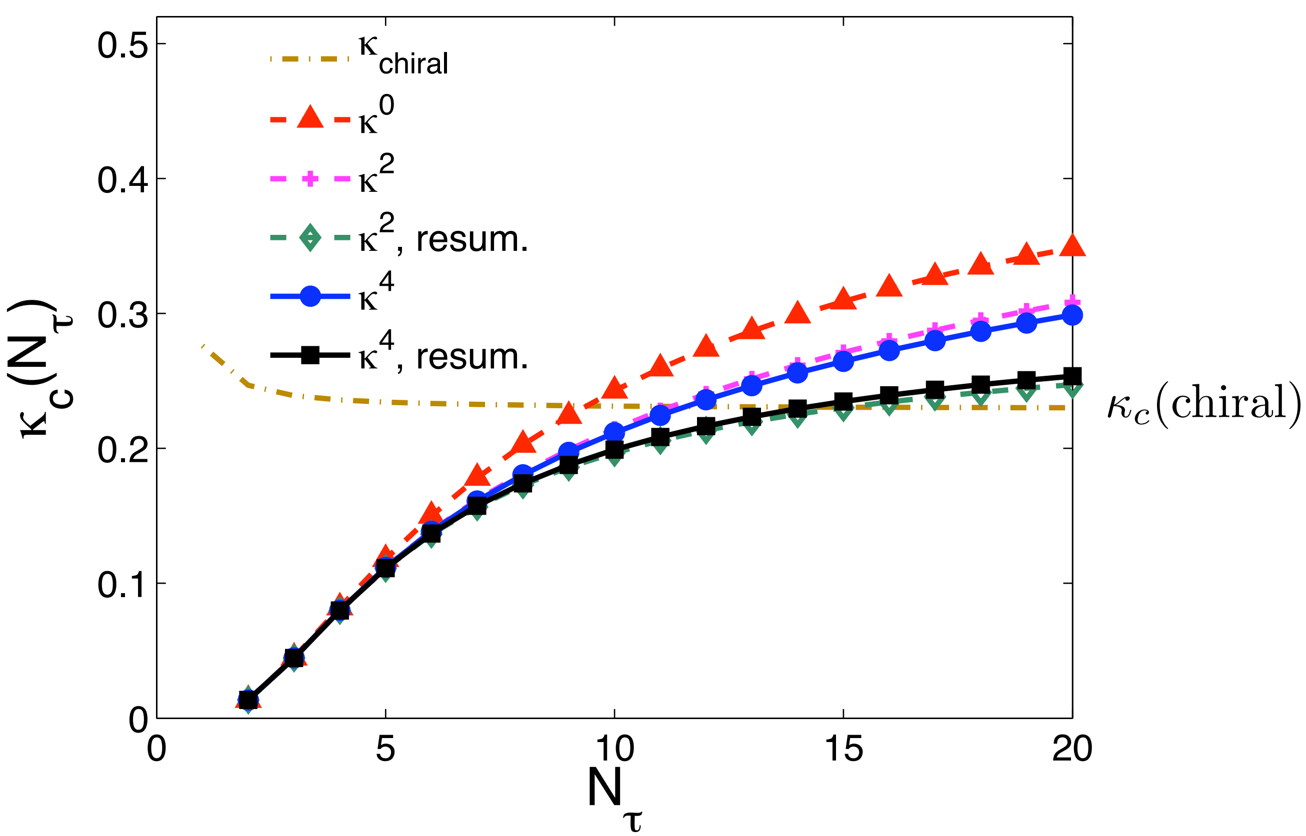
The deconfinement transition for heavy quarks



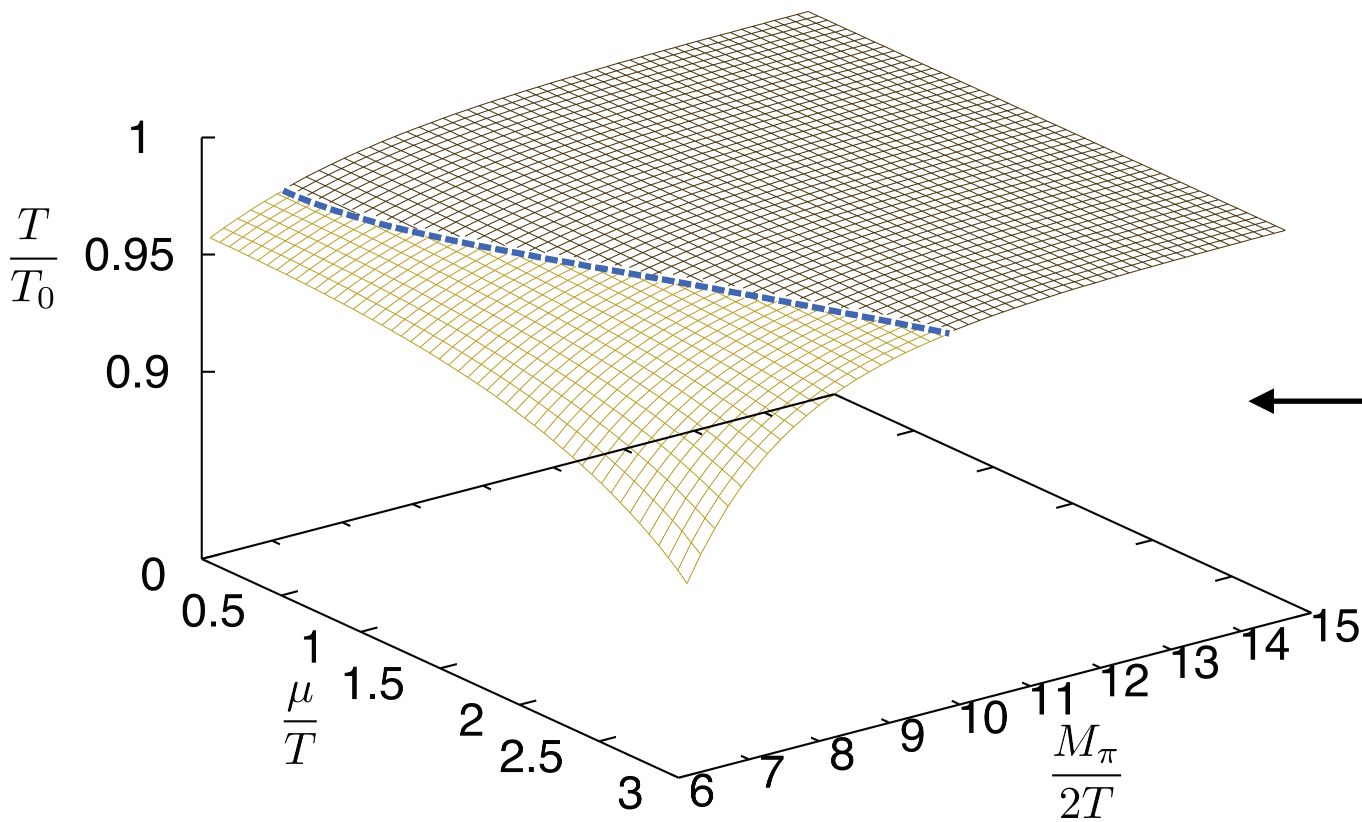
		eff. theory	4d MC, WHOT	4d MC, de Forcrand et al
N_f	M_c/T	$\kappa_c(N_\tau = 4)$	$\kappa_c(4)$, Ref. [23]	$\kappa_c(4)$, Ref. [22]
1	7.22(5)	0.0822(11)	0.0783(4)	~ 0.08
2	7.91(5)	0.0691(9)	0.0658(3)	–
3	8.32(5)	0.0625(9)	0.0595(3)	–

Accuracy ~5%, predictions for Nt=6,8,... available!

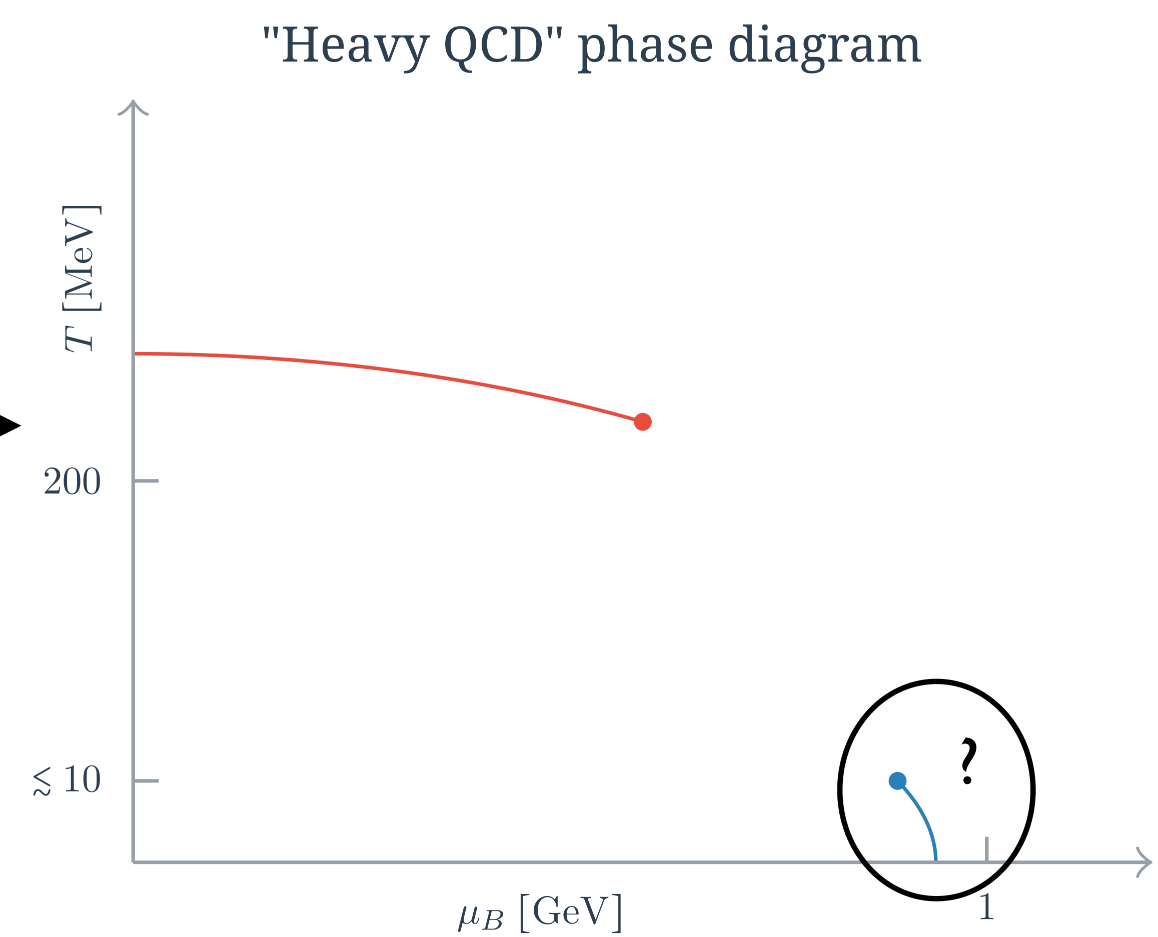
Not yet good enough for continuum extrapolation



The deconfinement transition at finite density



Fromm, Langelage, Lottini, O.P. 11

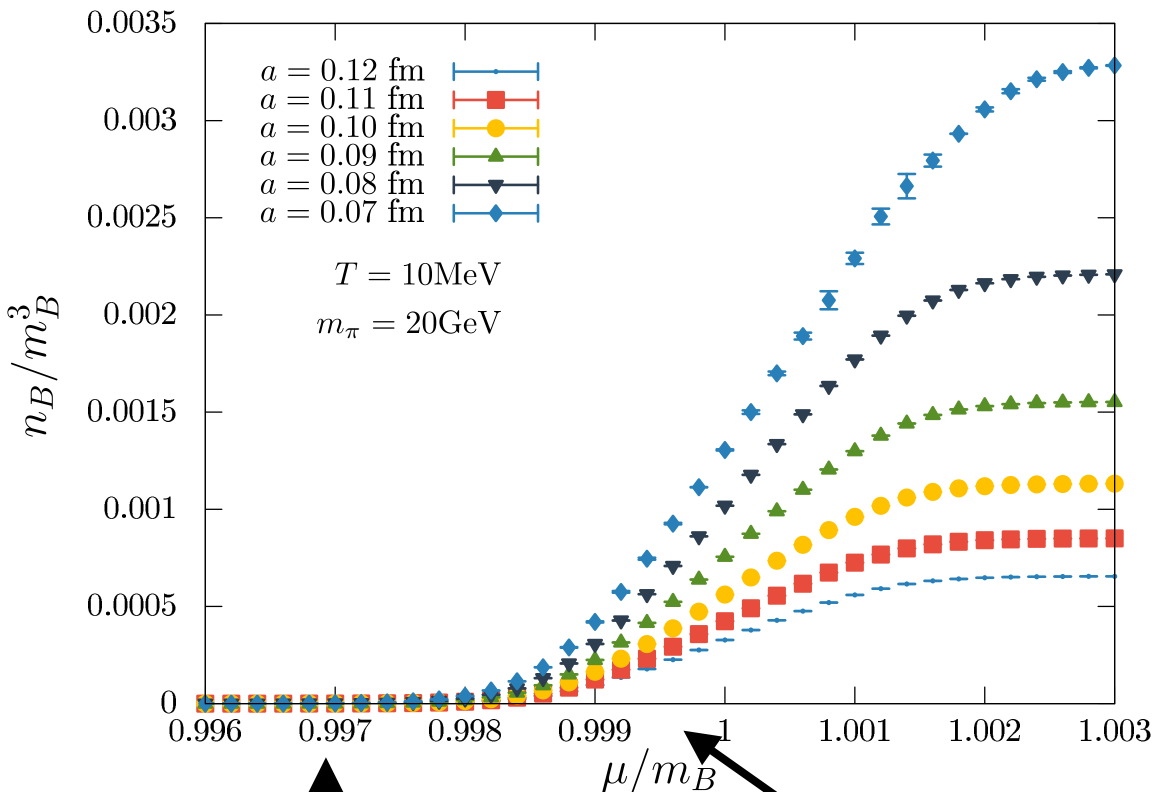


Continuum, functional methods:

Fischer, Lücker, Pawlowski 15

The cold and dense regime

Analytic calculation through $\sim u^5 \kappa^8$



Lattice saturation $a^3 n_B \sim 2N_c N_f$

same for isospin, quark matter

silver blaze phenomenon

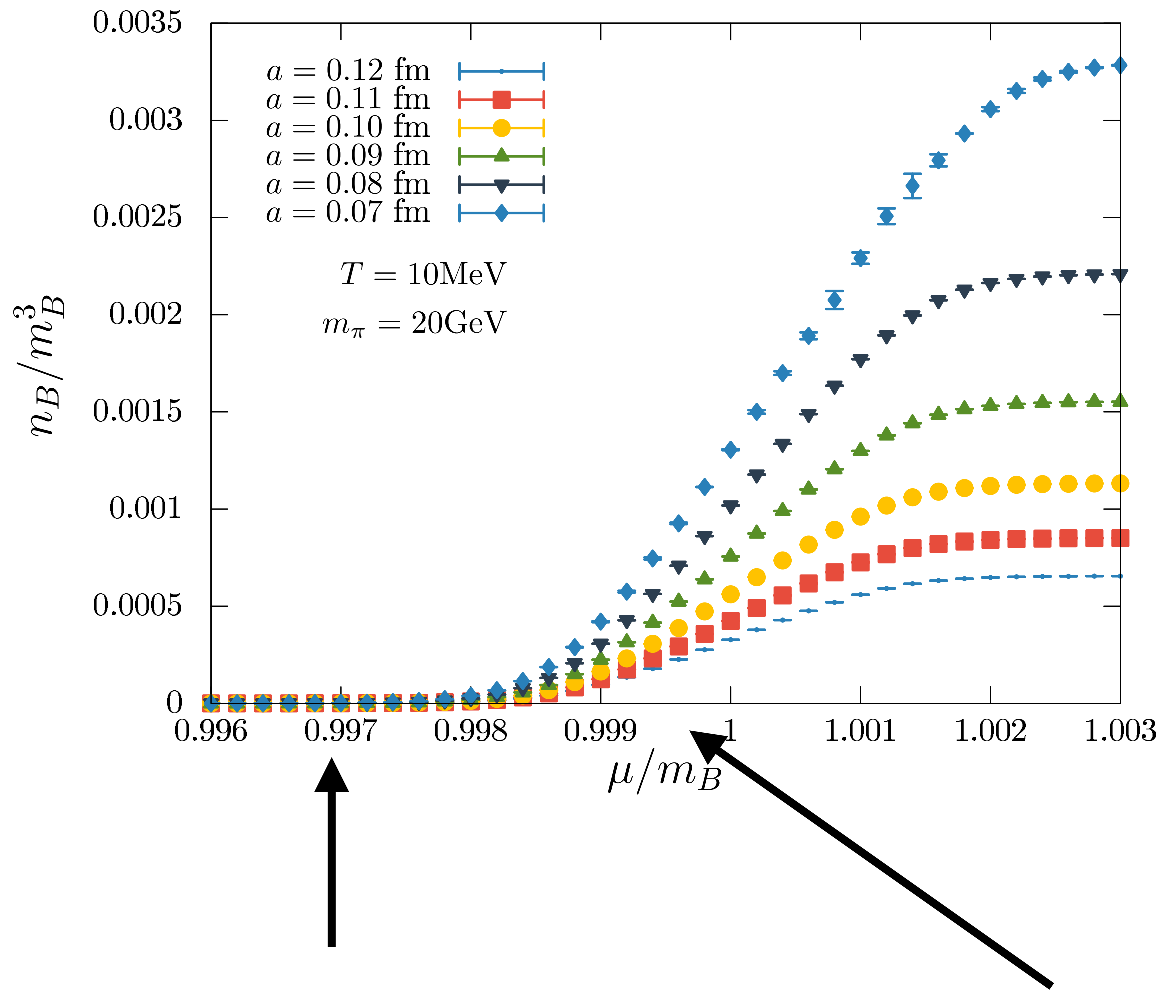
binding energy between nucleons

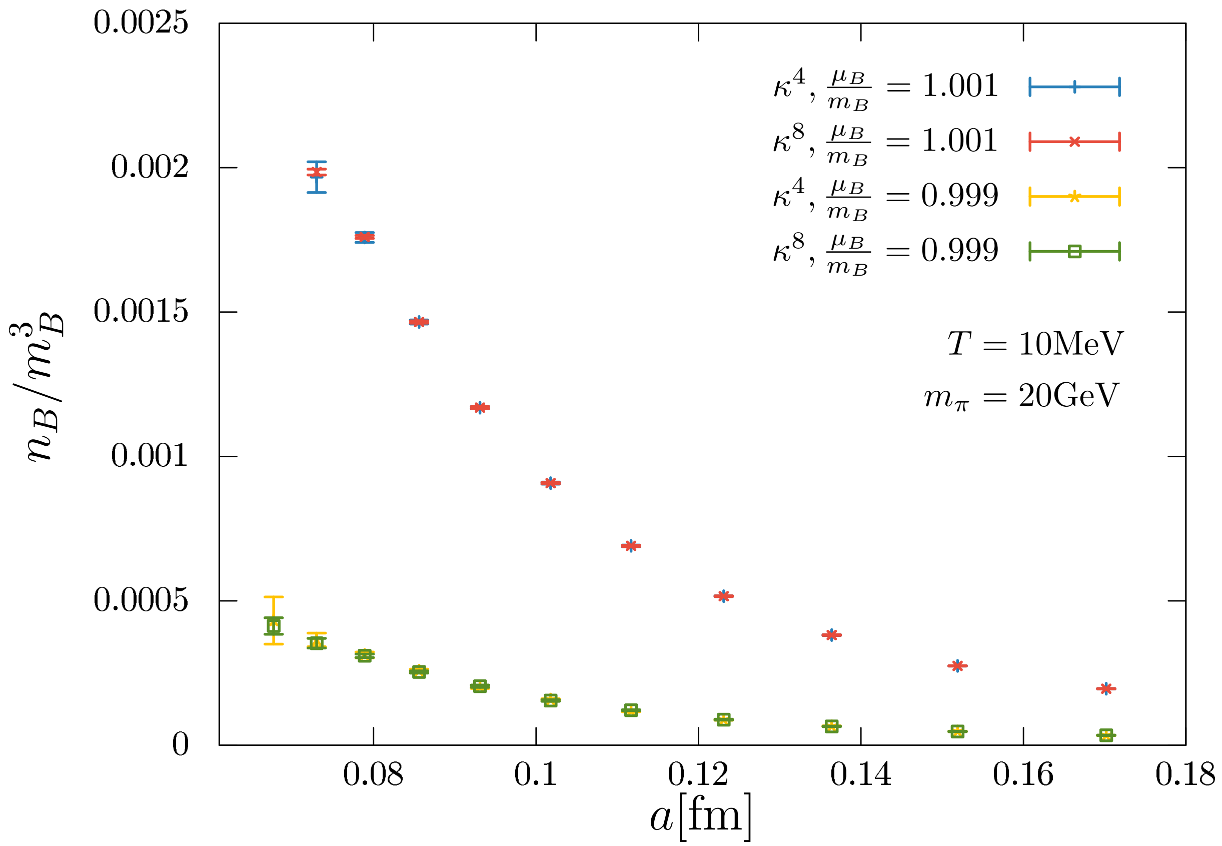
different for isospin, baryon matter

The cold and dense regime

Analytic calculation through $\sim u^5 \kappa^8$

Continuum extrapolation





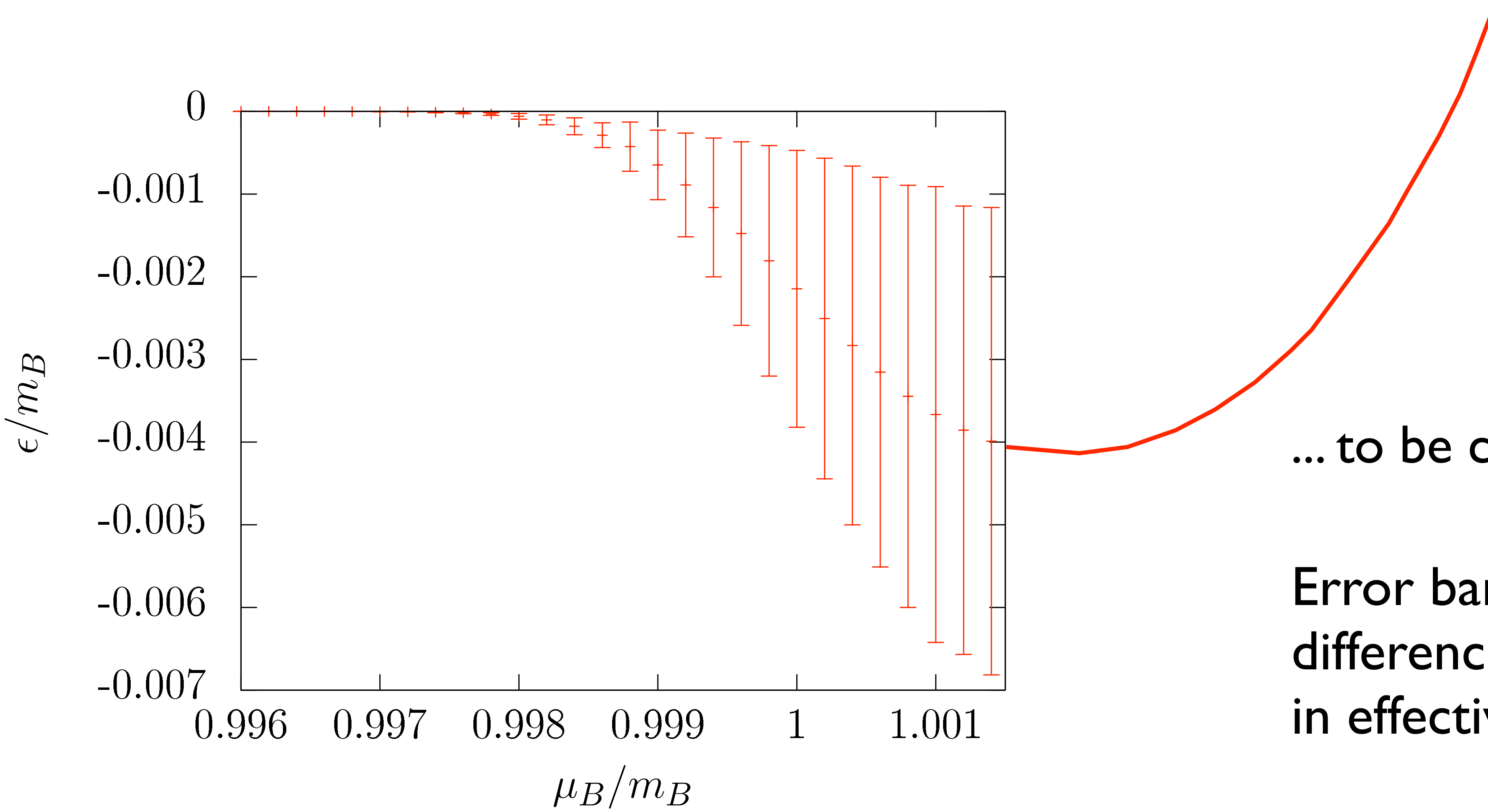
silver blaze phenomenon

binding energy between nucleons

different for isospin, baryon matter

Binding energy per nucleon

$$T \to 0: \qquad \epsilon = \frac{e - n_B m_B}{n_B m_B} = \frac{e}{n_B m_B} - 1 \overset{LO}{\sim} \kappa^2 \sim e^{-a m_{\rm meson}}$$



... to be continued...

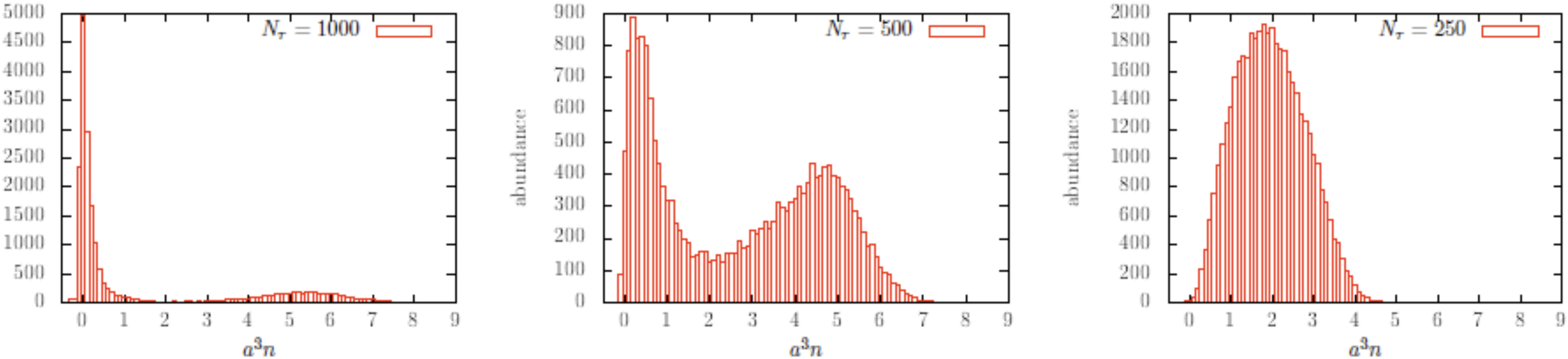
Error bars systematic: difference between orders in effective action

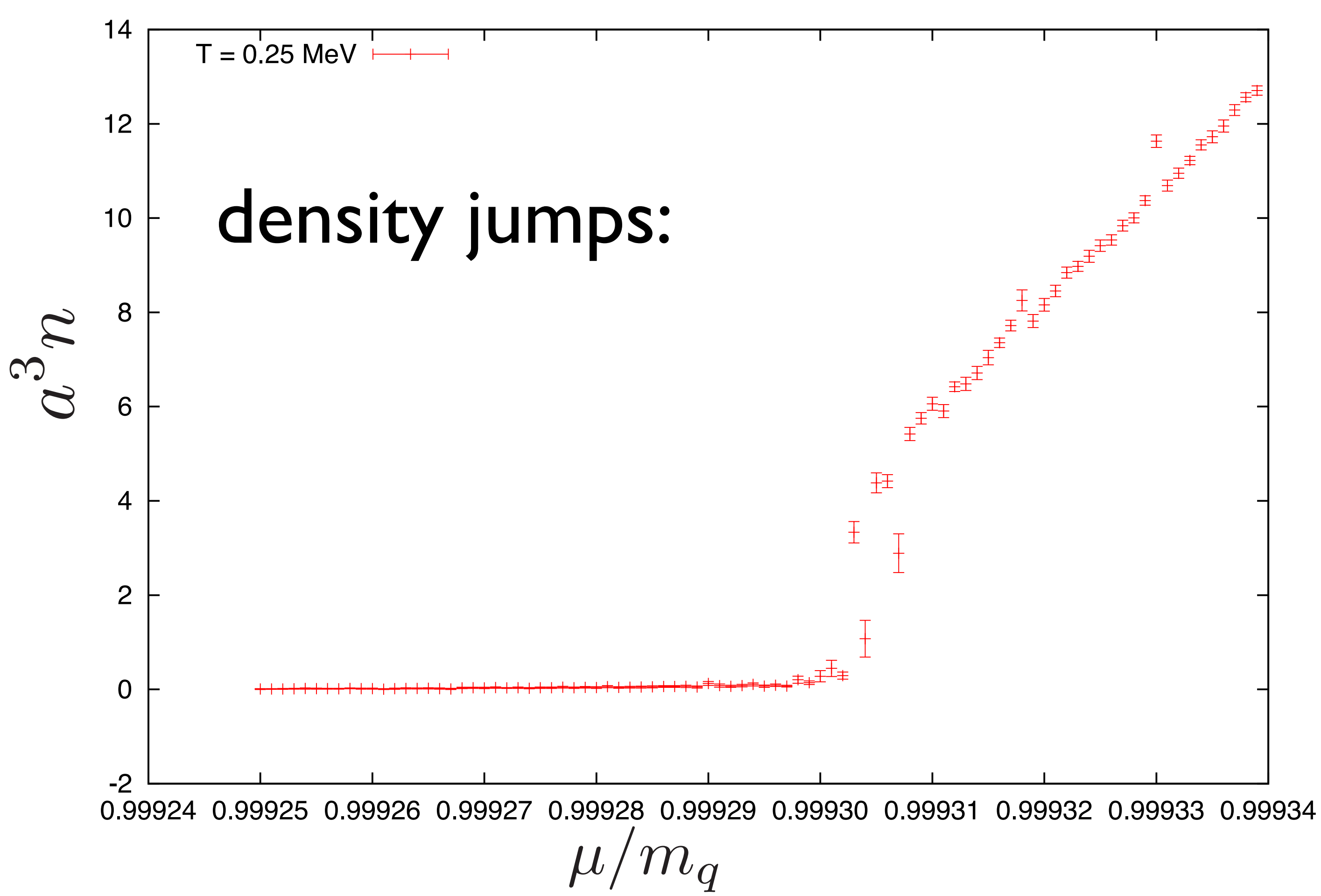
Minimum: access to nucl. binding energy, nucl. saturation density!

$\epsilon \sim 10^{-3}$ consistent with the location of the onset transition, heavy quarks

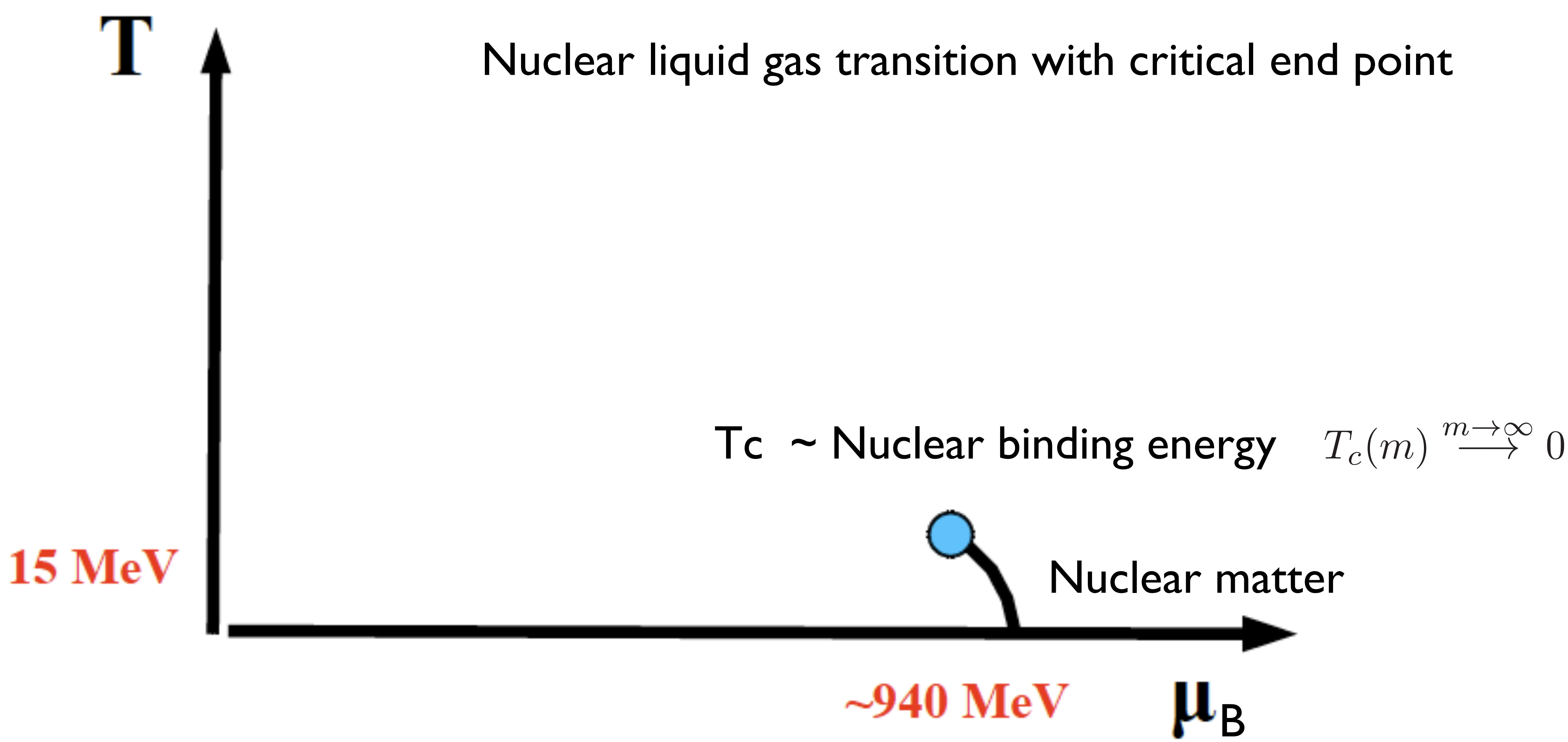
$\sim 10^{-2}$ in nature

Light quarks: first order transition + endpoint

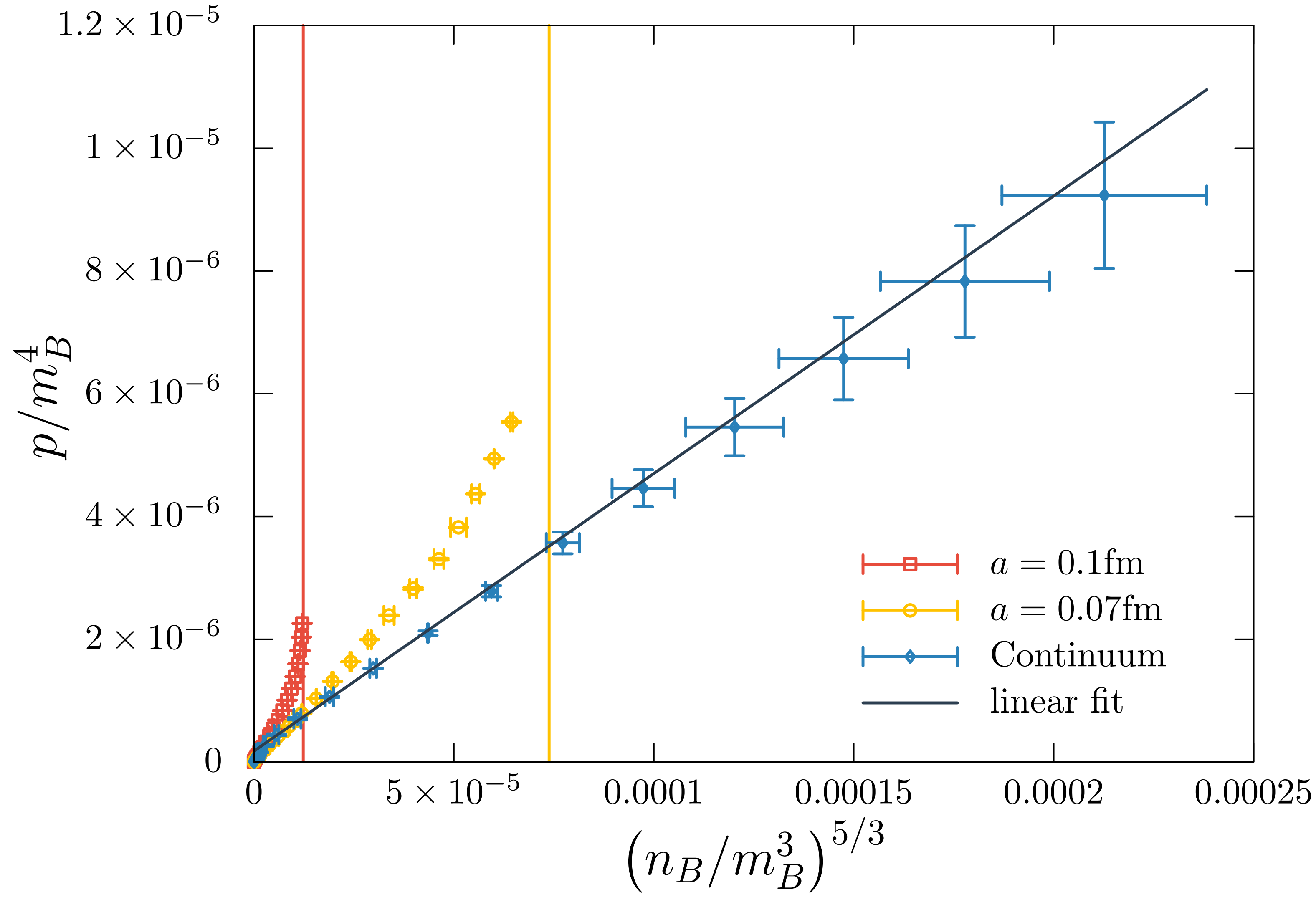




- phase coexistence: first order
- for higher $T = \frac{1}{aN_\tau}$ crossover
- nuclear liquid gas transition!


T
Nuclear liquid gas transition with critical end point
Tc ~ Nuclear binding energy $T_c(m) \stackrel{m\to\infty}{\longrightarrow} 0$
15 MeV
Nuclear matter
~940 MeV
μ_B

Equation of state of heavy nuclear matter, continuum



- EoS fitted by polytrope, non-relativistic fermions!
- Can we understand the pre-factor? Interactions, mass-dependence…

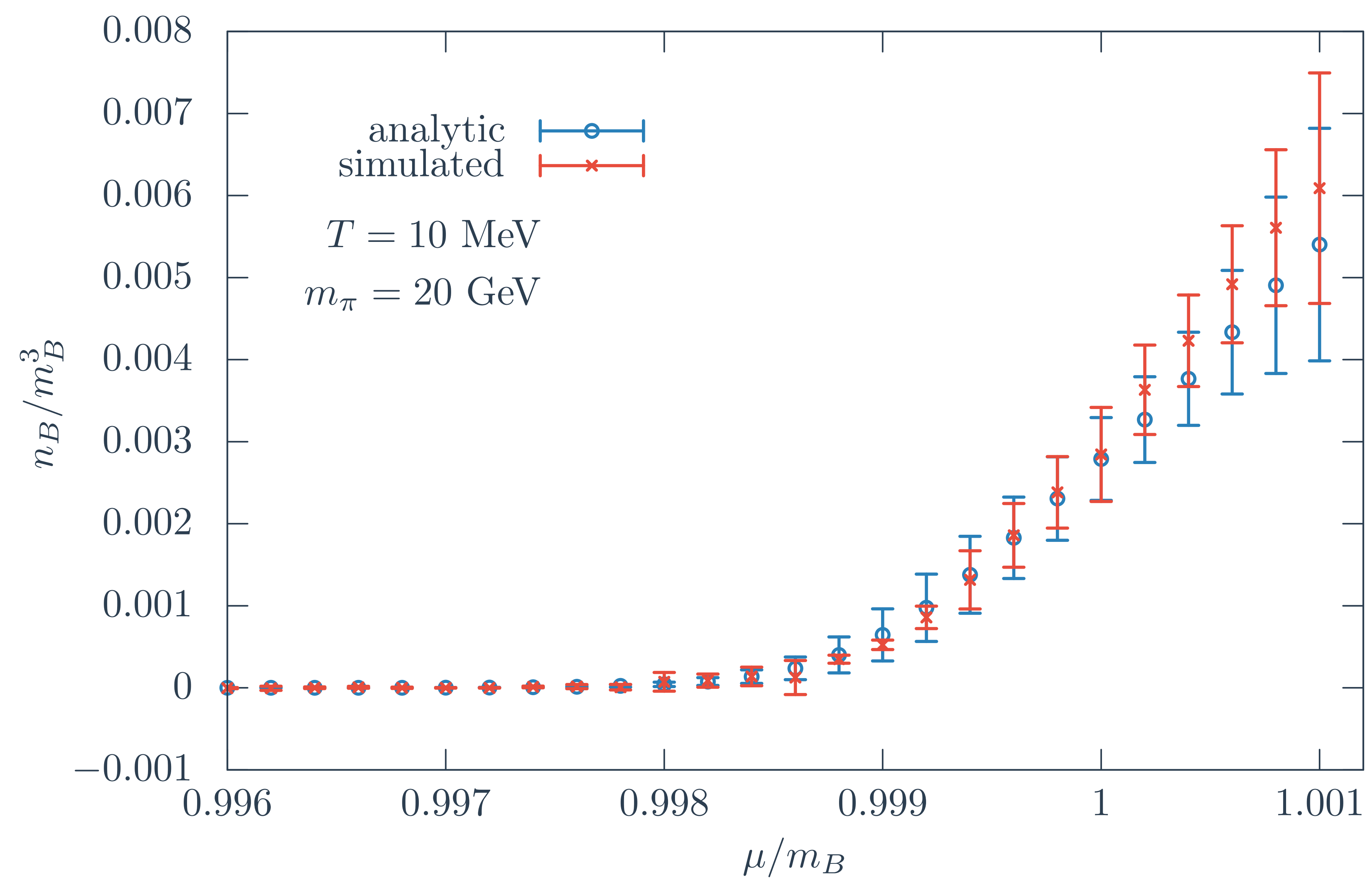
Linked cluster expansion of effective theory

$$\mathcal{Z} = \int \mathcal{D}\phi \, e^{-S_0[\phi] + \frac{1}{2}\sum v_{ij}(x,y)\phi_i(x)\phi_j(y) + \frac{1}{3!}\sum u_{ijk}(x,y,z)\phi_i(x)\phi_j(y)\phi_k(z) + \ldots}$$

"perturbation theory" in effective couplings

Glesaaen, Neuman, O.P. 15

through $u^5\kappa^8$



The effective lattice theory approach II

- Two-step treatment:

  I. Calculate effective theory analytically
  II. Simulate effective theory

- Step I.: integrate over gauge links in strong coupling expansion, leave fermions (staggered)

$$Z_{\rm QCD} = \int d\psi d\bar{\psi} dU e^{S_F+S_G} = \int d\psi d\bar{\psi} Z_F \left\langle e^{S_G} \right\rangle_{Z_F}$$

$$\left\langle e^{S_G} \right\rangle_{Z_F} \simeq 1 + \langle S_G \rangle_{Z_F} = 1 + \frac{\beta}{2N_c} \sum_P \left\langle \mathrm{tr}[U_P + U_P^\dagger] \right\rangle_{Z_F} \qquad Z_F(\psi,\bar{\psi}) = \int dU e^{S_F}$$

- Result: 4d "polymer" model of QCD (hadronic degrees of freedom!)
  Valid for all quark masses (also m=0!), at strong coupling (very coarse lattices)

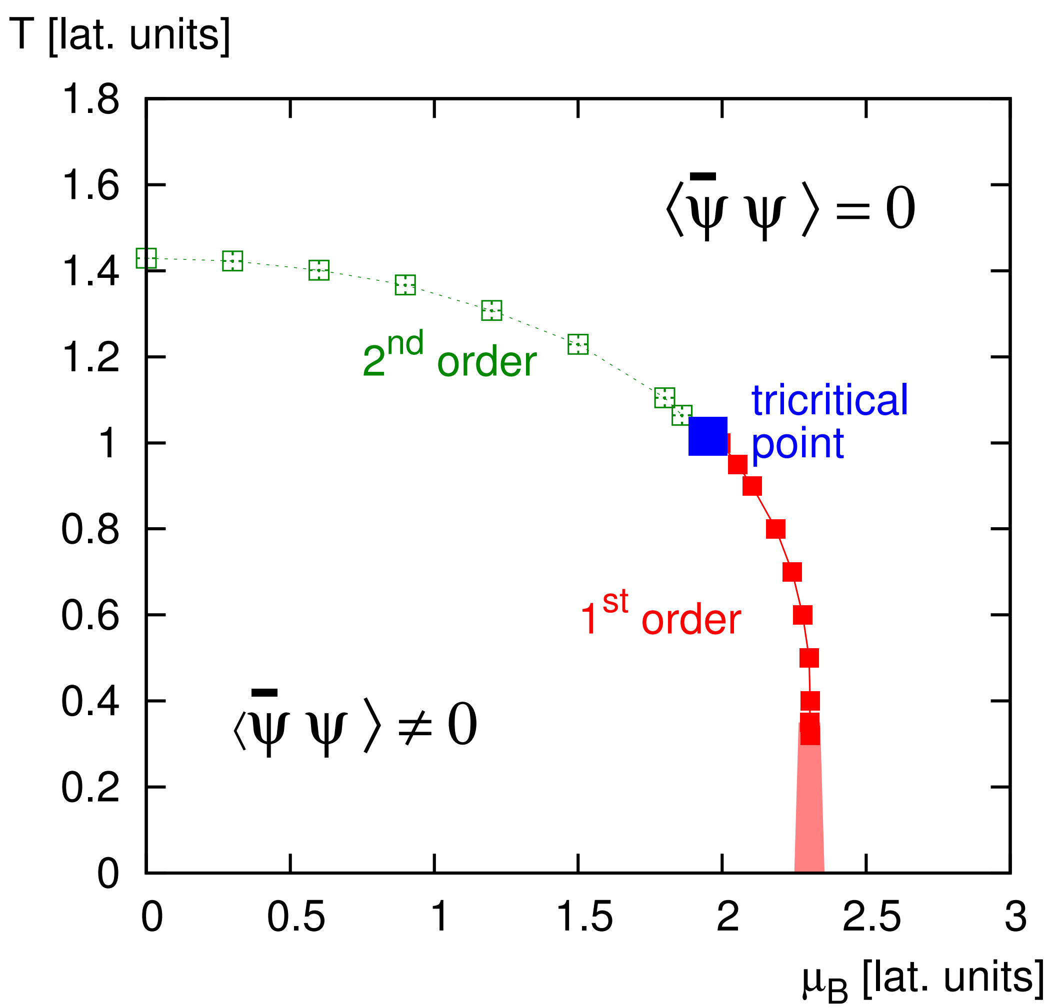
- Step II: sign problem milder: Monte Carlo with worm algorithm

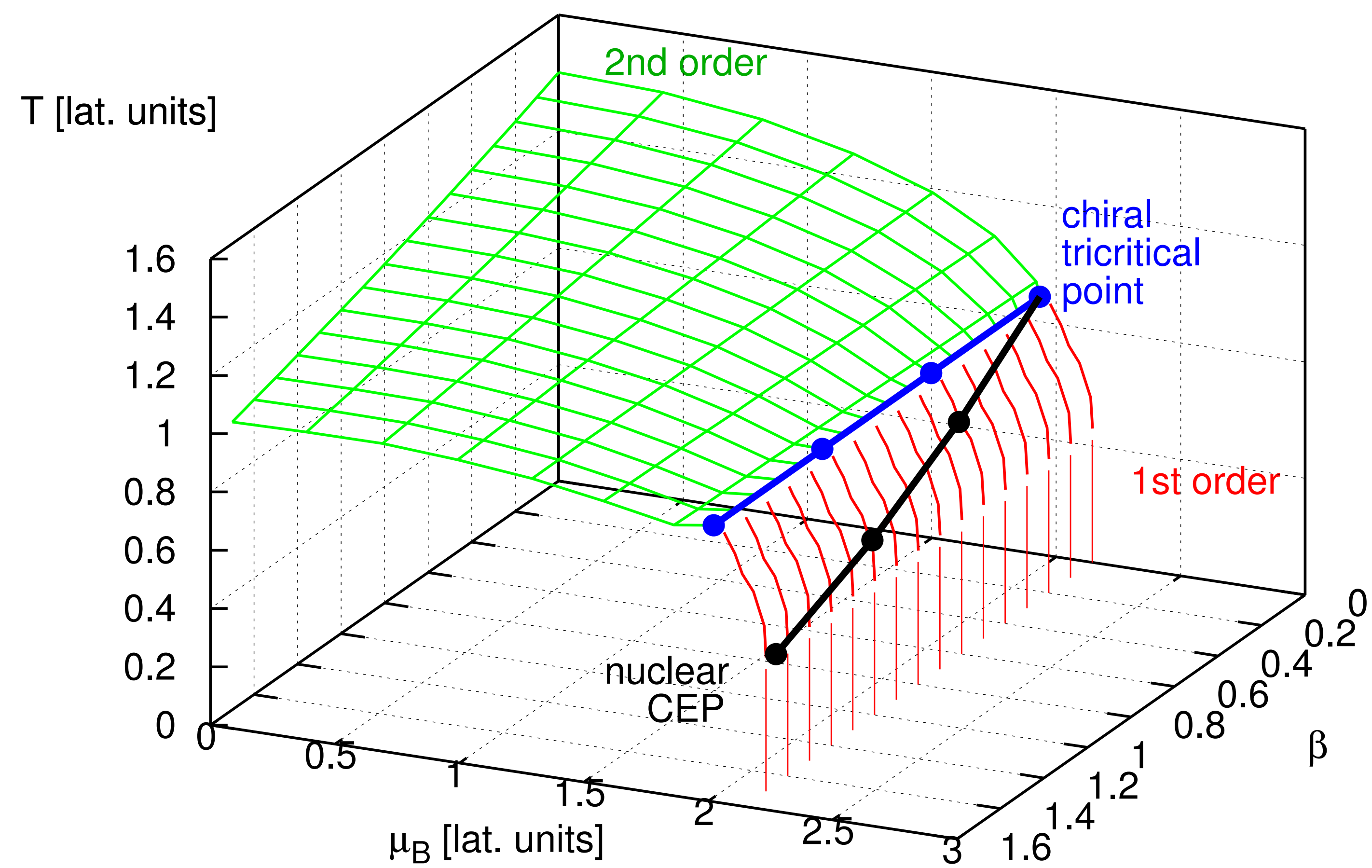
- Numerical simulations without fermion matrix inversion, very cheap!

From strong coupling limit to finite coupling

Unrooted staggered fermions: Nf=4

de Forcrand, Langelage, O.P., Unger 14





Strong coupling limit: $\beta = 0$

Chiral limit: m=0

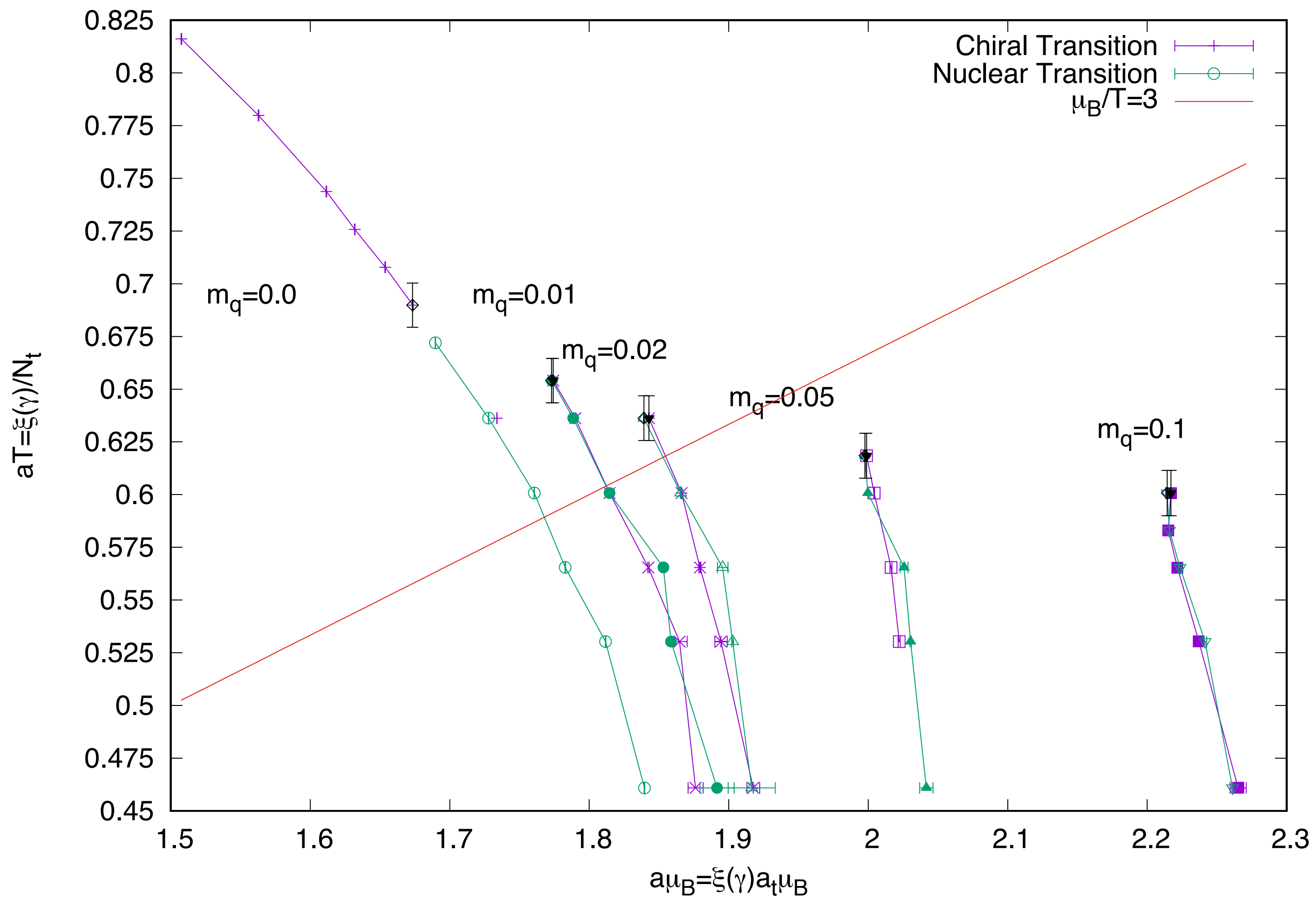
Including leading gauge corrections

Nucl. and chiral transition coincide!

[Kawamoto, Smit, NPB 81;…./Karsch, Mütter, NPB 89…]

From the chiral limit to finite mass

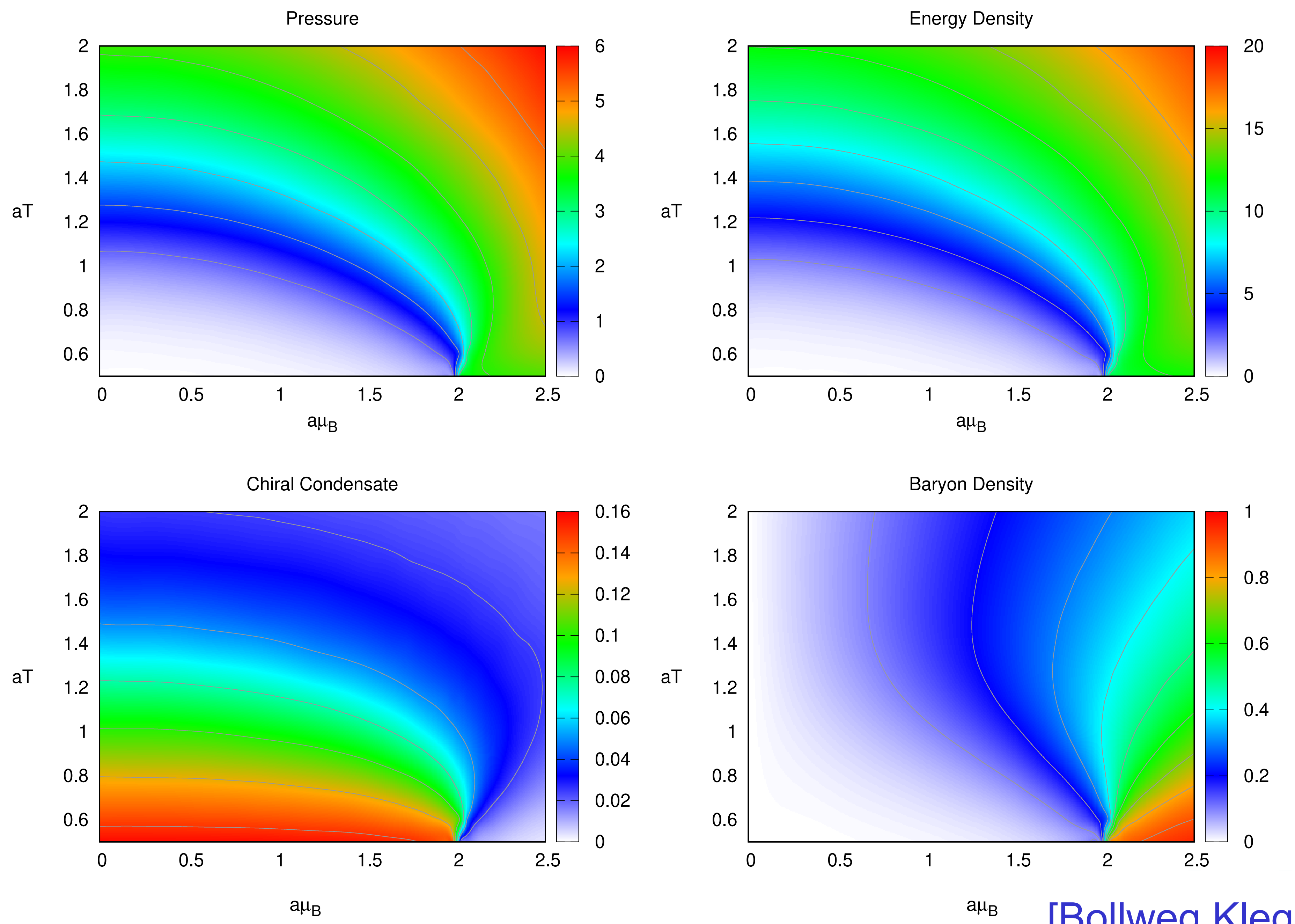
In the strong coupling limit: chiral and nuclear liquid gas tranistions coincide



Finite mass + finite coupling in progress within the CRC-TR 211

Equation of state

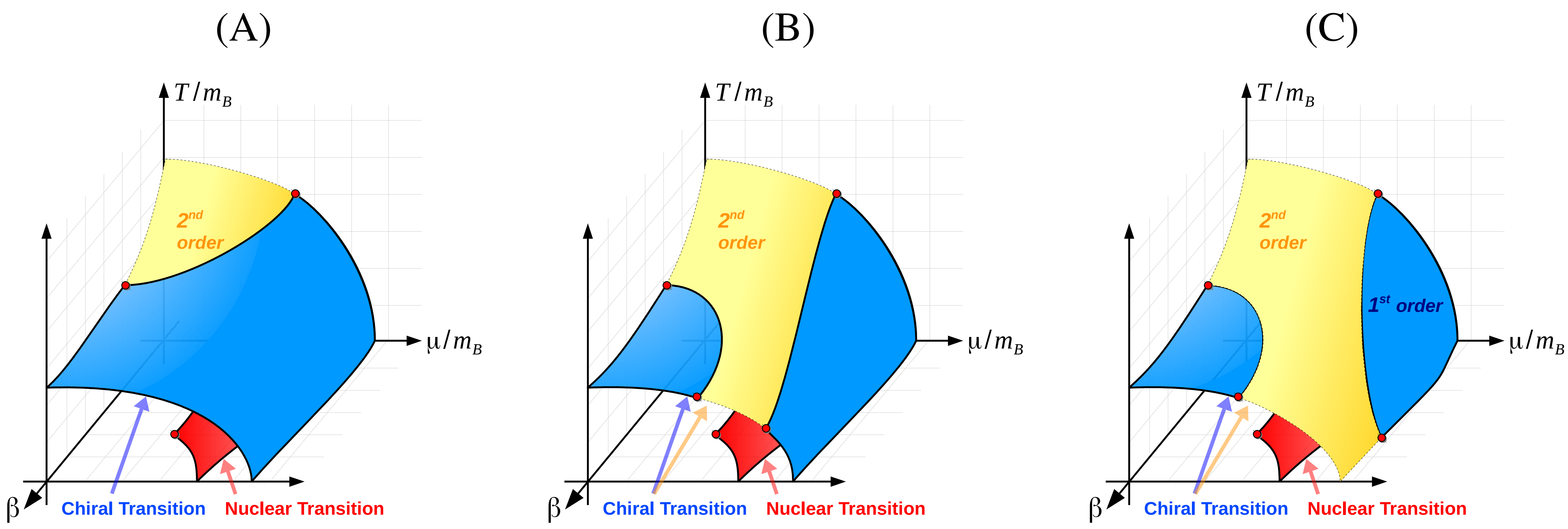
All observables measured via dual variables (here for $am_q = 0.1$), with $\frac{a}{a_t} = \xi(\gamma, m_q)$ non-perturbatively determined as a function of the bare anisotropy γ and quark mass



[Bollweg,Klegrewe,Unger PoS 2018]

Evolution towards the continuum

In the continuum, without rooting, this theory describes 4 quark flavours



Still many possibilities, but number of anchor points and constraints is growing

QCD at large N_c

Definition, 't Hooft 1974 : $N_c \longrightarrow \infty, \quad g^2 N_c = const.$

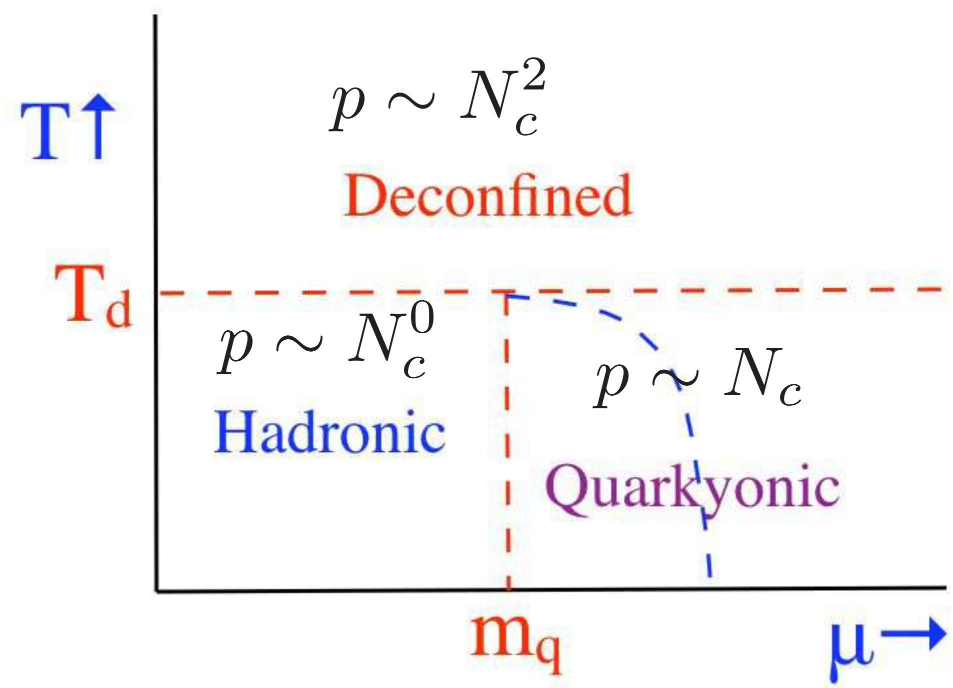
- suppresses quark loops in Feynman diagrams
- mesons are free;
  corrections: cubic interactions $\sim 1/\sqrt{N_c}$, quartic int. $\sim 1/N_c$
- meson masses $\sim \Lambda_{QCD}$
- baryons: N_c quarks, baryon masses $\sim N_c \Lambda_{QCD}$
- baryon interactions: $\sim N_c$

Witten 1979

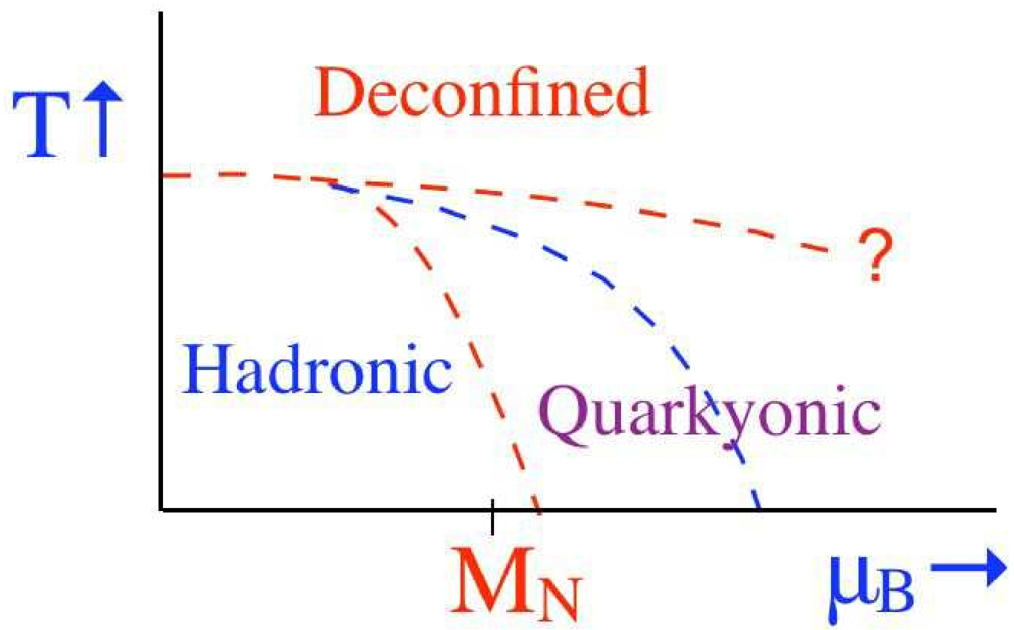
Implications on the phase diagram

McLerran, Pisarski 07: from Nucl. Phys. A 796 (2007)

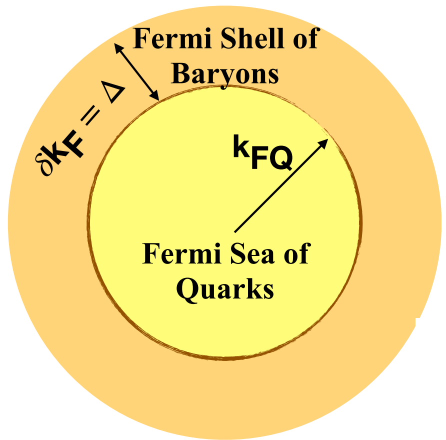
large Nc



QCD, conjectured



Quarkyonic matter



The effective theory for large $N_c(=N)$

Disclaimer: here we consider strong coupling limit, cannot yet keep $g^2 N_c = const$

1st step: recalculate previous results for general N_c

Static determinant:

$$\int_{SU(N)} \mathrm{d}U \det(1+h_1 U)^{2N_f} = \sum_{p=0}^{N_f} \left(\prod_{i=1}^{p} \frac{(i-1+2N_f-p+N)^{\underline{2N_f-p}}}{(i-1+2N_f-p)^{\underline{2N_f-p}}} \right) \left(h_1^{pN} + h_1^{(2N_f-p)N} \right) \left(1 - \frac{\delta_{p,N_f}}{2} \right)$$

And corrections: $$\int_{SU(N)} \mathrm{d}U \det(1+h_1 U)^{2N_f} \operatorname{tr}\left(\frac{(h_1 U)^n}{(1+h_1 U)^m} \right)$$

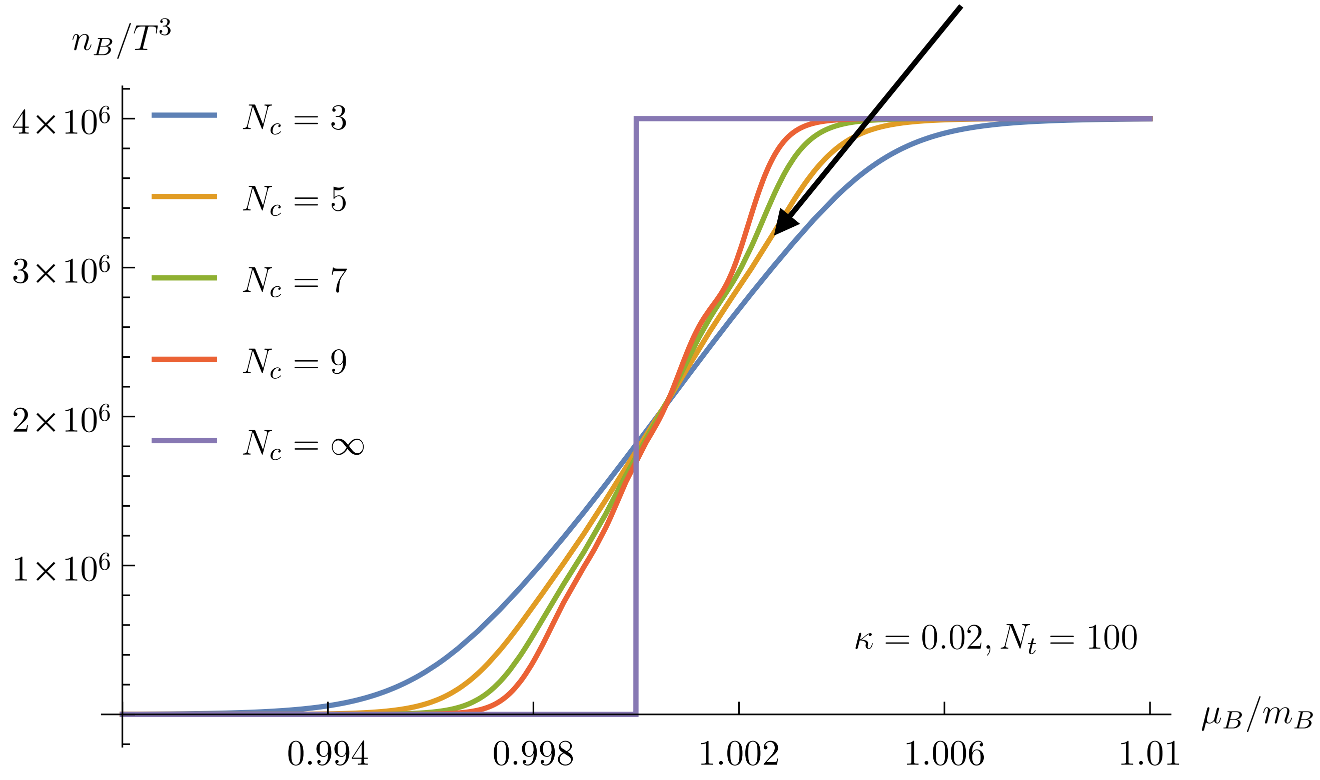
$$= h_1^{N(2N_f+1)} \sum_{r=\max(0,N-m)}^{2N_f+N-m} (-1)^{r+N+1} \binom{N+r-1}{r} (r+m-1)^{\underline{N-1}} \frac{(2N_f)^{\underline{2N_f+1-r-m}}}{(N+2N_f-r-m)}$$

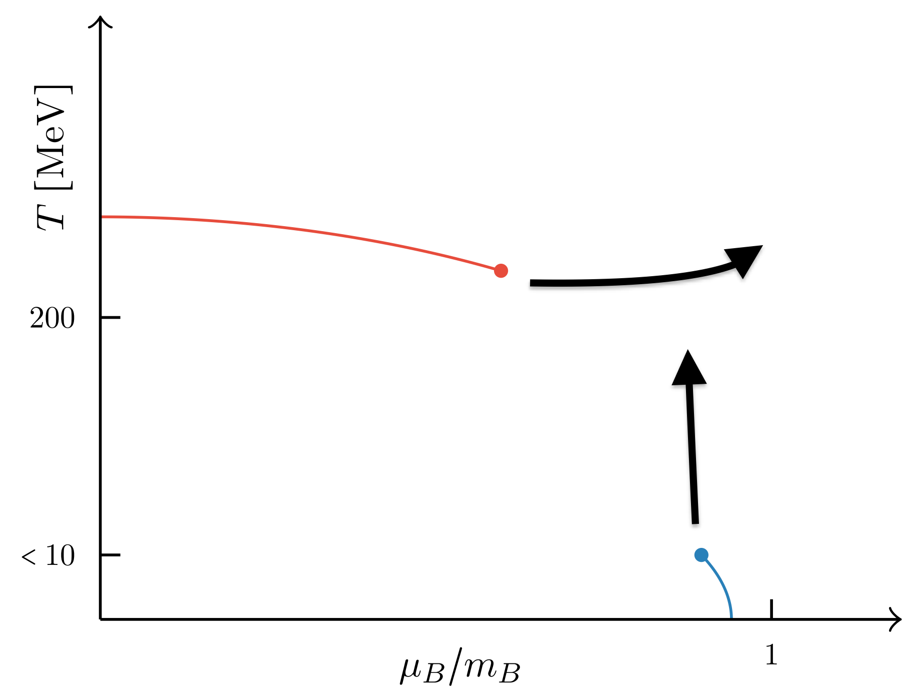
$$+ \sum_{p=0}^{2N_f} h_1^{Np} \det_{1\le i,j\le N} \left[\binom{2N_f}{i-j+p} \right] \sum_{\mu=1}^{N} \sum_{r=\max(0,\mu-m)}^{\mu+p-m} (-1)^r \binom{r+n-1}{r}$$

$$\times \frac{(-1)^{\mu+1}}{r+m} \frac{(r+m+N-\mu)^{\underline{r+m}}}{(r+m-\mu)!(\mu-1)!} \frac{(\mu+p-1)^{\underline{r+m}}}{(N+2N_f-p+r+m-\mu)^{\underline{r+m}}}.$$

Implications for phase diagram

$$\begin{aligned} p &= a \cdot N_c \cdot \kappa^0 + b \cdot N_c \cdot \kappa^2 \\ &\quad + c \cdot N_c \cdot \kappa^4 + \ldots \\ &\sim N_c \end{aligned}$$





The conjectured large N_c phase diagram seen to emerge gradually

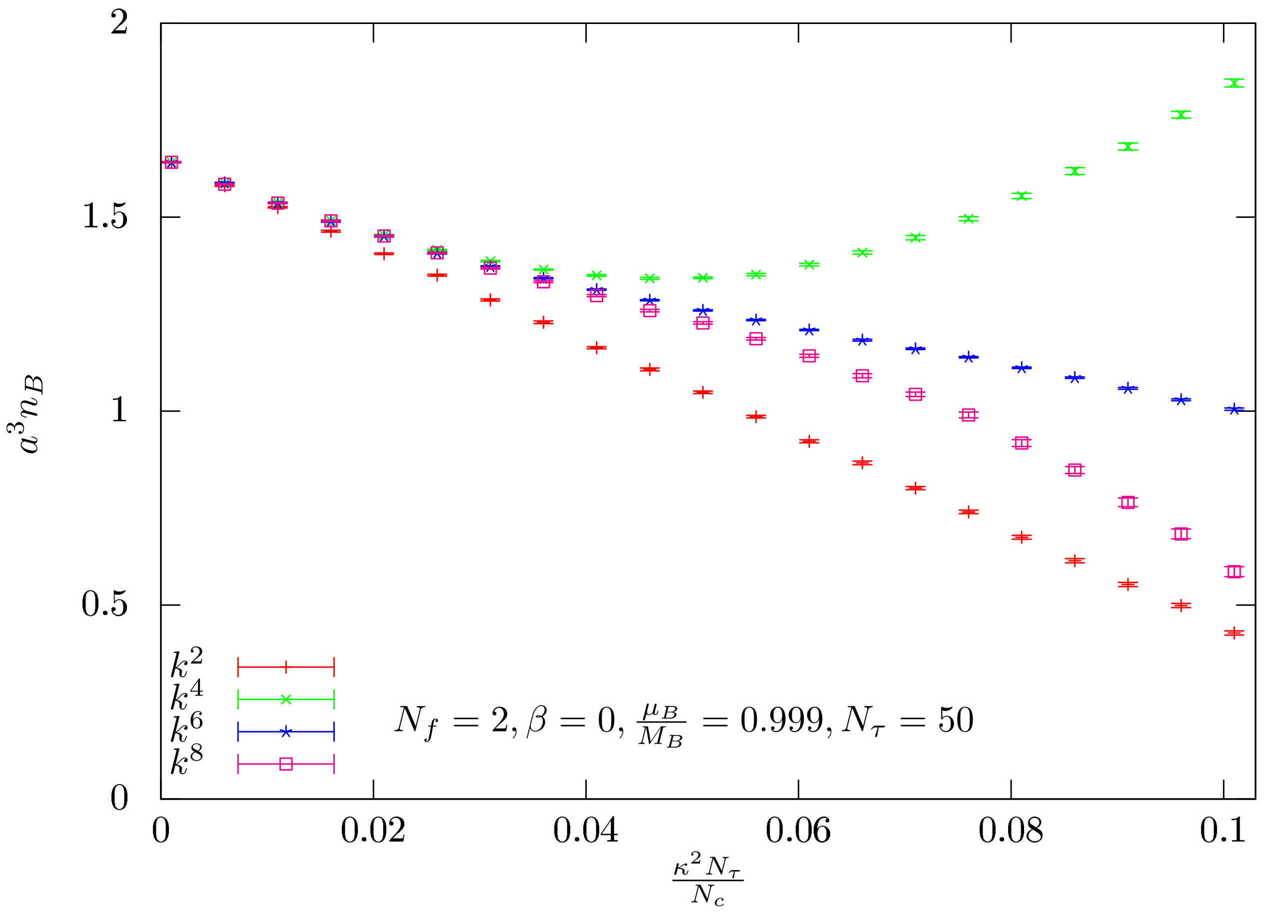
Conclusions

- QCD at finite density possible with effective lattice theories
- Full deconfinement transition for heavy quarks near continuum
- Chiral transition in strong coupling region
- Nuclear liquid gas transition, endpoint as function of quark mass

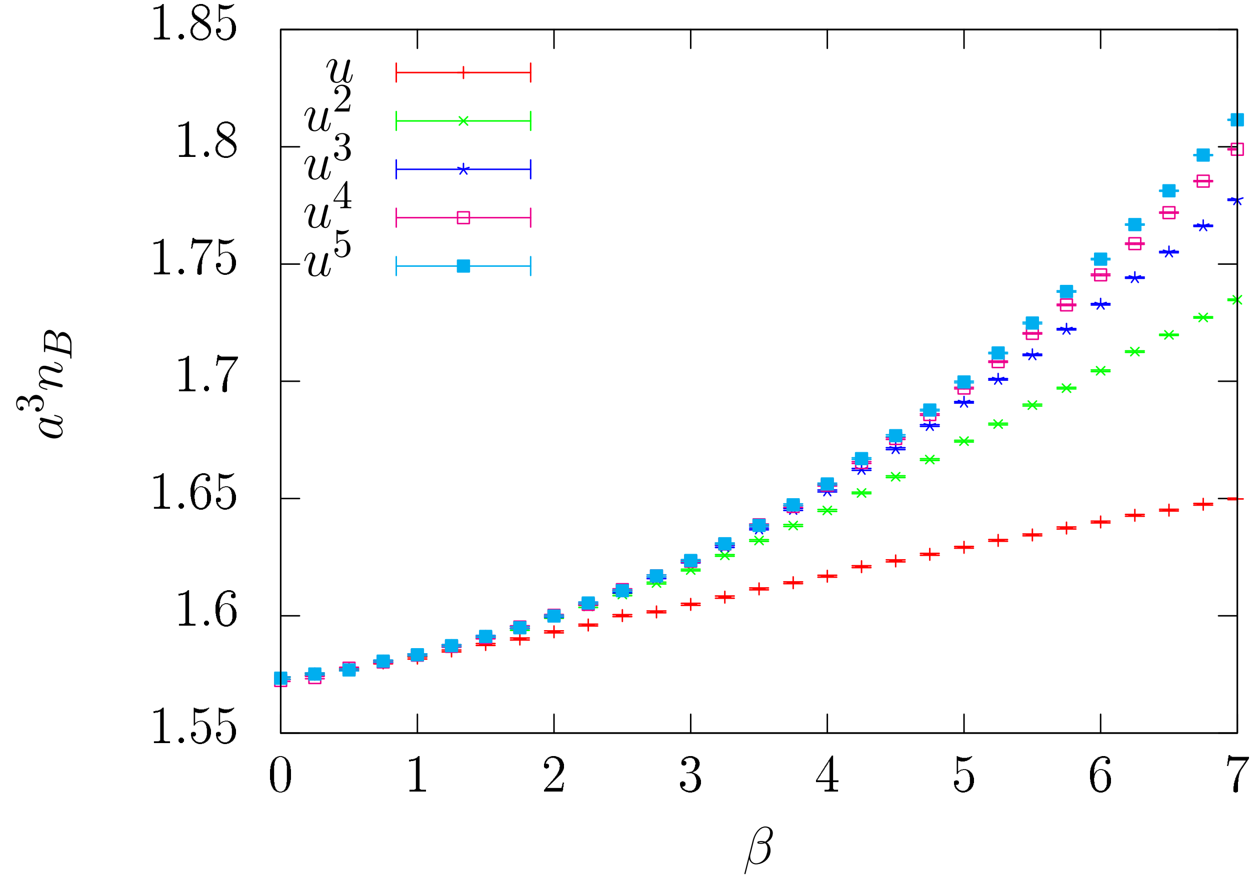
Tool development to move on to physically interesting parameter space

Backup slides

Convergence of the effective theory

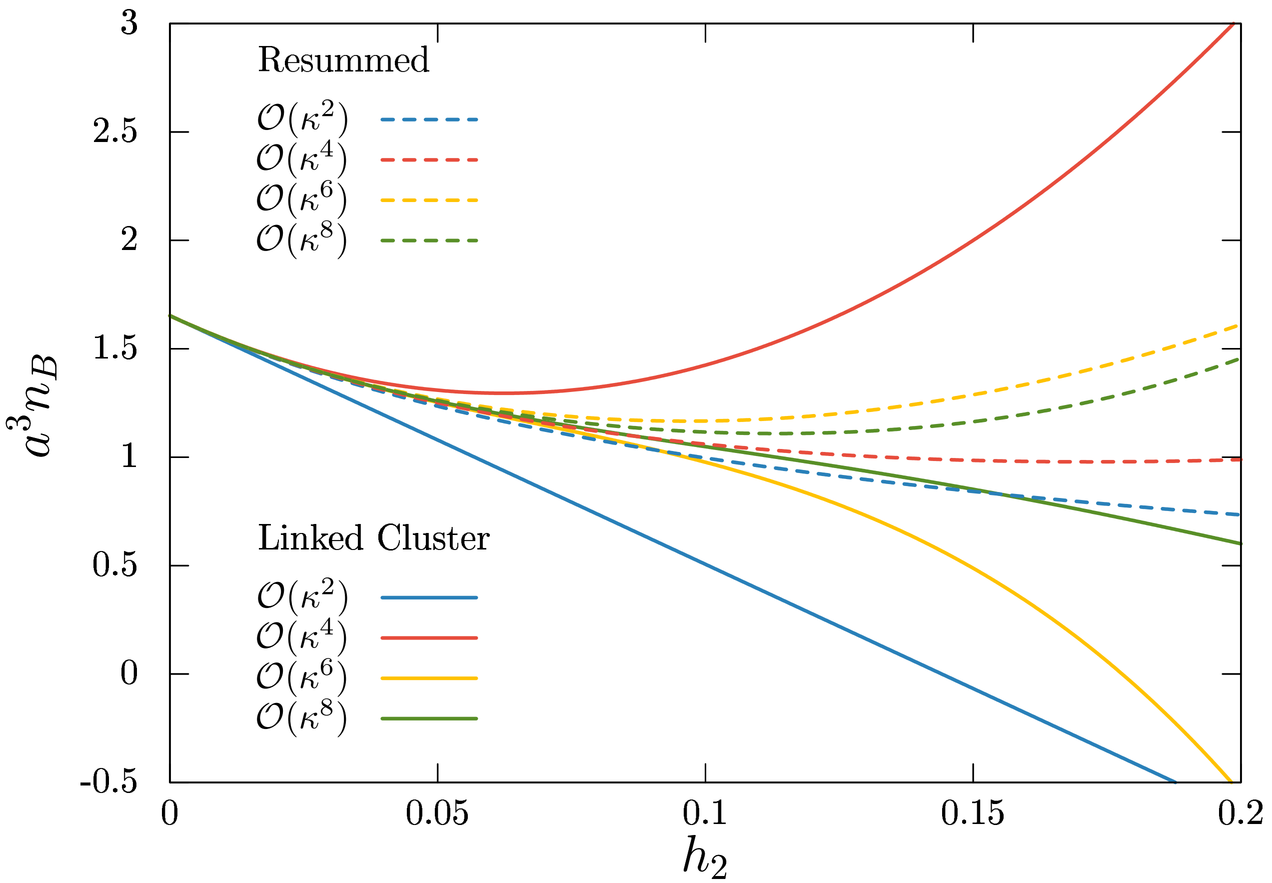


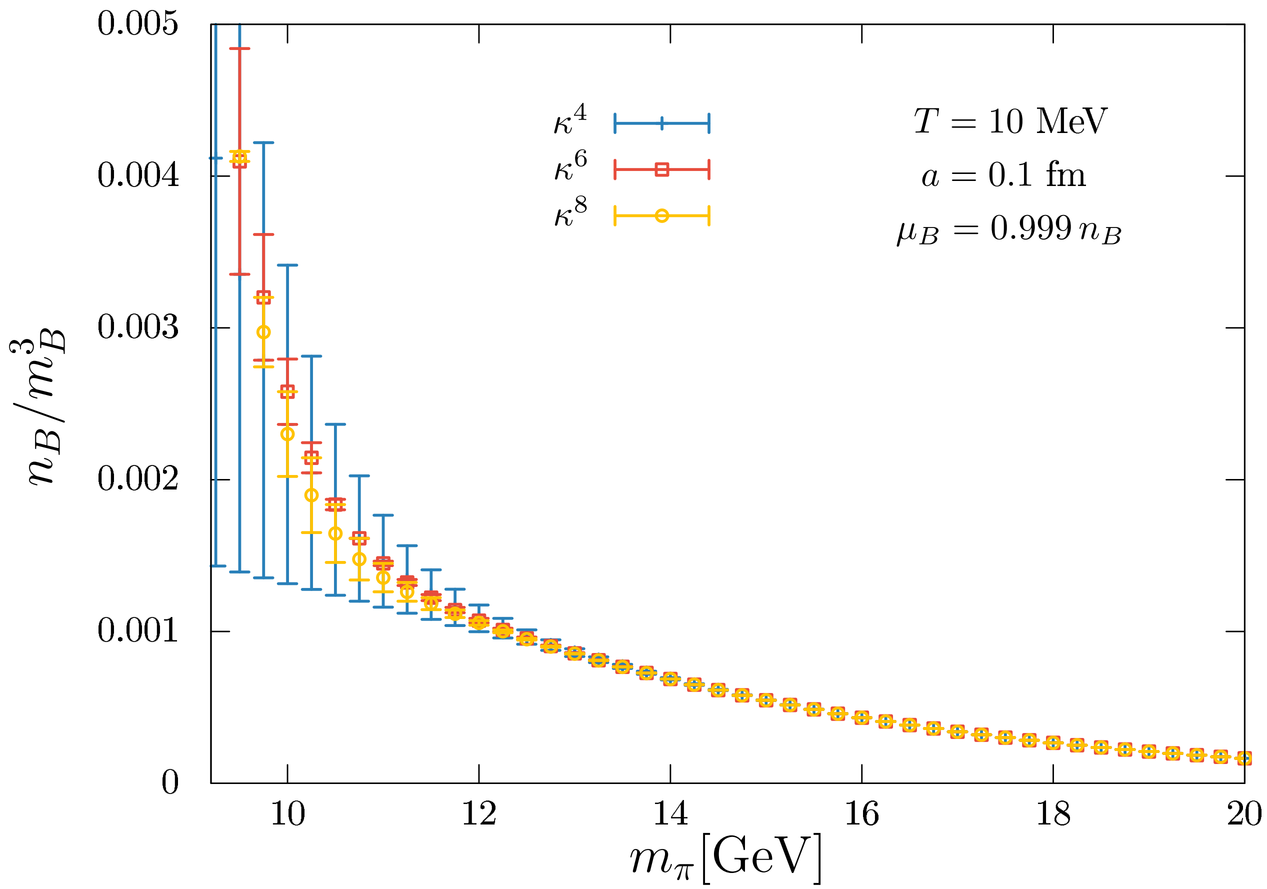
hopping expansion in strong coupling limit



strong coupling expansion at κ^8

Resummations + reach in mass range





Resumming long range non-overlapping chains, gain in mass range “sobering”

Subleading couplings

Subleading contributions for next-to-nearest neighbours:

$$\lambda_2 S_2 \quad \propto \quad u^{2N_\tau+2} \sum_{[kl]}' 2\mathrm{Re}(L_k L_l^*) \quad \text{distance} \; = \sqrt{2}$$

$$\lambda_3 S_3 \quad \propto \quad u^{2N_\tau+6} \sum_{\{mn\}}'' 2\mathrm{Re}(L_m L_n^*) \quad \text{distance} \; = 2$$

as well as terms from loops in the *adjoint* representation:

$$\lambda_a S_a \propto u^{2N_\tau} \sum_{<ij>} \mathrm{Tr}^{(a)} W_i \mathrm{Tr}^{(a)} W_j \quad ; \quad \mathrm{Tr}^{(a)} W = |L|^2 - 1$$

Cold and dense QCD: static strong coupling limit

Fromm, Langelage, Lottini, Neuman, O.P. , PRL 13

For T=0 (at finite density) anti-fermions decouple $N_f = 1, h_1 = C, h_2 = 0$

$$C \equiv (2\kappa_f e^{a\mu_f})^{N_\tau} = e^{(\mu_f - m_f)/T}, \; \bar{C}(\mu_f) = C(-\mu_f)$$

$$Z(\beta = 0) \xrightarrow{T\to 0} \left[\prod_f \int \mathrm{d}W \, (1 + C L + C^2 L^* + C^3)^2 \right]^{N_s^3}$$

$$= [1 + 4C^{N_c} + C^{2N_c}]^{N_s^3}$$

Free gas of baryons!

Quarkyonic?

$$n = \frac{T}{V} \frac{\partial}{\partial \mu} \ln Z = \frac{1}{a^3} \frac{4N_c C^{N_c} + 2N_c C^{2N_c}}{1 + 4C^{N_c} + C^{2N_c}}$$

$$\lim_{\mu \to \infty} (a^3 n) = 2N_c$$

Sivler blaze property + saturation!

$$\lim_{T\to 0} a^3 n = \begin{cases} 0, & \mu < m \\ 2N_c, & \mu > m \end{cases}$$

$N_f = 2$

$$z_0 = (1 + 4h_d^3 + h_d^6) + (6h_d^2 + 4h_d^5)h_u + (6h_d + 10h_d^4)h_u^2 + (4 + 20h_d^3 + 4h_d^6)h_u^3 + (10h_d^2 + 6h_d^5)h_u^4 + (4h_d + 6h_d^4)h_u^5 + (1 + 4h_d^3 + h_d^6)h_u^6 \ . \quad (3.11)$$

Free gas of baryons: complete spin flavor structure of vacuum states!