Lattice-based Equation of State of QCD matter with a critical point

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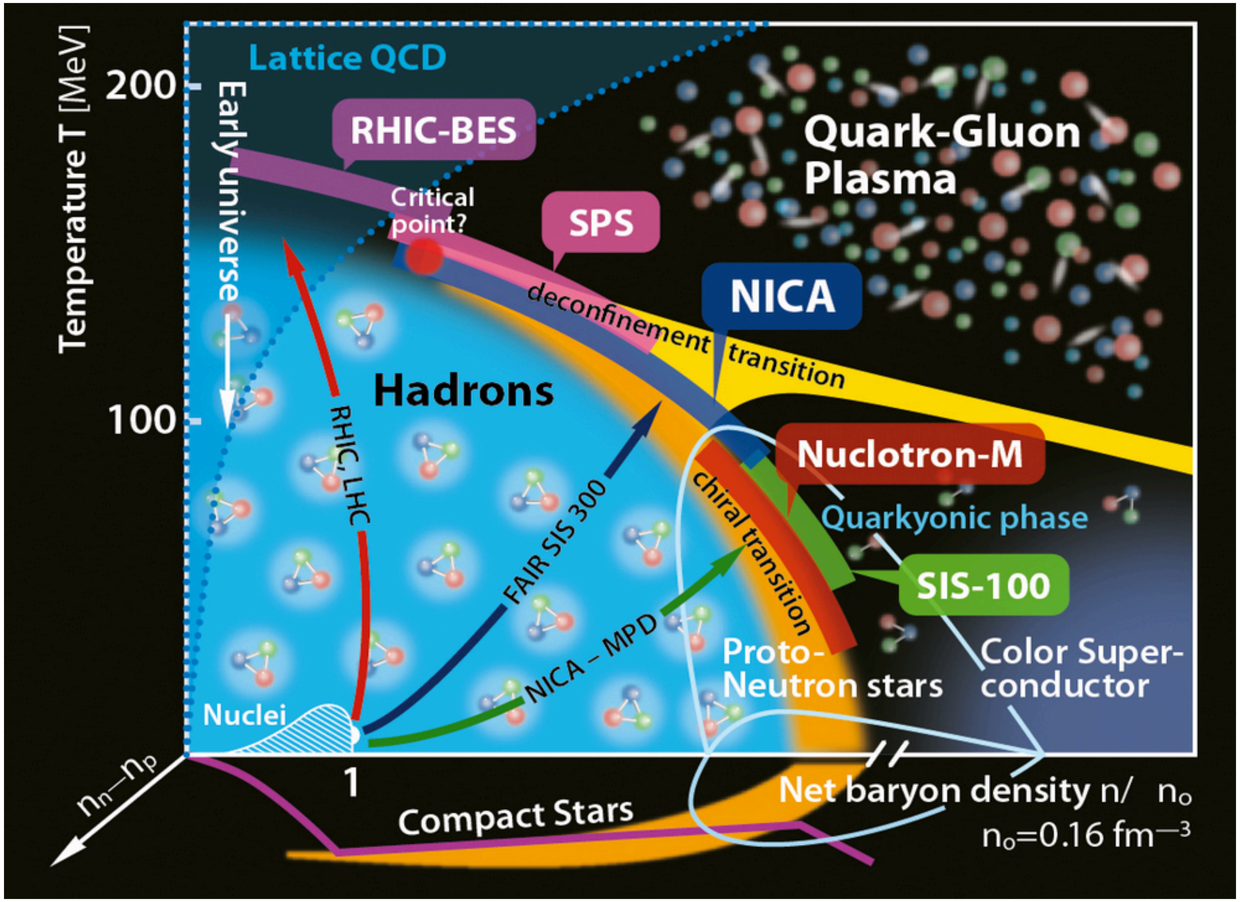
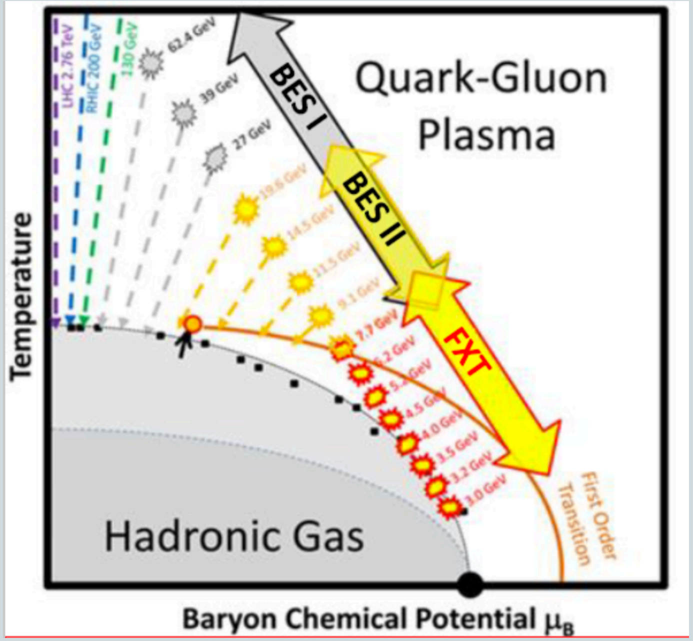
Open Questions

- Is there a critical point in the QCD phase diagram?
- What are the degrees of freedom in the vicinity of the phase transition?
- Where is the transition line at high density?
- What are the phases of QCD at high density?
- Are we creating a thermal medium in experiments?



Second Beam Energy Scan (BESII) at RHIC

- Planned for 2019-2020
- 24 weeks of runs each year
- Beam Energies have been chosen to keep the μ_B step ~50 MeV
- Chemical potentials of interest: μ_B/T~1.5…4



√s (GeV)	19.6	14.5	11.5	9.1	7.7	6.2	5.2	4.5
μ_B (MeV)	205	260	315	370	420	487	541	589
# Events	400M	300M	230M	160M	100M	100M	100M	100M

Collider Fixed Target

Comparison of the facilities

Compilation by D. Cebra

Facilty	RHIC BESII	SPS	NICA	SIS-100 SIS-300	J-PARC HI
Exp.:	**STAR** +FXT	**NA61**	**MPD** **+ BM@N**	**CBM**	**JHITS**
Start:	**2019-20** 2018	**2009**	**2020** 2017	**2022**	**2025**
Energy: $\sqrt{s_{NN}}$ (GeV)	**7.7– 19.6** 2.5-7.7	**4.9-17.3**	**2.7 - 11** 2.0-3.5	**2.7-8.2**	**2.0-6.2**
Rate: At 8 GeV	**100 HZ** 2000 Hz	**100 HZ**	**<10 kHz**	**<10 MHZ**	**100 MHZ**
Physics:	**CP&OD**	**CP&OD**	**OD&DHM**	**OD&DHM**	**OD&DHM**
	Collider Fixed target	Fixed target Lighter ion collisions	Collider Fixed target	Fixed target	Fixed target

CP=Critical Point OD= Onset of Deconfinement DHM=Dense Hadronic Matter

Purpose of this work

- Build an equation of state which:
  - Reproduces the one from lattice QCD up to $O(\mu_B^4)$
  - Contains a critical point in the 3D Ising model universality class
  - Can be used as input for hydrodynamic simulations to test the effect of the critical point on observables
- Future hydro simulations and comparison with BESII data might help to constrain the position of the critical point

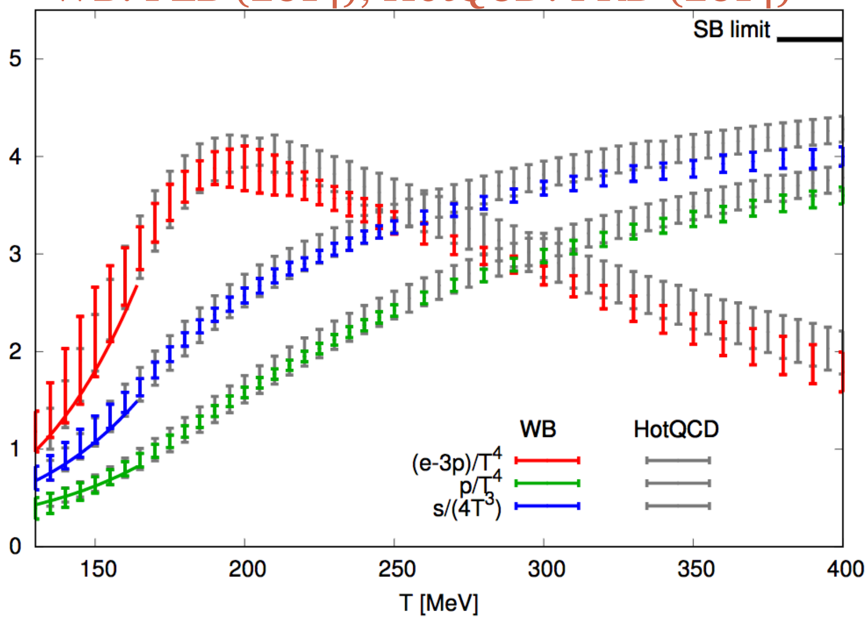
QCD Equation of State at finite density from the lattice

TAYLOR EXPANSION

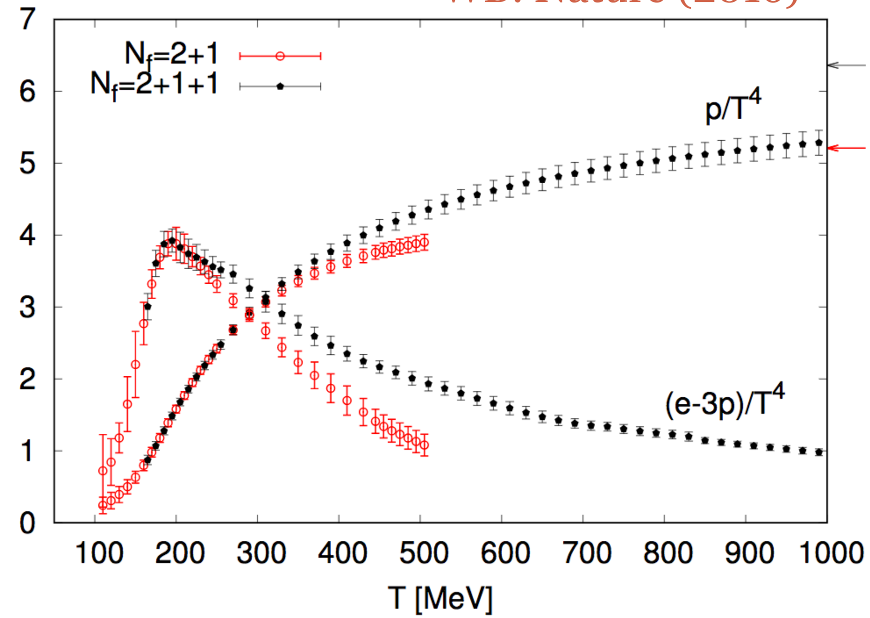
ANALYTICAL CONTINUATION FROM IMAGINARY CHEMICAL POTENTIAL

QCD EoS at μ_B=0

WB: PLB (2014); HotQCD: PRD (2014)



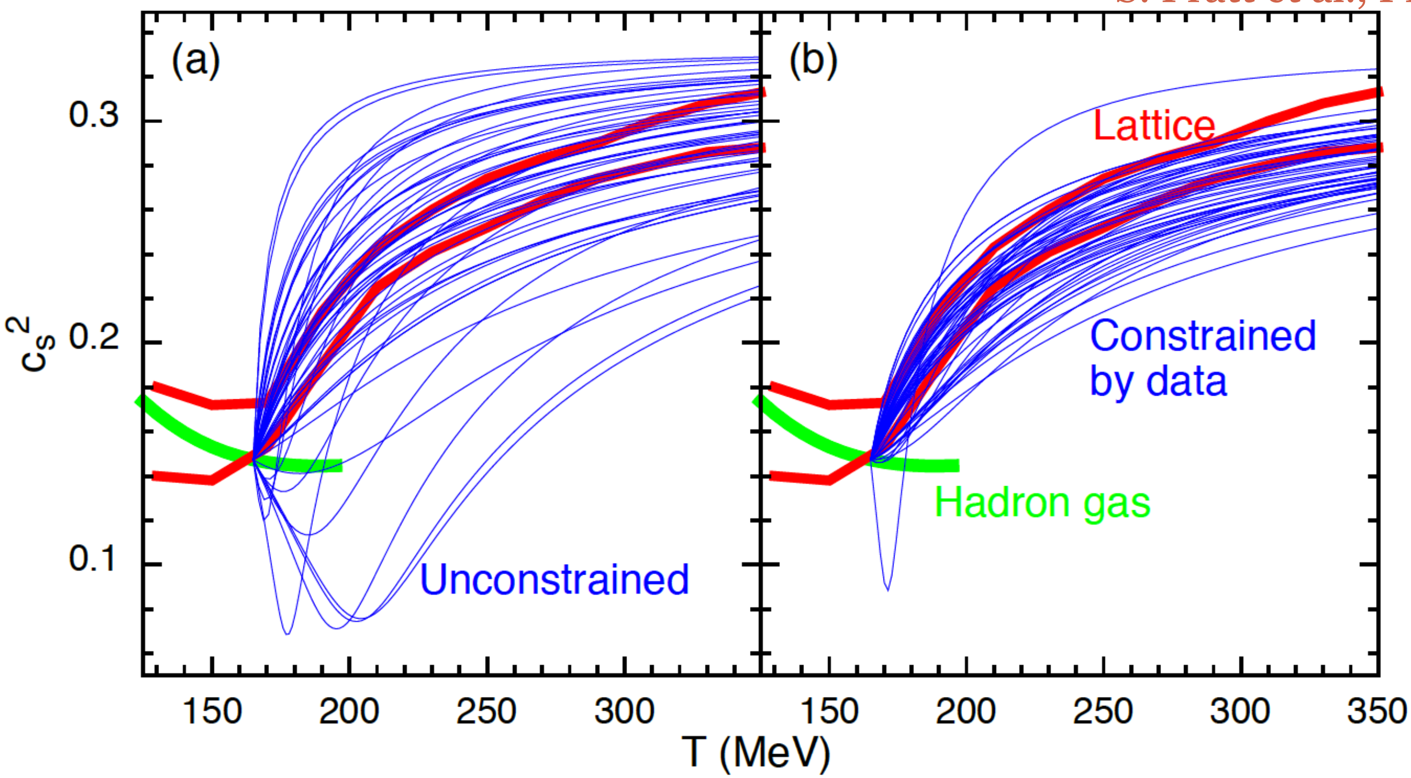
WB: Nature (2016)



- EoS for N_f=2+1 known in the continuum limit since 2013
- Good agreement with the HRG model at low temperature
- Charm quark relevant degree of freedom already at T~250 MeV

Constraints on the EoS from the experiments

S. Pratt et al., PRL (2015)



- Comparison of data from RHIC and LHC to theoretical models through Bayesian analysis
- The posterior distribution of EoS is consistent with the lattice QCD one

Taylor expansion of EoS

- Taylor expansion of the pressure:

$$\frac{p(T,\mu_B)}{T^4} = \frac{p(T,0)}{T^4} + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \left.\frac{d^{2n}(p/T^4)}{d(\frac{\mu_B}{T})^{2n}}\right|_{\mu_B=0} \left(\frac{\mu_B}{T}\right)^{2n} = \sum_{n=0}^{\infty} c_{2n}(T) \left(\frac{\mu_B}{T}\right)^{2n}$$

- Two ways of extracting the Taylor expansion coefficients:
  - Direct simulation
  - Simulations at imaginary μ_B

- Two physics choices:
  - $\mu_B \neq 0$, $\mu_S = \mu_Q = 0$
  - μ_S and μ_Q are functions of T and μ_B to match the experimental constraints:

$\langle n_S \rangle = 0$ $\langle n_Q \rangle = 0.4 \langle n_B \rangle$

Pressure coefficients: direct simulation

<u>Direct simulation</u>:

- Calculate derivatives of *lnZ*, where *Z* in the staggered formulation is given by:

$$Z = \int \mathcal{D}U \; e^{-S_g} (\det M_1)^{1/4} (\det M_2)^{1/4} (\det M_3)^{1/4} = \int \mathcal{D}U \; e^{-S_{\text{eff}}}$$

where M_i is the fermionic determinant of flavor *i* and *Sg* the gauge action

- The derivatives with respect to the chemical potential of flavor *i* are

$$\begin{aligned}
A_j &= \frac{d}{d\mu_j}(\det M_j)^{1/4} = \tilde{\mathrm{tr}} M_j^{-1} M_j' \,, \\
B_j &= \frac{d^2}{(d\mu_j)^2}(\det M_j)^{1/4} = \tilde{\mathrm{tr}} \left(M_j'' M_j^{-1} - M_j' M_j^{-1} M_j' M_j{-1} \right) , \\
C_j &= \frac{d^3}{(d\mu_j)^3}(\det M_j)^{1/4} = \tilde{\mathrm{tr}} \left(M_j' M_j^{-1} - 3 M_j'' M_j^{-1} M_j' M_j^{-1} \right. \\
&\qquad \left. + 2 M_j' M_j^{-1} M_j' M_j^{-1} M_j' M_j^{-1} \right) , \\
D_j &= \frac{d^4}{(d\mu_j)^4} \log(\det M_j)^{1/4} = \tilde{\mathrm{tr}} \left(M_j'' M_j^{-1} - 4 M_j' M_j^{-1} M_j' M_j^{-1} - 3 M_j'' M_j^{-1} M_j'' M_j^{-1} \right. \\
&\qquad + 12 M_j'' M_j^{-1} M_j' M_j^{-1} M_j' M_j^{-1} \\
&\qquad \left. - 6 M_j' M_j^{-1} M_j' M_j^{-1} M_j' M_j^{-1} M_j' M_j^{-1} \right) ,
\end{aligned}$$

From which:

$$\begin{aligned}
\partial_i^4 \log Z &= \left\langle A_i^4 \right\rangle - 3 \left\langle A_i^2 \right\rangle^2 + 3 \left(\left\langle B_i^2 \right\rangle - \langle B_i \rangle^2 \right) \\
&\quad + 6 \left(\left\langle A_i^2 B_i \right\rangle - \left\langle A_i^2 \right\rangle \langle B_i \rangle \right) + 4 \langle A_i C_i \rangle + \langle D_i \rangle
\end{aligned}$$

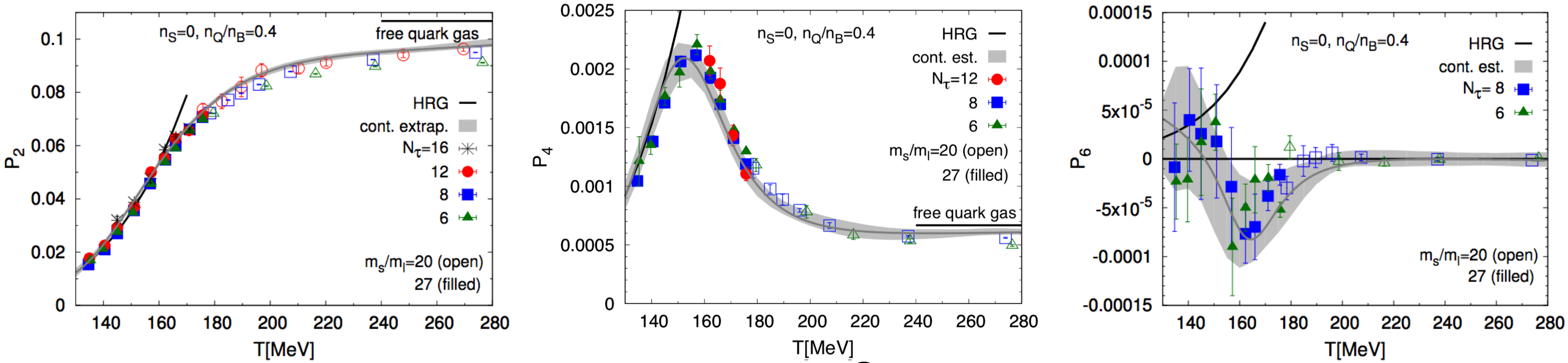
and so on…

Pressure coefficients

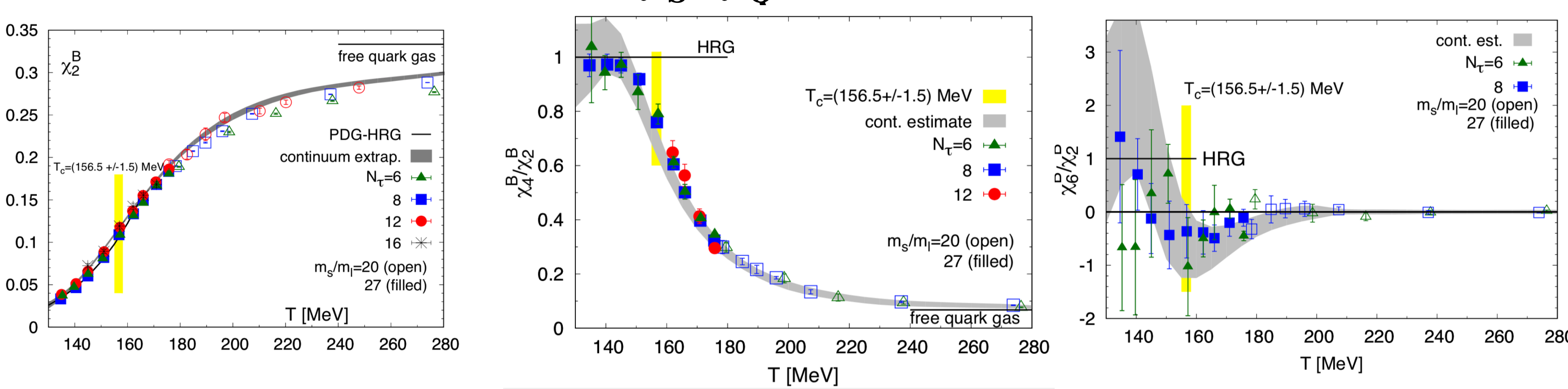
<u>Direct simulation</u>:

O(10^5) configurations (hotQCD: PRD (2017) and update 06/2018)

Strangeness neutrality

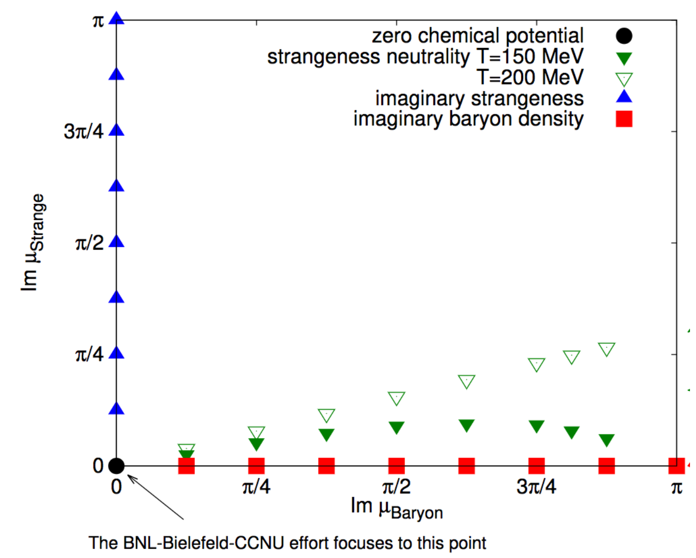


$\mu_S=\mu_Q=0$



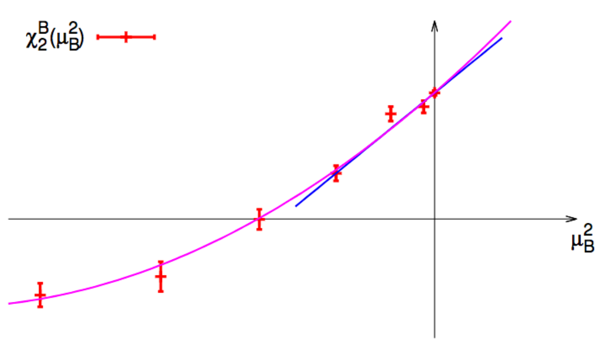
Pressure coefficients: simulations at imaginary μ_B

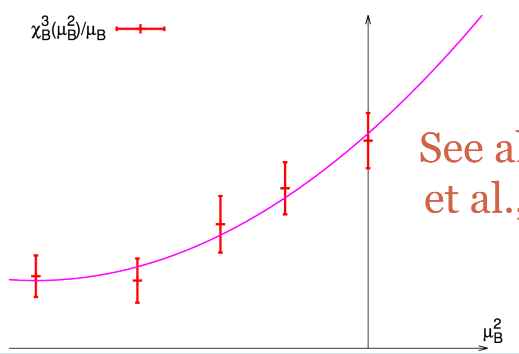
Simulations at imaginary μ_B:



Common technique: [de Forcrand, Philipsen (2002)], [D'Elia and Lombardo, (2002)], [Bonati et al., (2015), (2018)], [Cea et al., (2015)]

Strategy: simulate lower-order fluctuations and use them in a combined, correlated fit





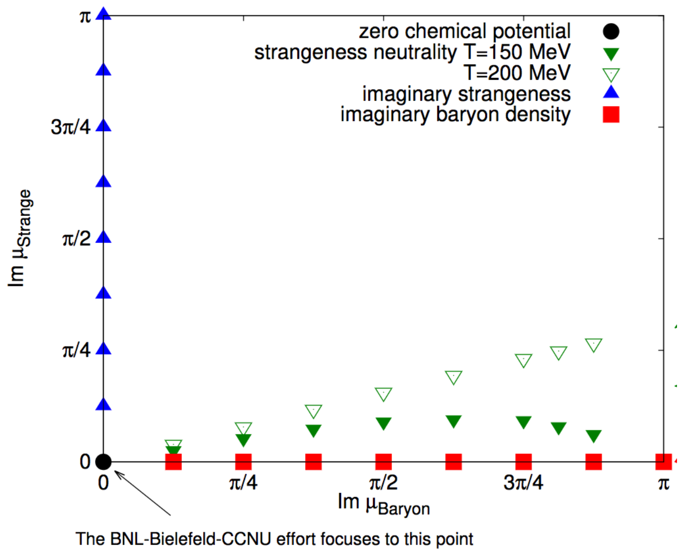
See also M. D'Elia et al., PRD (2017)

$$\chi_2^B(\mu_B^2) \approx \chi_2^B(0) + \frac{1}{2}\mu_B^2\chi_4^B(0) + \frac{1}{24}\mu_B^4\chi_6^B(0) + \dots$$

$$\frac{\chi_3^B(\mu_B^2)}{\mu_B} \approx \chi_4^B(0) + \frac{1}{6}\mu_B^2\chi_6^B(0) + \frac{1}{120}\mu_B^4\chi_8^B(0)$$

Pressure coefficients: simulations at imaginary μ_B

Simulations at imaginary μ_B:



Common technique: [de Forcrand, Philipsen (2002)], [D'Elia and Lombardo, (2002)], [Bonati et al., (2015), (2018)], [Cea et al., (2015)]

Strategy: simulate lower-order fluctuations and use them in a combined, correlated fit

$$\chi_1^B(\hat{\mu}_B) = 2c_2\hat{\mu}_B + 4c_4\hat{\mu}_B^3 + 6c_6\hat{\mu}_B^5 + \frac{4!}{7!}c_4\epsilon_1\hat{\mu}_B^7 + \frac{4!}{9!}c_4\epsilon_2\hat{\mu}_B^9$$

$$\chi_2^B(\hat{\mu}_B) = 2c_2 + 12c_4\hat{\mu}_B^2 + 30c_6\hat{\mu}_B^4 + \frac{4!}{6!}c_4\epsilon_1\hat{\mu}_B^6 + \frac{4!}{8!}c_4\epsilon_2\hat{\mu}_B^8$$

$$\chi_3^B(\hat{\mu}_B) = 24c_4\hat{\mu}_B + 120c_6\hat{\mu}_B^3 + \frac{4!}{5!}c_4\epsilon_1\hat{\mu}_B^5 + \frac{4!}{7!}c_4\epsilon_2\hat{\mu}_B^7$$

$$\chi_4^B(\hat{\mu}_B) = 24c_4 + 360c_6\hat{\mu}_B^2 + c_4\epsilon_1\hat{\mu}_B^4 + \frac{4!}{6!}c_4\epsilon_2\hat{\mu}_B^6.$$

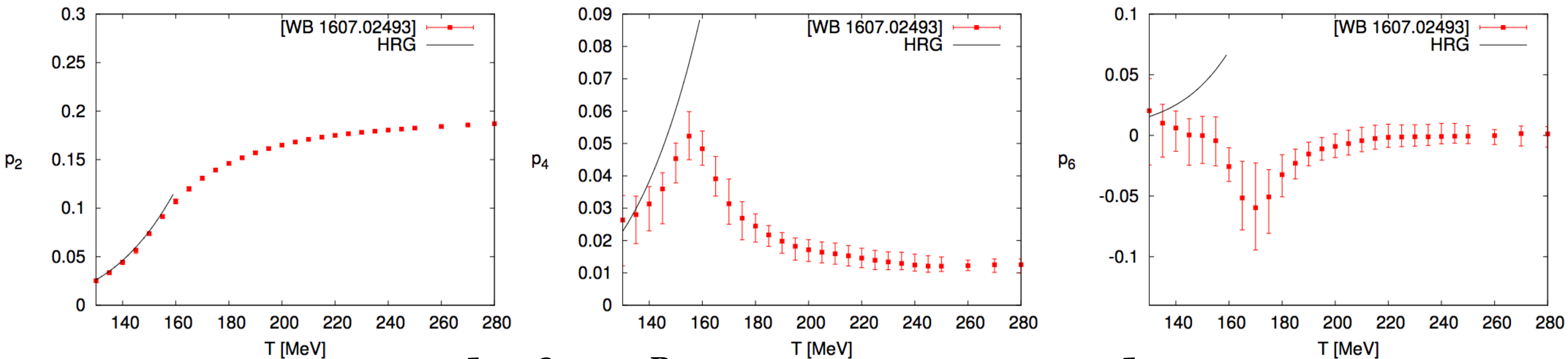
See also M. D'Elia et al., PRD (2017)

Pressure coefficients

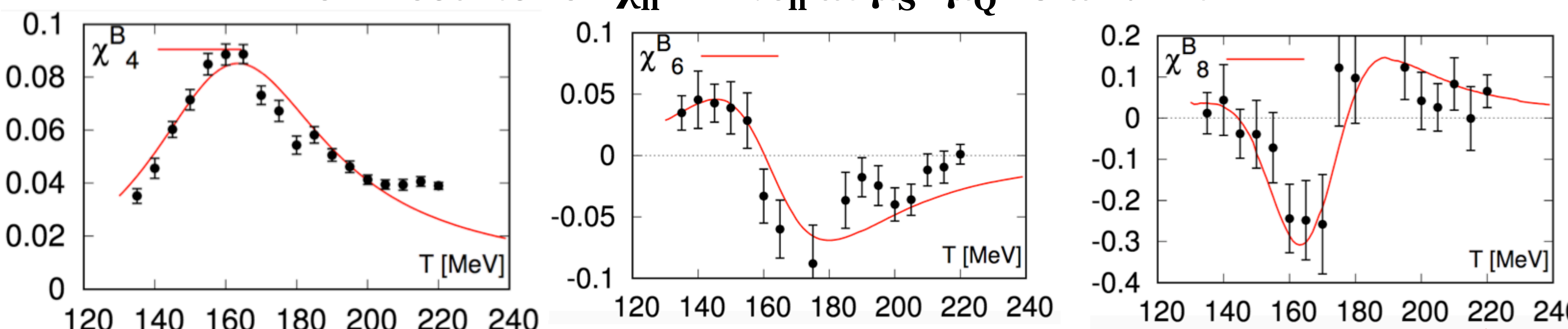
Simulations at imaginary μ_B:

Continuum, O(10^4) configurations, errors include systematics (WB: NPA (2017))

Strangeness neutrality



New results for $\chi_n^B = n! c_n$ at $\mu_S=\mu_Q=0$ and Nt=12

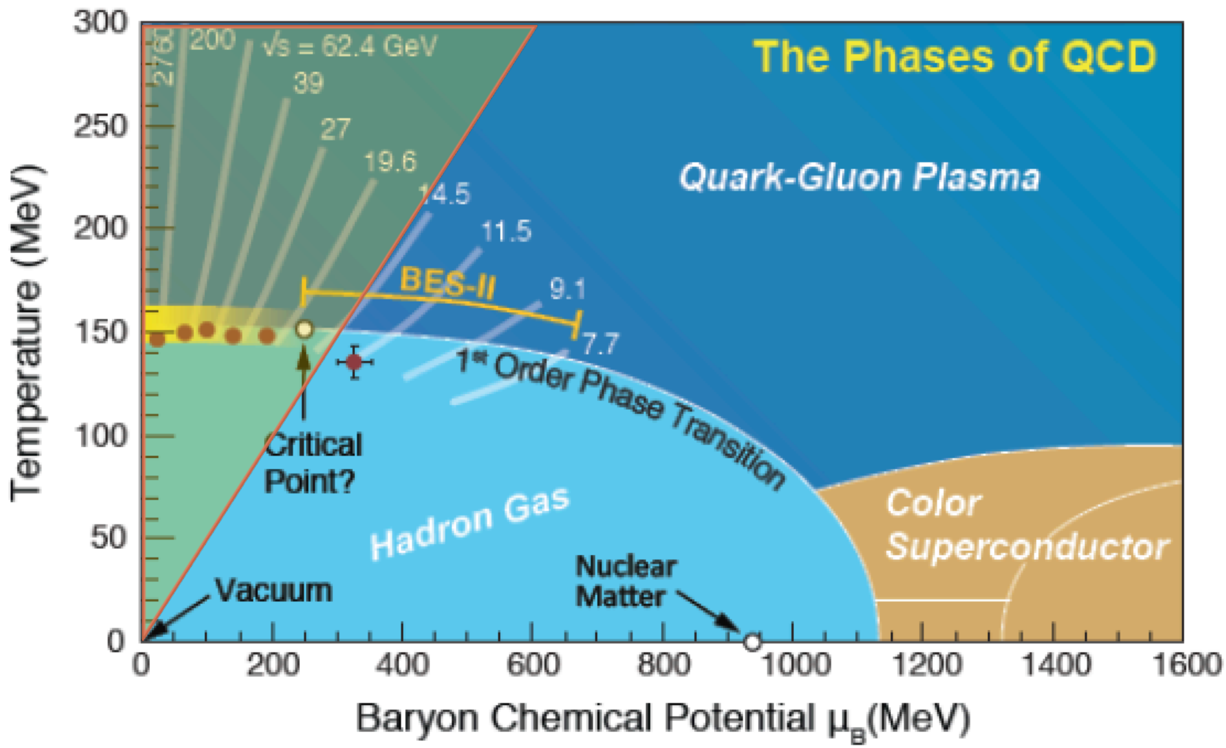


WB, JHEP (2018)

Range of validity of equation of state

- We now have the equation of state for $\mu_B/T \leq 2$ or in terms of the RHIC energy scan:

$$\sqrt{s} = 200,\ 62.4,\ 39,\ 27,\ 19.6,\ 14.5\text{GeV}$$

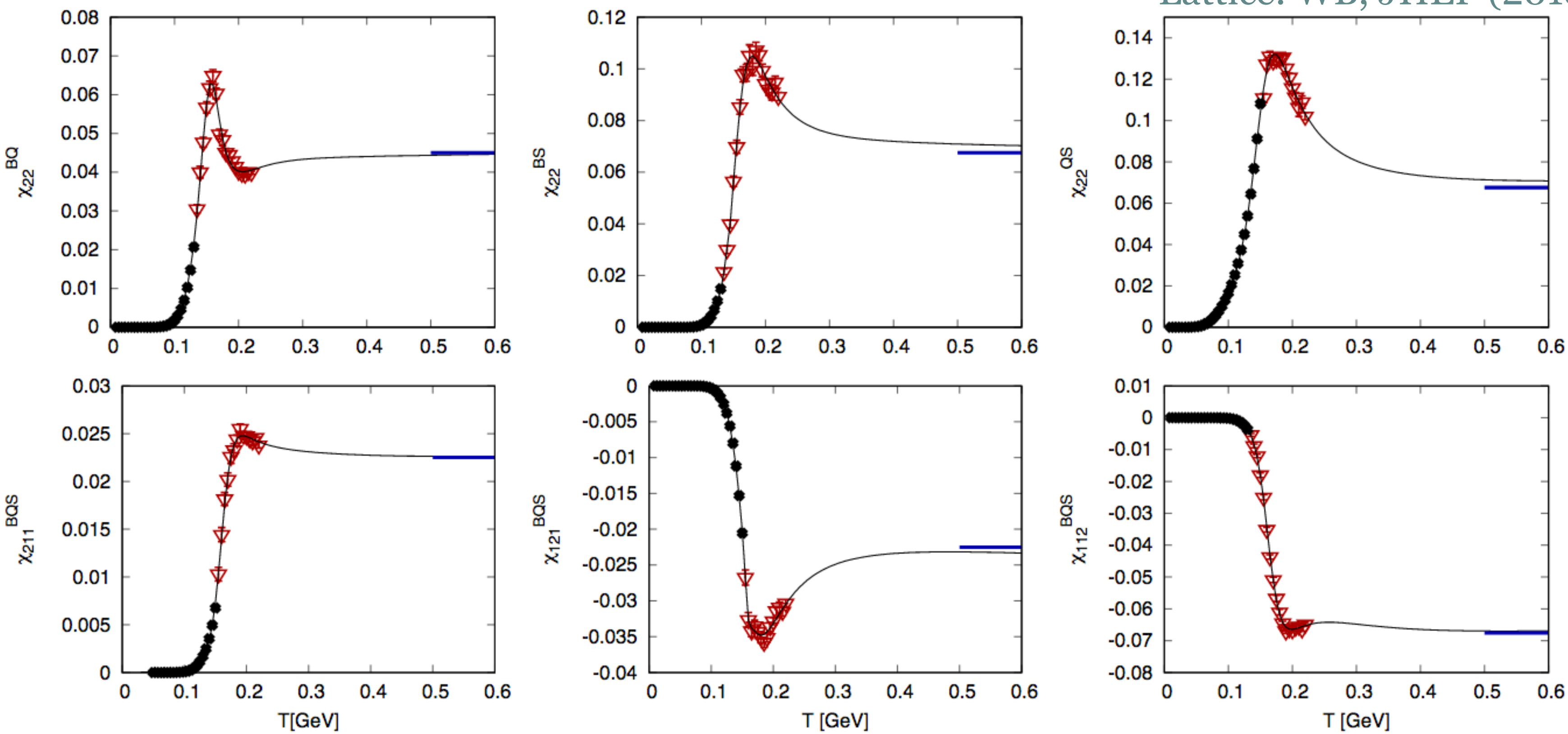


QCD Equation of state for $\mu_B, \mu_S, \mu_Q > 0$

J. Noronha-Hostler, C.R. et al., 1902.06723

$$\frac{p(T,\mu_B,\mu_Q,\mu_S)}{T^4} = \sum_{i,j,k} \frac{1}{i!j!k!} \chi_{ijk}^{BQS} \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k$$

Lattice: WB, JHEP (2018)

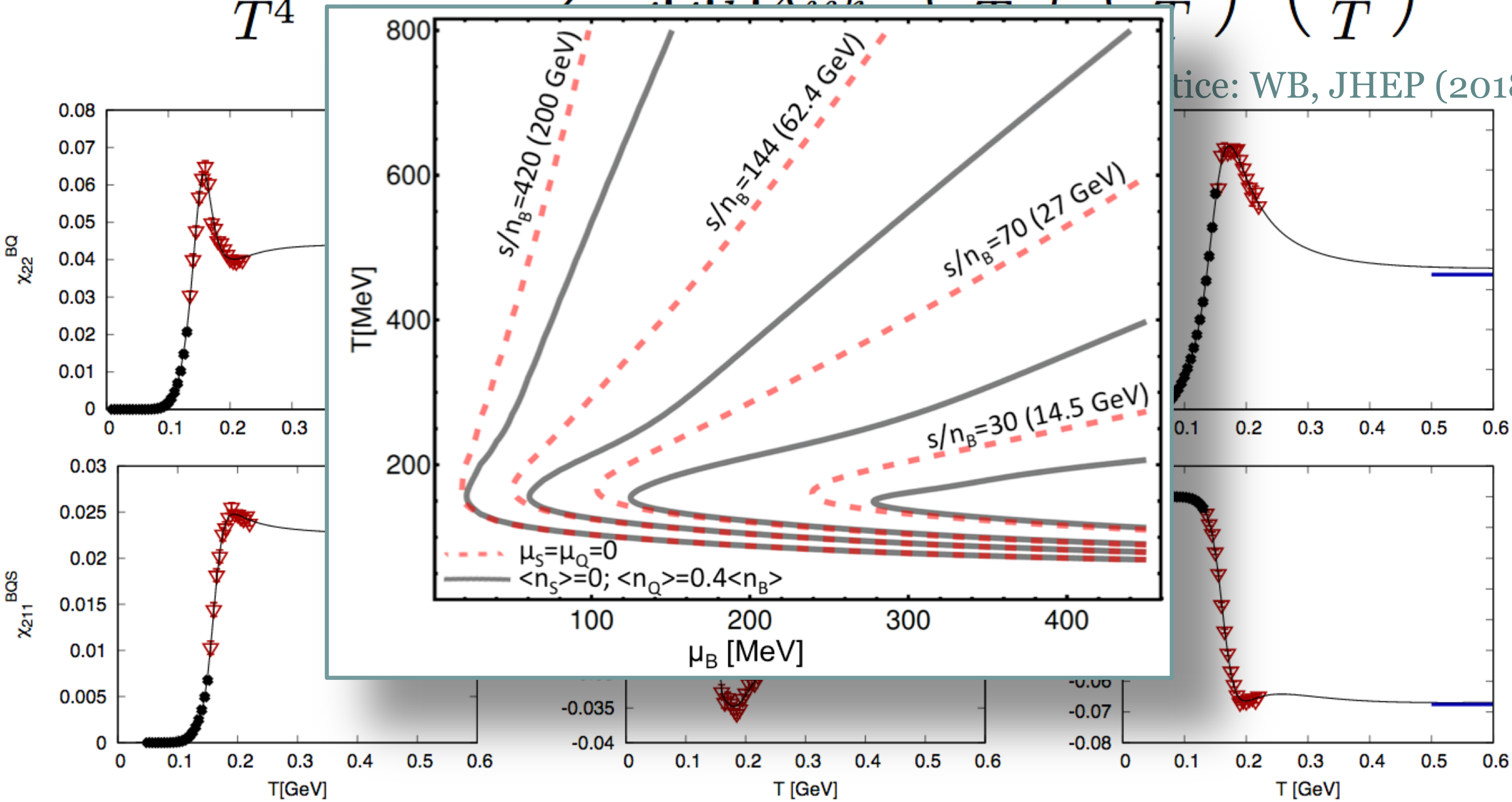


QCD Equation of state for $\mu_B, \mu_S, \mu_Q > 0$

J. Noronha-Hostler, C.R. et al., 1902.06723

$$\frac{p(T,\mu_B,\mu_Q,\mu_S)}{T^4} = \sum \frac{1}{i!j!k!}\chi_{ijk}^{BQS}\left(\frac{\mu_B}{T}\right)^i\left(\frac{\mu_Q}{T}\right)^j\left(\frac{\mu_S}{T}\right)^k$$

tice: WB, JHEP (2018)



Introducing a 3D-Ising critical point

IMPLEMENT THE SCALING BEHAVIOR OF THE 3D ISING MODEL EOS

MAP THE 3D ISING MODEL PHASE DIAGRAM ONTO THE QCD ONE

ESTIMATE THE CONTRIBUTION FROM THE CRITICAL POINT

RECONSTRUCT THE FULL PRESSURE OVER THE WHOLE PHASE DIAGRAM

P. Parotto, M. Bluhm, D. Mroczek, M. Nahrgang, J. Noronha-Hostler, K. Rajagopal, C. Ratti, T. Schaefer, M. Stephanov: hep-ph/1805.05249

Implement the scaling EoS for 3D Ising

Parametrization of the scaling form of the EoS can be given for magnetization M, magnetic field h and reduced temperature $r = (T - T_c)/T_c$ in 3D Ising model:

$$(\mathbf{R}, \theta) \longmapsto (\mathbf{r}, \mathbf{h}) : \qquad \begin{aligned} M &= M_0 R^{\beta} \theta \\ h &= h_0 R^{\beta\delta} \tilde{h}(\theta) \\ r &= R(1 - \theta^2) \end{aligned}$$

where:

- $M_0 \simeq 0.605$, $h_0 \simeq 0.394$ are normalization constants;
- $\tilde{h}(\theta) = \theta(1 + a\,\theta^2 + b\,\theta^4)$ with $(a = -0.76201, b = 0.00804)$;
- $R \geq 0$ and $|\theta| \leq 1.154$ (second zero of $\tilde{h}(\theta)$);
- $\beta \simeq 0.326$, $\delta \simeq 4.80$ are critical exponents.

C. Nonaka and M. Asakawa, Phys.Rev. C71 (2005) 044904, R. Guida and J. Zinn-Justin, Nucl.Phys. B489 (1997) 626-652, P. Schofield, Phys. Rev. Lett. 23 (1969) 109

Implement the scaling EoS for 3D Ising

Construct (Helmoltz) and thus Gibbs free energy densities:

$$F(M,r) = h_0 M_0 R^{2-\alpha} g(\theta) \quad \longrightarrow \quad G(r,h) = F(M,r) - Mh$$

Thanks to the map:

$$(\mathbf{R},\theta) \longmapsto (\mathbf{r},\mathbf{h}) \longleftrightarrow (\mathbf{T},\mu_{\mathbf{B}})$$

we can write the pressure in QCD as:

$$\boxed{P_{\text{Ising}}(T,\mu_B) = -G(T(R,\theta),\mu_B(R,\theta)) = h_0 M_0 R^{2-\alpha}\left[g(\theta) - \theta\tilde{h}(\theta)\right]}$$

NOTE: Explicit functional form of $G(T(R,\theta),\mu_B(R,\theta))$ ONLY as a function of (R,θ). Evaluation will require numerical inversion of :

$$T(R,\theta) = T^* \qquad \mu_B(R,\theta) = \mu_B^*$$

C. Nonaka and M. Asakawa, Phys.Rev. C71 (2005) 044904, R. Guida and J. Zinn-Justin, Nucl.Phys. B489 (1997) 626-652

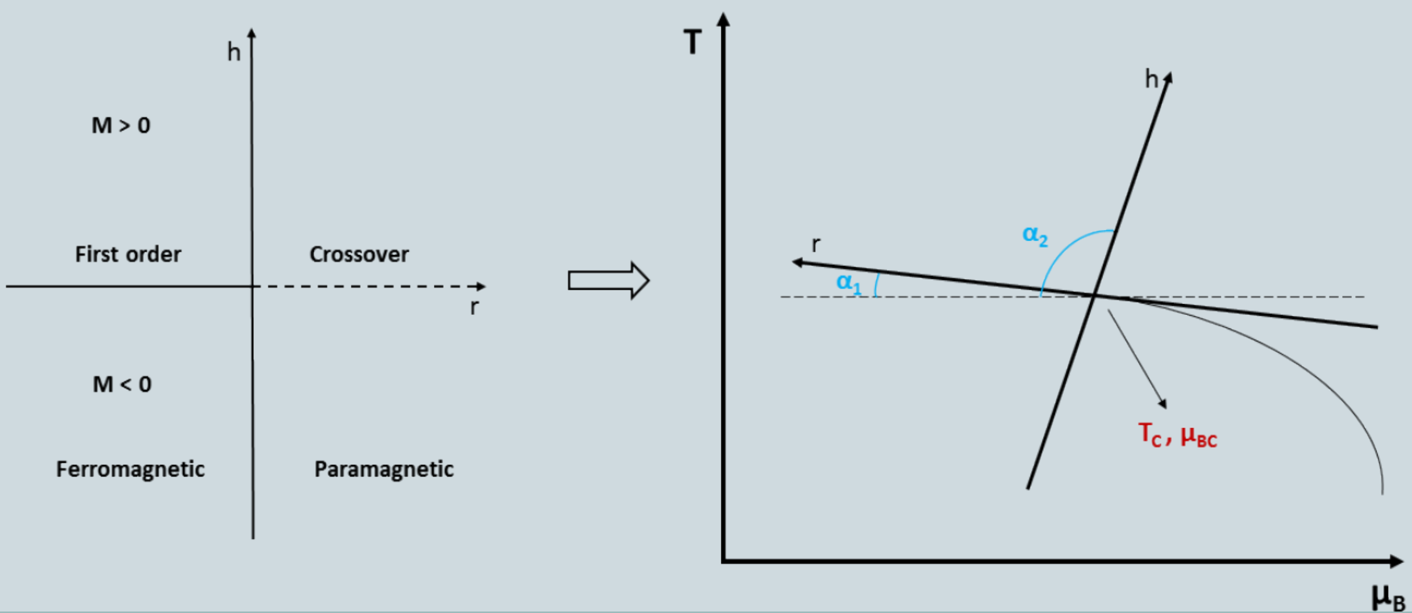
Map the phase diagram

The relation between the Ising model scaling variables (h, r) and the QCD thermodynamic coordinates (T, μ_B), can be expressed in linear form, with the use of **six parameters**:

$$(\mathbf{r}, \mathbf{h}) \longleftrightarrow (\mathbf{T}, \mu_{\mathbf{B}})$$

$$\frac{T - \mathbf{T_C}}{\mathbf{T_C}} = \mathbf{w}\,(r\rho\,\sin\alpha_1 + h\,\sin\alpha_2)$$

$$\frac{\mu_B - \mu_{\mathbf{BC}}}{\mathbf{T_C}} = \mathbf{w}\,(-r\rho\,\cos\alpha_1 - h\,\cos\alpha_2)$$



Map the phase diagram

- Exploit the parametric nature of the EoS to constrain the values of the parameters:
  - Theoretical (a priori) arguments (i.e. require **thermodynamic stability and causality**)
  - Comparison (a posteriori) to future BES-II program experimental data
- How is the choice of the parameters driven?
  - From Lattice QCD calculations: T_C<150 *MeV*, $\mu_{BC} > 2T_C$
  - place the critical point in the region of the phase diagram accessible to the BESII
  - we want to investigate the role of w and ρ

Map the phase diagram

- The number of free parameters is reduced:
  - Assume the shape of transition line is a parabola (good approximation at BES-like energies) → reduce to four parameters:

$$\frac{T_C}{T_C(\mu_B=0)} = 1 + \kappa\left(\frac{\mu_B}{T_C(\mu_B=0)}\right)^2 + \mathcal{O}(\mu_B^4)$$

with the values $T_C(\mu_B=0)=155$ *MeV*, $\kappa=-0.0149$.

- For a chosen value of μ_{BC}, one gets

$$T_C = T_0 + \frac{\kappa}{T_0}\mu_{BC}^2 \qquad \alpha_1 = \tan^{-1}\left(2\frac{\kappa}{T_0}\mu_{BC}\right)$$

- In the following:

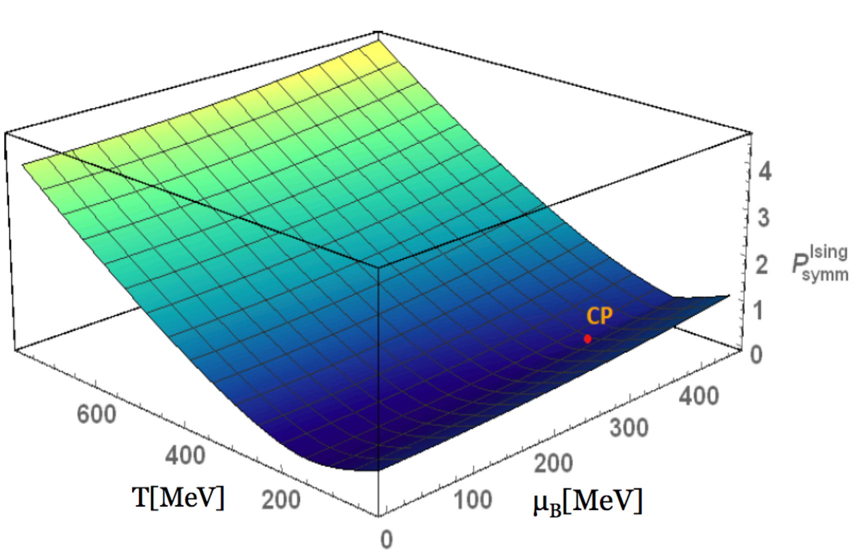
$$\mu_{BC} = 350\,\mathrm{MeV} \qquad w = 1 \qquad T_C \simeq 143.2\,\mathrm{MeV}$$

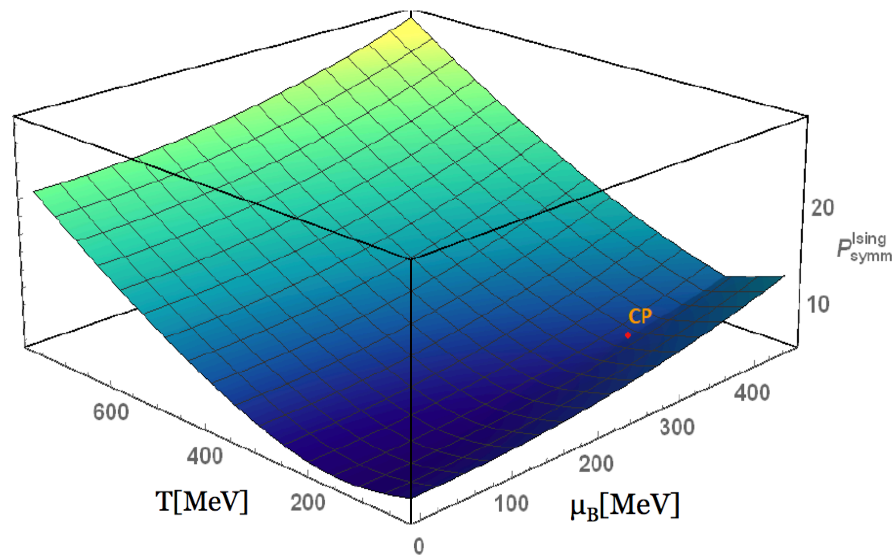
$$\alpha_2 - \alpha_1 = \pi/2 \qquad \rho = 2 \qquad \alpha_1 \simeq 4^\circ$$

Critical pressure

The critical pressure for this parameter choice (left) and for a smaller value of $w = 0.25$ (right)

$\Rightarrow$ **smaller w results in larger critical contribution**





NOTE: the critical pressure is symmetrized around $\mu_B = 0$ to ensure $c^n(T) = 0, \forall n$ odd

Expansion coefficients

The Taylor coefficients from Lattice QCD contain an **"Ising"** contribution from the critical point and a **"Non-Ising"** one:

$$T^4 c_n^{\mathrm{LAT}}(T) = T^4 c_n^{\mathrm{Non\text{-}Ising}}(T) + f(T,\mu_B) c_n^{\mathrm{Ising}}(T)$$

where $f(T,\mu_B)$ is a regular function of T and μ_B, with dimension 4.
In the following, we will choose $f(T,\mu_B) = T_C^4$.

The Ising coefficients are just:

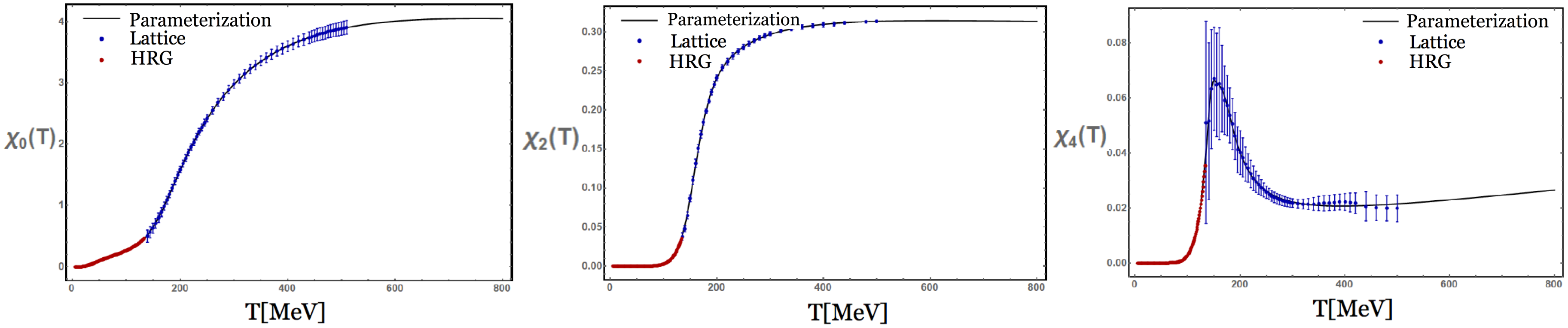
$$c^n_{\mathrm{Ising}}(T) = \frac{1}{n!}\chi_B^n(T,\mu_B=0) = -\frac{1}{n!}T^n \left.\frac{\partial^n G}{\partial \mu_B^n}\right|_{\mu_B=0}$$

and their explicit expression can be derived with multiple use of the chain rule.

Parameterization of lattice coefficients

In order to be used in our procedure, Lattice QCD coefficients need to:

- Be extended to low temperatures
  - extend with HRG model
- Be smooth enough to allow for derivatives
  - parameterization

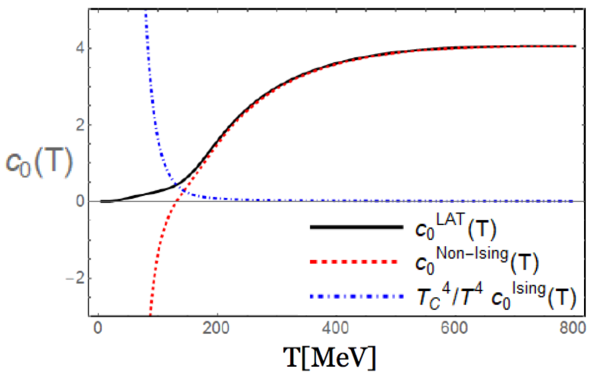


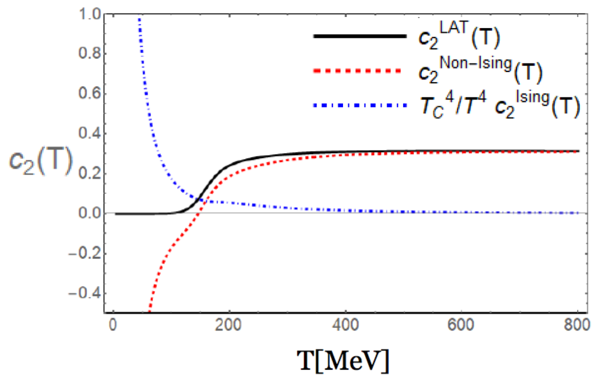
Lattice results from: WB: JHEP (20110), PRD (2015)

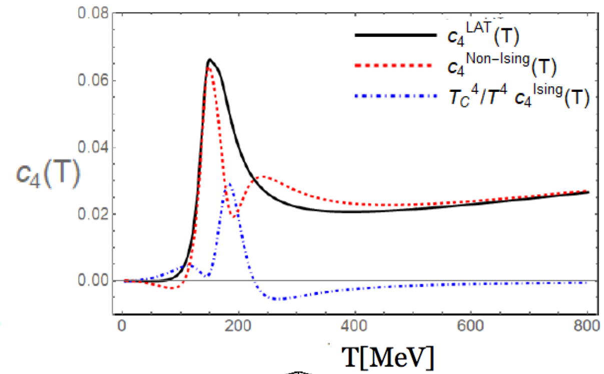
Expansion coefficients

Extract the "regular" contribution as the difference between the lattice and Ising ones

$$T^4 c_n^{\text{LAT}}(T) = T^4 c_n^{\text{Non-Ising}}(T) + T_C^4 c_n^{\text{Ising}}(T)$$

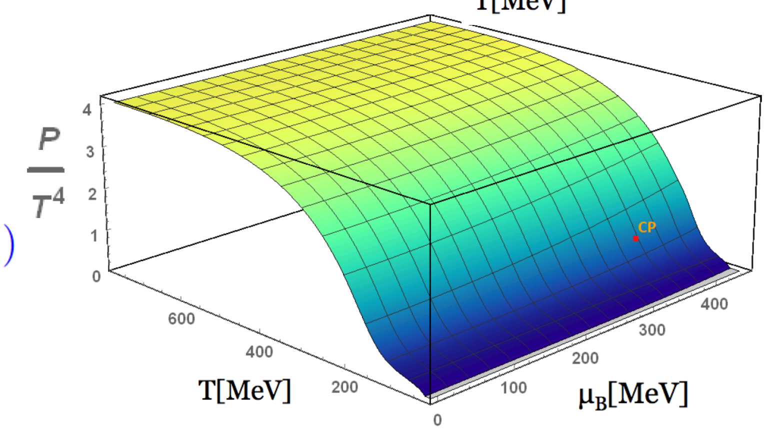






Total pressure becomes:

$$P(T, \mu_B) = T^4 \sum_n c^n_{\text{Non-Ising}}(T) \left(\frac{\mu_B}{T}\right)^n + T_C^4 P_{\text{Ising}}(T, \mu_B)$$



Final EoS: Thermodynamic quantities

Once the pressure is determined, all thermodynamic quantities can be calculated:

Entropy density : $$\frac{S}{T^3} = \frac{1}{T^3}\frac{\partial P}{\partial T}$$

Baryon density : $$\frac{n_B}{T^3} = \frac{1}{T^3}\frac{\partial P}{\partial \mu_B}$$

Energy density : $$\frac{\epsilon}{T^4} = \frac{S}{T^3} - \frac{P}{T^4} + \frac{\mu_B}{T}\frac{n_B}{T^3}$$

Speed of sound : $$c_s^2 = \left(\frac{\partial P}{\partial \epsilon}\right)_{s/n_B} = \frac{n_B^2 \partial_T^2 P - 2 S n_B \partial_T \partial_{\mu_B} P + S^2 \partial_{\mu_B}^2 P}{(\epsilon + P)\left(\partial_T^2 P \partial_{\mu_B}^2 P - (\partial_T \partial_{\mu_B} P)^2\right)}$$

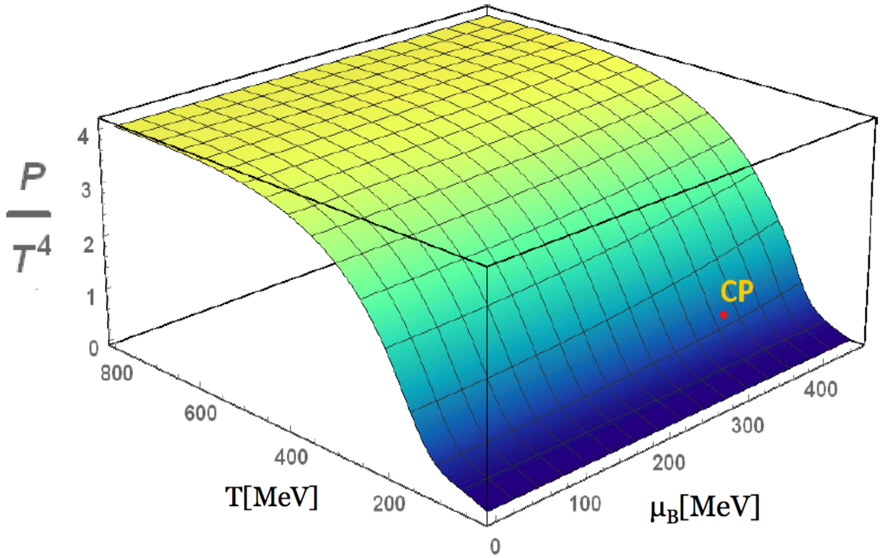
Baryon susceptibilities : $$\chi_n^B = \frac{\partial (P/T^4)}{\partial (\mu_B/T)}$$ (only the second one in the paper)

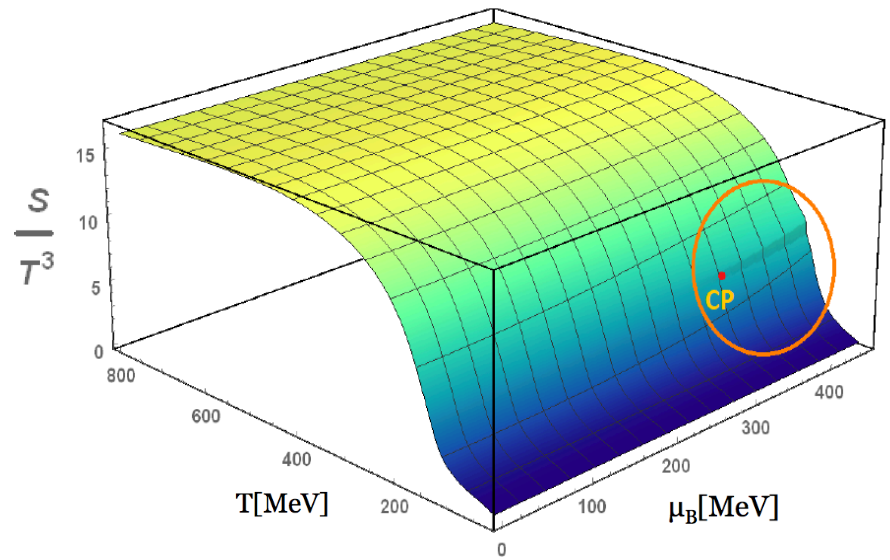
Final EoS: Pressure and Entropy Density

P. Parotto et al.,: hep-ph/1805.05249

The final EoS covers the range:

$$T = 30 - 800\,\text{MeV} \qquad \mu_B = 0 - 450\,\text{MeV}$$





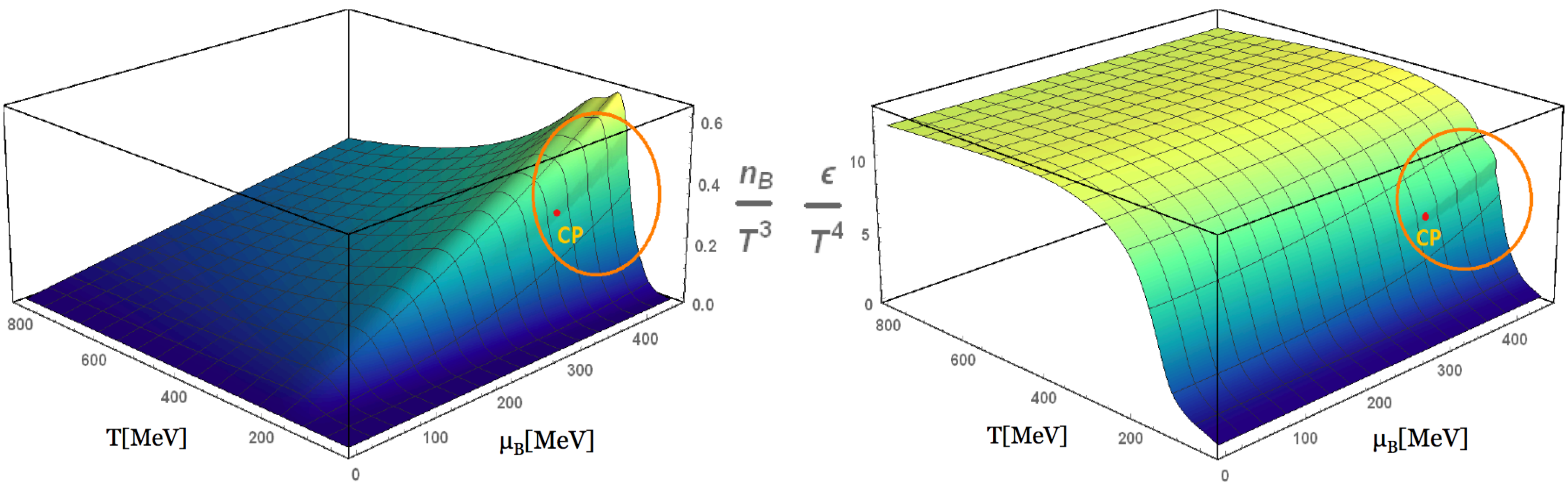
Although the effect is barely visible in the pressure, the entropy density shows a discontinuity in the first order transition region.

Final EoS: Baryon and Energy Density

P. Parotto et al.,: hep-ph/1805.05249

The final EoS covers the range:

$$T = 30 - 800\,\text{MeV} \qquad \mu_B = 0 - 450\,\text{MeV}$$



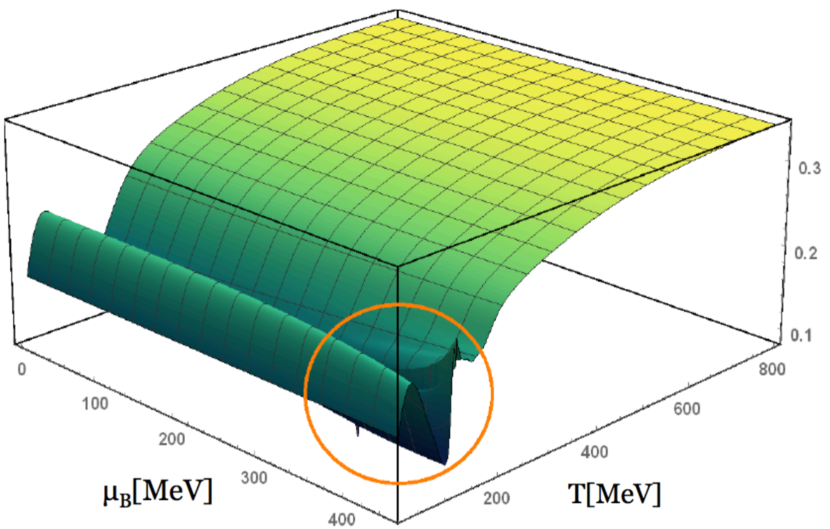
Baryon and energy density also show a discontinuity in the first order transition region.

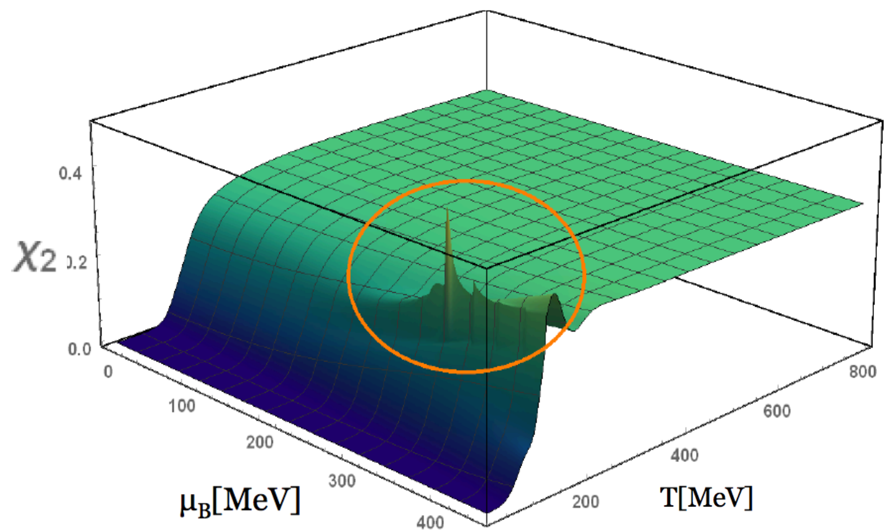
Final EoS: Speed of sound and baryon number χ_2

P. Parotto et al.,: hep-ph/1805.05249

The final EoS covers the range:

$$T = 30 - 800\,\text{MeV} \qquad \mu_B = 0 - 450\,\text{MeV}$$



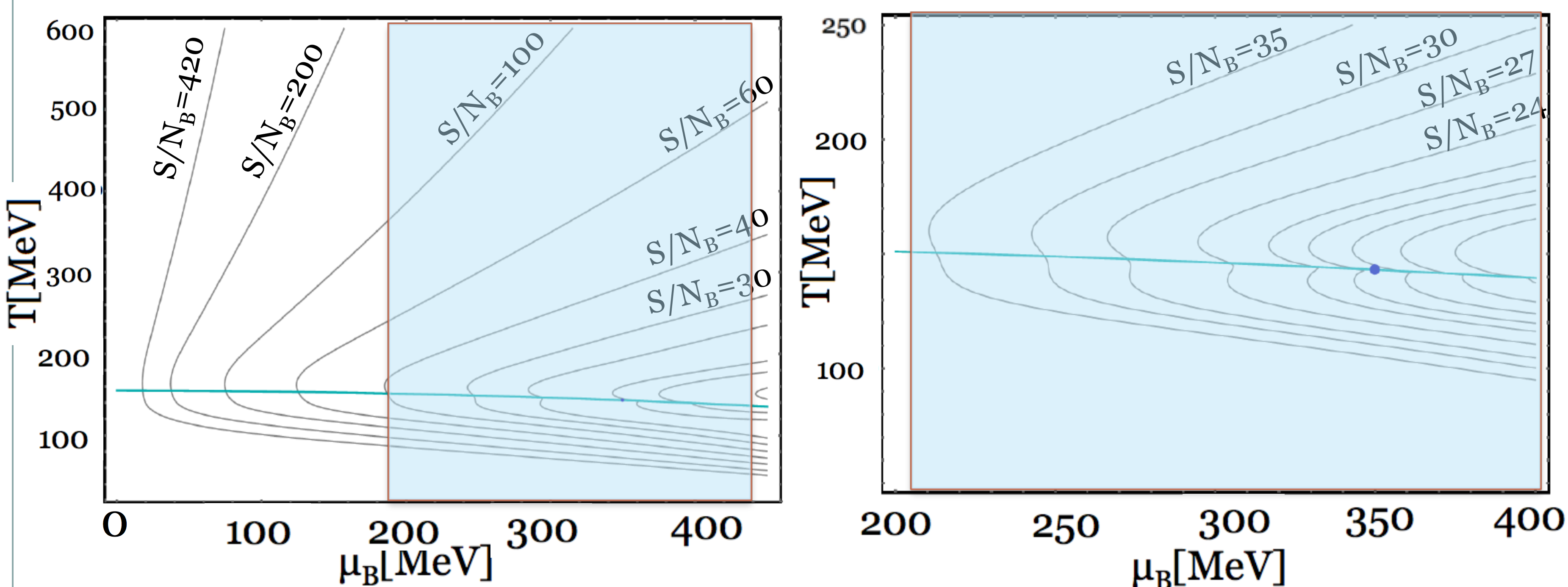


The speed of sound and the second baryon number cumulant show a (weak) dip and a (strong) peak at the critical point respectively.

Final EoS: Isentropic trajectories

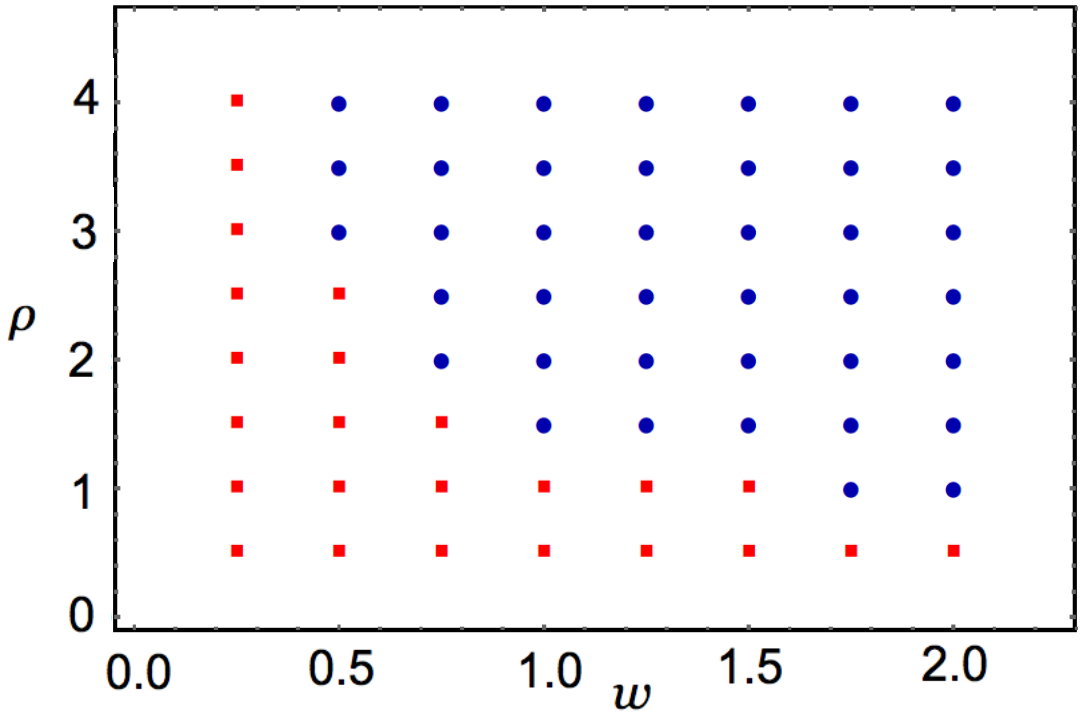
P. Parotto et al.,: hep-ph/1805.05249

- Relevant for hydrodynamic evolution are the lines of $s/n_B = \text{const}$:
  - Low-μ_B: match behavior from Lattice QCD
  - Close to the CP: some structure appears



Final EoS: explore parameter space

Keeping the position of the critical point fixed, as well as the orientation of the axes mapped from the 3D Ising model phase diagram, we varied the parameters w, ρ, and required thermodynamic stability ($P, S, n_B, \epsilon, c_s^2 > 0$) and causality ($c_s^2 \leq 1$)



In blue (dots) the values corresponding to acceptable choices, in red (squares) the values leading to pathological EoS's

Conclusions

- Need for quantitative results at finite-density to support the experimental programs
- Current lattice results for thermodynamics up to $\mu_B/T \leq 2$
- We created a family of EoS matching lattice QCD calculations up to $O(\mu_B{}^4)$
- These equations of state can be readily used in hydrodynamic simulations
- The code is available at:

https://www.bnl.gov/physics/best/resources.php

Backup slides

Merging with HRG model at low T

$\Rightarrow$ **Smooth merging with Hadron Resonance Gas (HRG) model through:**

$$\frac{P_{\mathrm{Final}}(T,\mu_B)}{T^4} = \frac{P(T,\mu_B)}{T^4}\frac{1}{2}\left[1+\tanh\left(\frac{T-T'(\mu_B)}{\Delta T}\right)\right] + \frac{P_{\mathrm{HRG}}(T,\mu_B)}{T^4}\frac{1}{2}\left[1-\tanh\left(\frac{T-T'(\mu_B)}{\Delta T}\right)\right]$$

where:

- $T'(\mu_B)$ is the "transition" temperature, depending on μ_B:

$$T'(\mu_B) = T_0 + \frac{\kappa}{T_0}\mu_B^2 - T^*$$

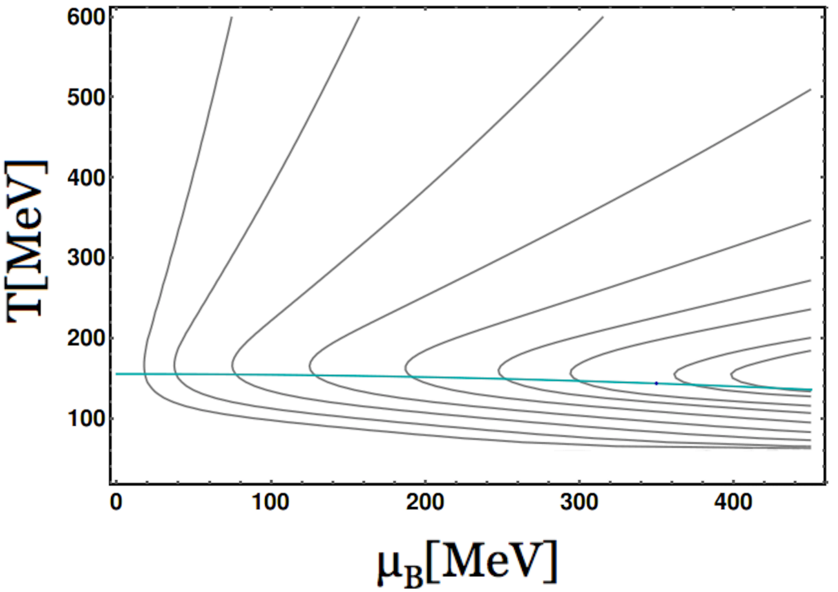
- ΔT is a measure of the overlap region size

$\Rightarrow$ **In the following:** $T^* = 23\,\mathrm{MeV}$, $\Delta T = 17\,\mathrm{MeV}$

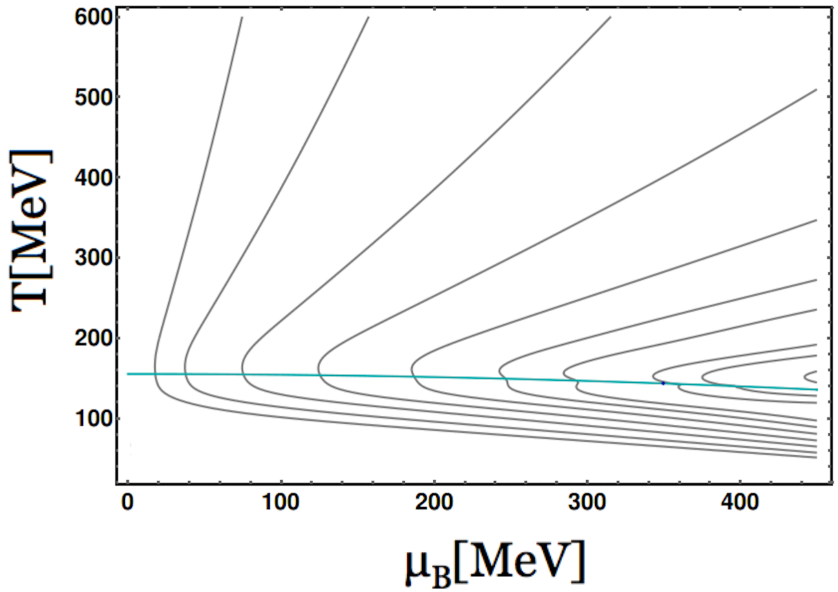
Final EoS: Isentropic trajectories

P. Parotto et al.,: hep-ph/1805.05249

Isentropic trajectories from lattice QCD



Our isentropic trajectories



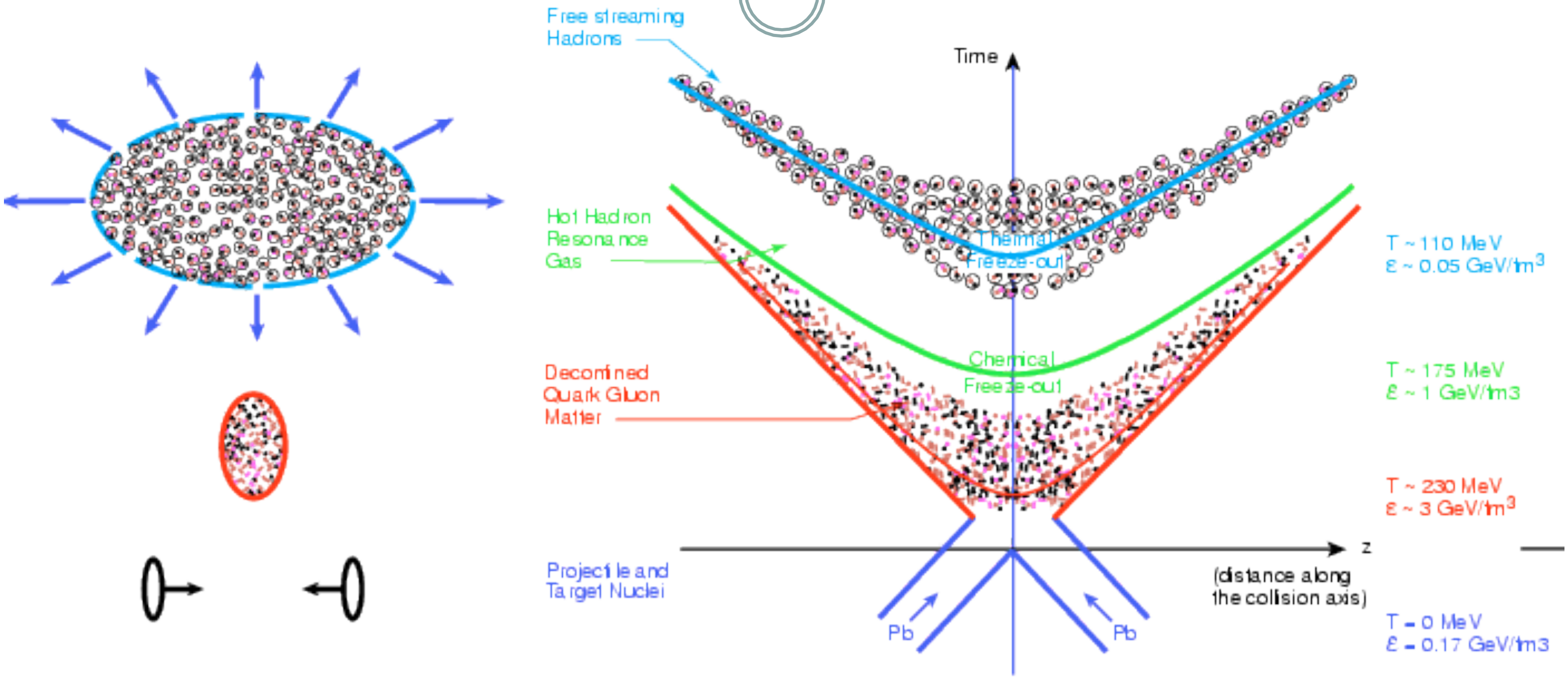
μ_B[MeV]

Fluctuations of conserved charges

COMPARISON TO EXPERIMENT: CHEMICAL FREEZE-OUT PARAMETERS

COMPARISON TO HRG MODEL: SEARCH FOR THE CRITICAL POINT

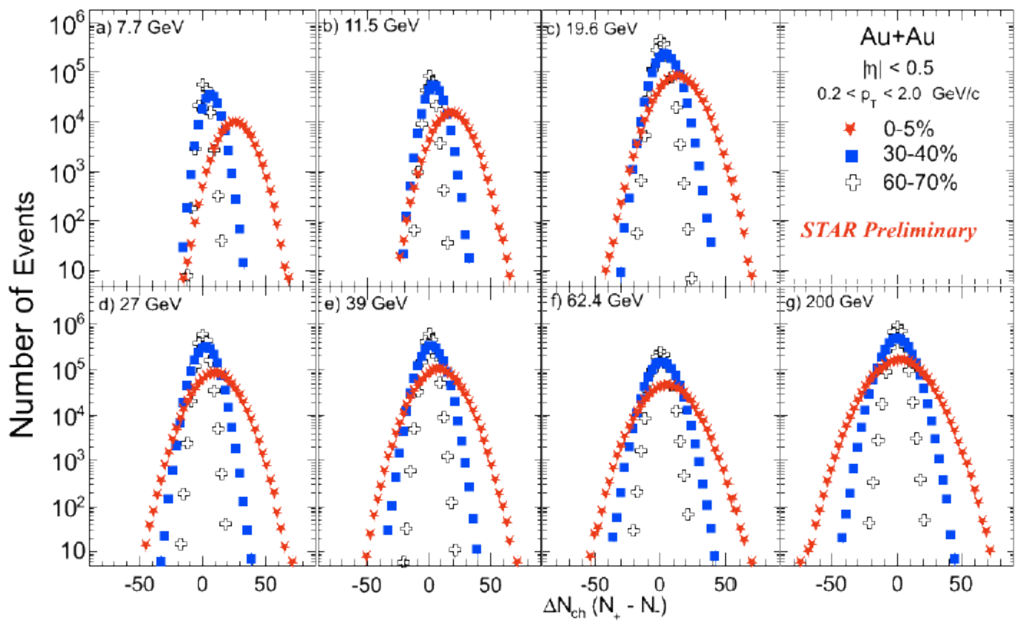
Evolution of a heavy-ion collision



- Chemical freeze-out: inelastic reactions cease: the chemical composition of the system is fixed (particle yields and fluctuations)
- Kinetic freeze-out: elastic reactions cease: spectra and correlations are frozen (free streaming of hadrons)
- Hadrons reach the detector

Distribution of conserved charges

- Consider the number of electrically charged particles N_Q
- Its average value over the whole ensemble of events is $\langle N_Q \rangle$
- In experiments it is possible to measure its event-by-event distribution



STAR Collab.: PRL (2014)

Fluctuations on the lattice

- Fluctuations of conserved charges are the cumulants of their event-by-event distribution
- Definition: $$\chi_{lmn}^{BSQ} = \frac{\partial^{\,l+m+n} p/T^4}{\partial(\mu_B/T)^l \partial(\mu_S/T)^m \partial(\mu_Q/T)^n}.$$
- They can be calculated on the lattice and compared to experiment
- variance: $\sigma^2=\chi_2$ Skewness: $S=\chi_3/(\chi_2)^{3/2}$ Kurtosis: $\kappa=\chi_4/(\chi_2)^2$

$$S\sigma = \chi_3/\chi_2 \qquad \kappa\sigma^2 = \chi_4/\chi_2$$

$$M/\sigma^2 = \chi_1/\chi_2 \qquad S\sigma^3/M = \chi_3/\chi_1$$

Things to keep in mind

- Effects due to volume variation because of finite centrality bin width
  - Experimentally corrected by centrality-bin-width correction method

V. Skokov et al., PRC (2013), P. Braun-Munzinger et al., NPA (2017), V. Begun and M. Mackowiak-Pawlowska (2017)

- Finite reconstruction efficiency
  - Experimentally corrected based on binomial distribution

A.Bzdak,V.Koch, PRC (2012)

- Spallation protons
  - Experimentally removed with proper cuts in p_T
- Canonical vs Gran Canonical ensemble
  - Experimental cuts in the kinematics and acceptance

V. Koch, S. Jeon, PRL (2000)

- Baryon number conservation
  - Experimental data need to be corrected for this effect

P. Braun-Munzinger et al., NPA (2017)

- Proton multiplicity distributions vs baryon number fluctuations
  - Recipes for treating proton fluctuations

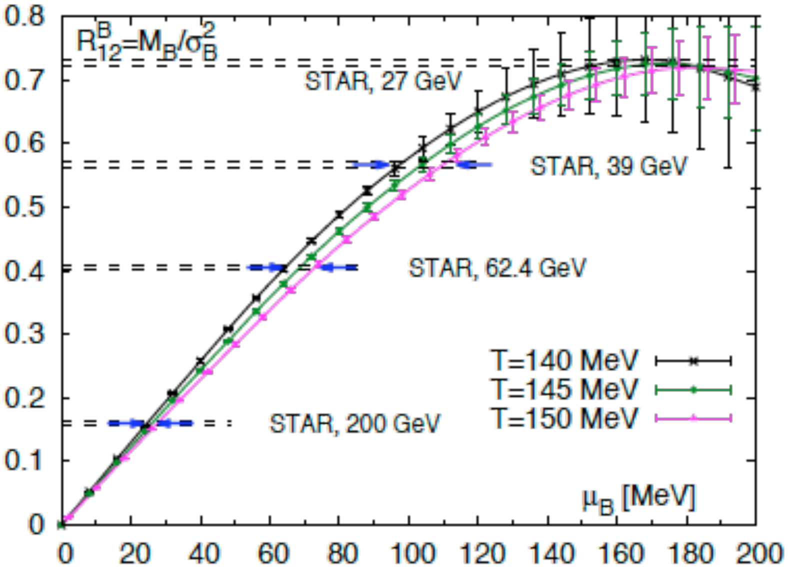
M. Asakawa and M. Kitazawa, PRC(2012), M. Nahrgang et al., 1402.1238

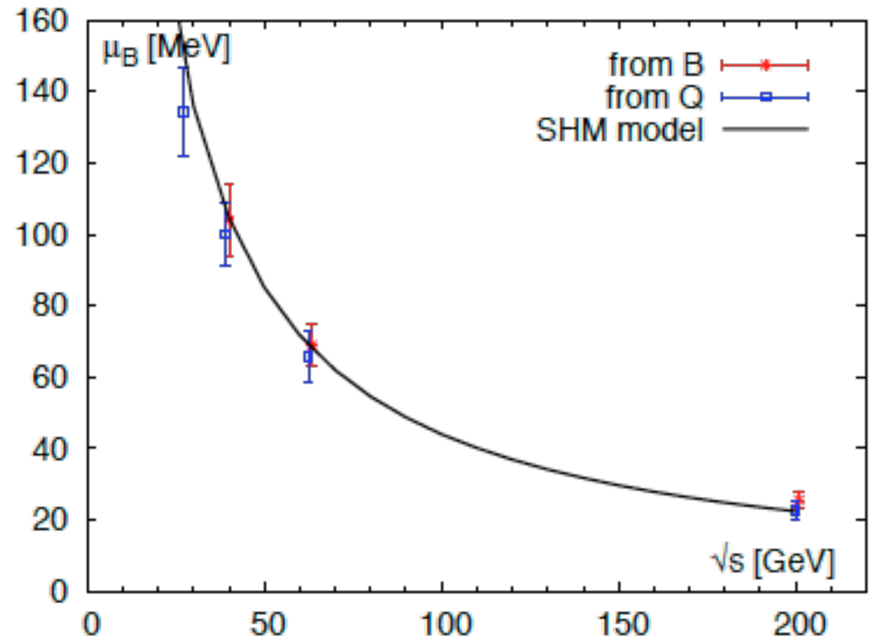
- Final-state interactions in the hadronic phase
  - Consistency between different charges = fundamental test

J.Steinheimer et al., PRL (2013)

Consistency between freeze-out of B and Q

- Independent fit of of R_{12}^{Q} and R_{12}^{B}: consistency between freeze-out chemical potentials





$\sqrt{s}[GeV]$	μ_B^f [MeV] (from B)	μ_B^f [MeV] (from Q)
200	25.8±2.7	22.8±2.6
62.4	69.7±6.4	66.6±7.9
39	105±11	101±10
27	-	136±13.8

WB: PRL (2014)
STAR collaboration, PRL (2014)

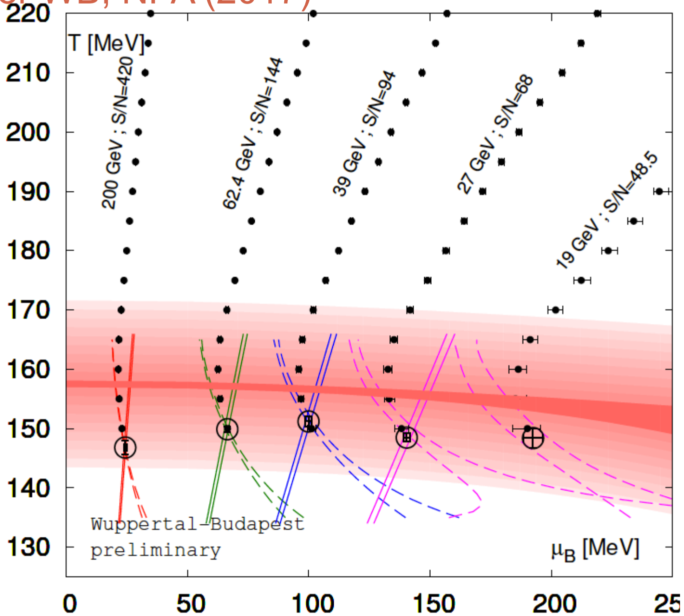
Freeze-out line from first principles

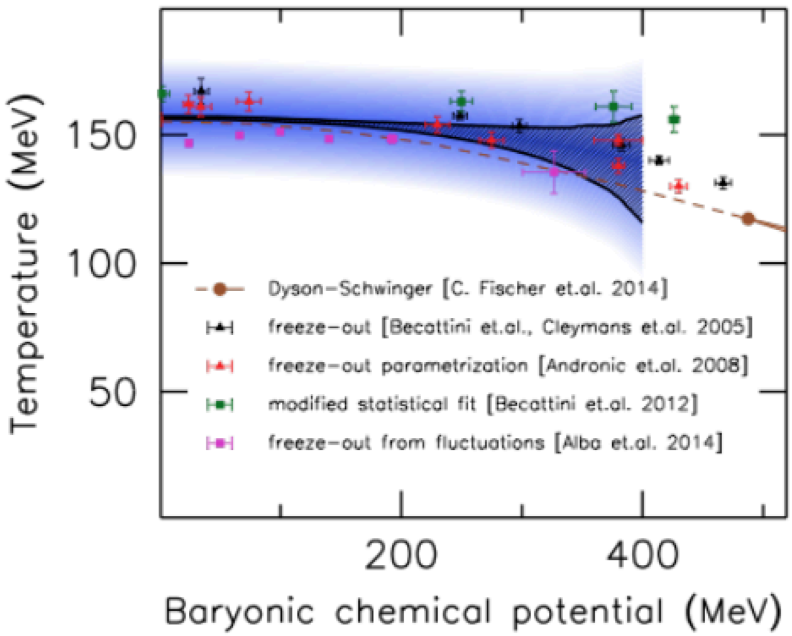
- Use T- and μ_B-dependence of R_{12}^Q and R_{12}^B for a combined fit:

$$R_{12}^Q(T,\mu_B) = \frac{\chi_1^Q(T,\mu_B)}{\chi_2^Q(T,\mu_B)} = \frac{\chi_{11}^{QB}(T,0) + \chi_2^Q(T,0)q_1(T) + \chi_{11}^{QS}(T,0)s_1(T)}{\chi_2^Q(T,0)}\frac{\mu_B}{T} + \mathcal{O}(\mu_B^3).$$

$$R_{12}^B(T,\mu_B) = \frac{\chi_1^B(T,\mu_B)}{\chi_2^B(T,\mu_B)} = \frac{\chi_2^B(T,0) + \chi_{11}^{BQ}(T,0)q_1(T) + \chi_{11}^{BS}(T,0)s_1(T)}{\chi_2^B(T,0)}\frac{\mu_B}{T} + \mathcal{O}(\mu_B^3)$$

C. Ratti for WB, NPA (2017)





Kaon fluctuations on the lattice

J. Noronha-Hostler, C.R. et al., 1607.02527

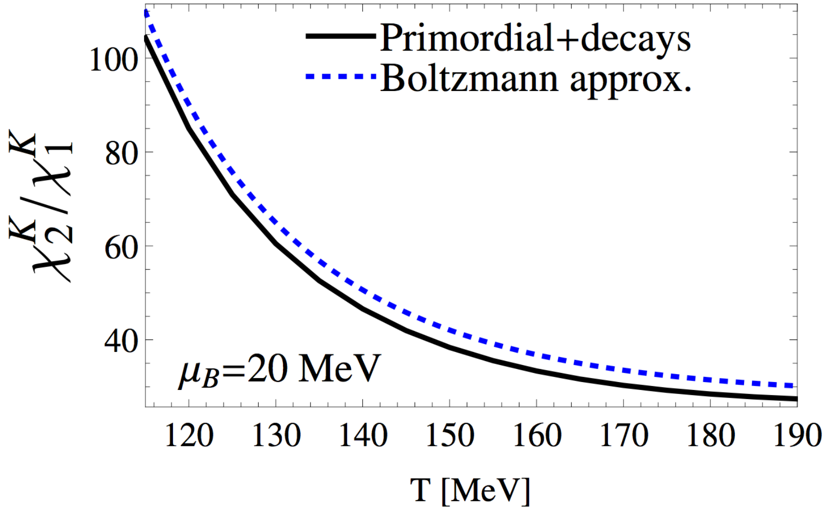
- Lattice QCD works in terms of conserved charges
- Challenge: isolate the fluctuations of a given particle species
- Assuming an HRG model in the Boltzmann approximation, it is possible to write the pressure as:

$$P(\hat{\mu}_B, \hat{\mu}_S) = P_{00}^{BS} + P_{10}^{BS}\cosh(\hat{\mu}_B) + P_{01}^{BS}\cosh(\hat{\mu}_S)$$
$$+ P_{11}^{BS}\cosh(\hat{\mu}_B - \hat{\mu}_S) + P_{12}^{BS}\cosh(\hat{\mu}_B - 2\hat{\mu}_S) + P_{13}^{BS}\cosh(\hat{\mu}_B - 3\hat{\mu}_S)$$

- Kaons in heavy ion collisions: primordial + decays
- Idea: calculate χ_2^K/χ_1^K in the HRG model for the two cases: only primordial kaons in the Boltzmann approximation vs primordial + resonance decay kaons

Kaon fluctuations on the lattice

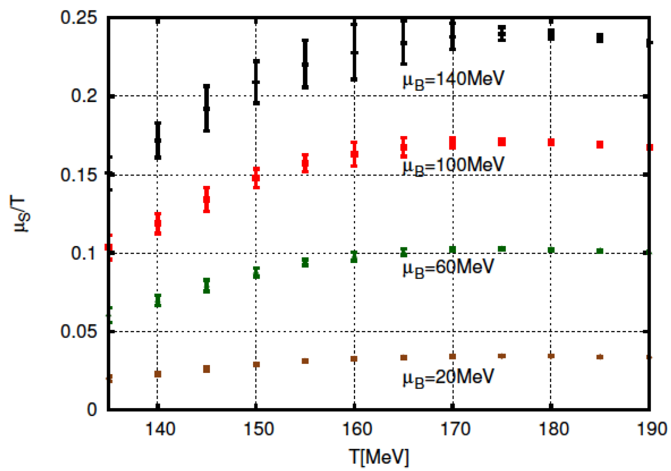
J. Noronha-Hostler, C.R. et al., forthcoming

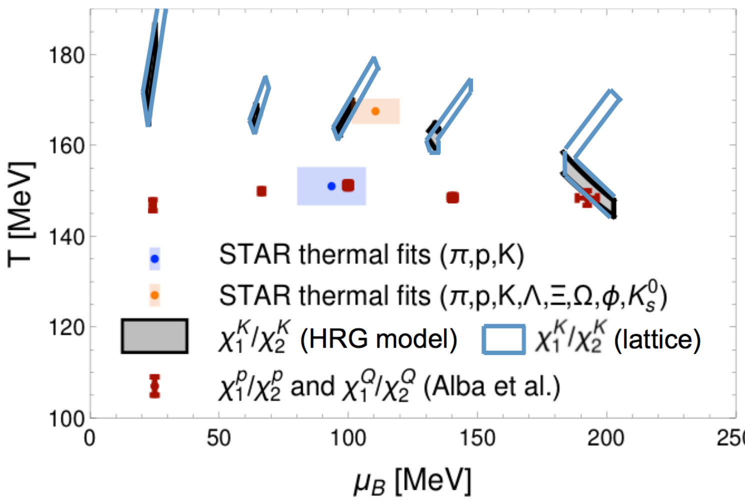


- Boltzmann approximation works well for lower order kaon fluctuations

$$\frac{\chi_2^K}{\chi_1^K} = \frac{\cosh(\hat{\mu}_S + \hat{\mu}_Q)}{\sinh(\hat{\mu}_S + \hat{\mu}_Q)}$$

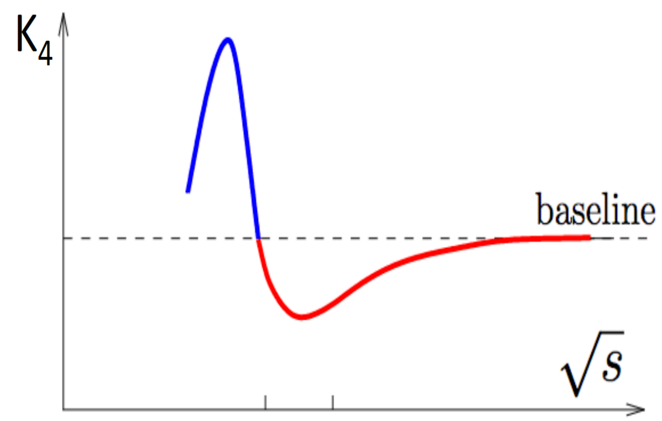
- χ_2^K/χ_1^K from primordial kaons + decays is very close to the Boltzmann approximation
- μ_S and μ_Q are functions of T and μ_B to match the experimental constraints:

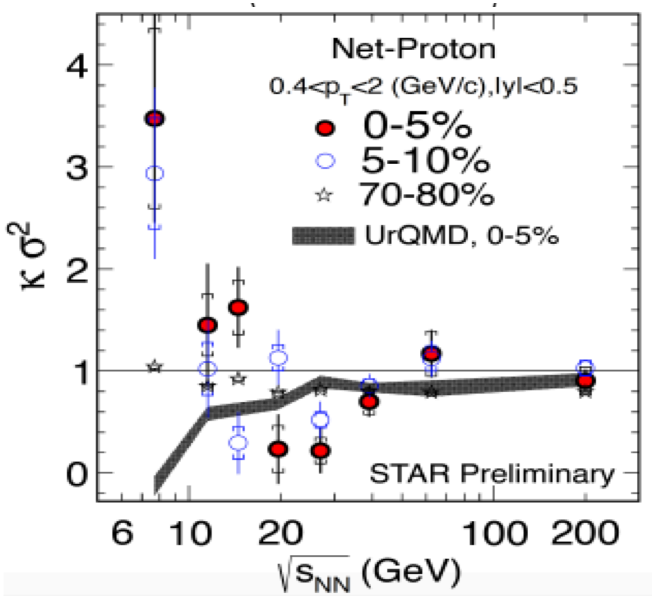




Fluctuations at the critical point

M. Stephanov, PRL (2009).





- Correlation length near the critical point

$$\xi \sim |T - T_c|^{-\nu} \text{ where } \nu > 0$$

$$\chi_2 = VT\xi^2$$
$$\chi_3 = 2VT^{3/2}\hat{\lambda}_3\xi^{9/2}$$
$$\chi_4 = 6VT^2[2\hat{\lambda}_3^2 - \hat{\lambda}_4]\xi^7$$

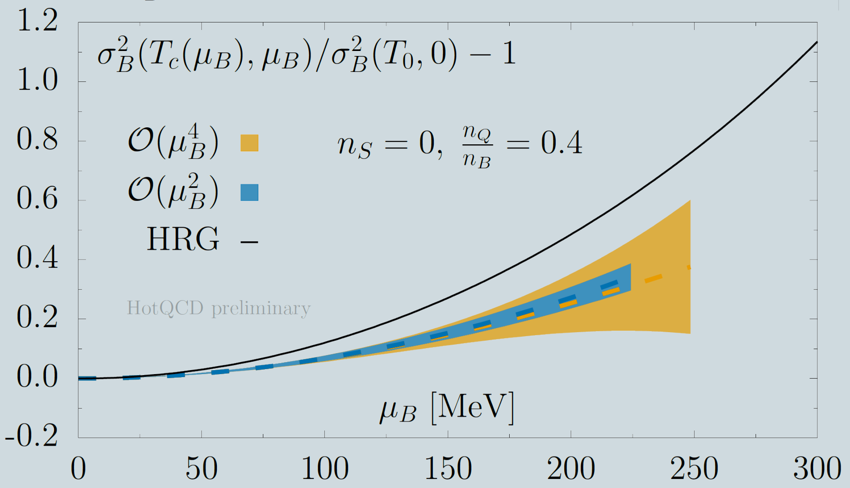
- Fluctuations are expected to diverge at the critical point
- Fourth-order fluctuations should have a non-monotonic behavior
- Preliminary STAR data seem to confirm this
- Can we describe this trend with lattice QCD?

Fluctuations along the QCD crossover

P. Steinbrecher for HotQCD, 1807.05607

Net-baryon variance

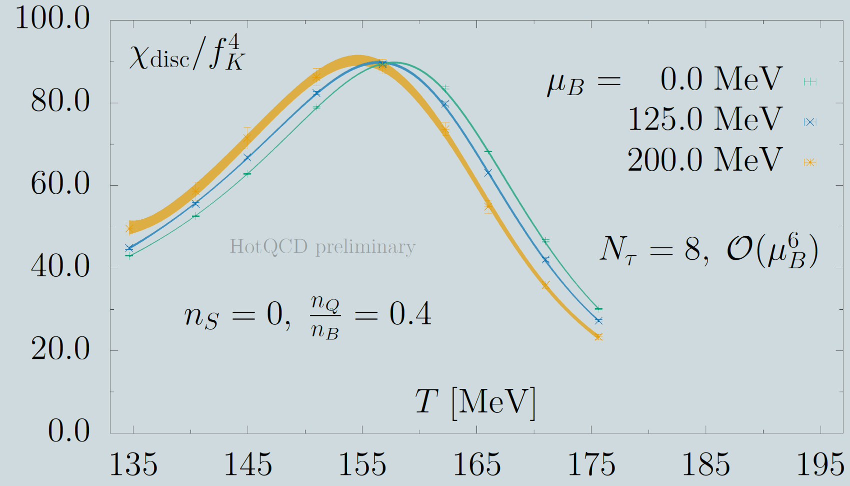
$$\frac{\sigma_B^2(T_c(\mu_B),\mu_B)-\sigma_B^2(T_0,0)}{\sigma_B^2(T_0,0)} = \lambda_2\left(\frac{\mu_B}{T_0}\right)^2 + \lambda_4\left(\frac{\mu_B}{T_0}\right)^4 + \mathcal{O}(\mu_B^6)$$



- Expected to be larger than HRG model result near the CP
- No sign of criticality

Disconnected chiral susceptibility

$$\chi_{\rm sub} \equiv \frac{T}{V} m_s\left(\frac{\partial}{\partial m_u}+\frac{\partial}{\partial m_d}\right)\Big[m_s(\Sigma_u+\Sigma_d)-(m_u+m_d)\Sigma_s\Big]$$



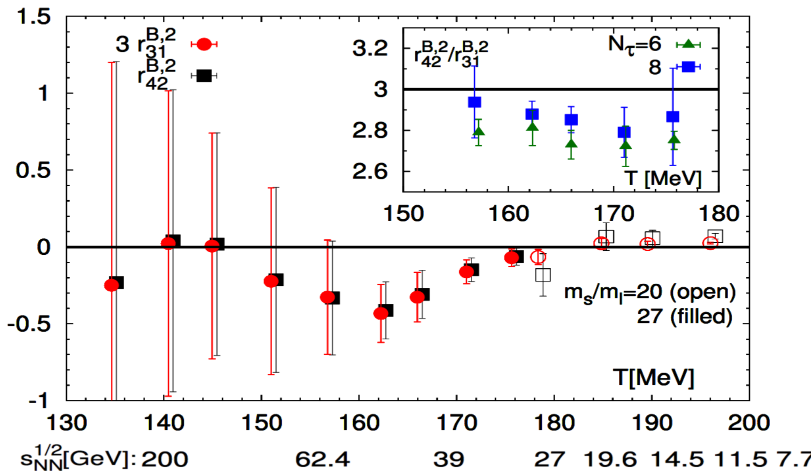
- Peak height expected to increase near the CP
- No sign of criticality

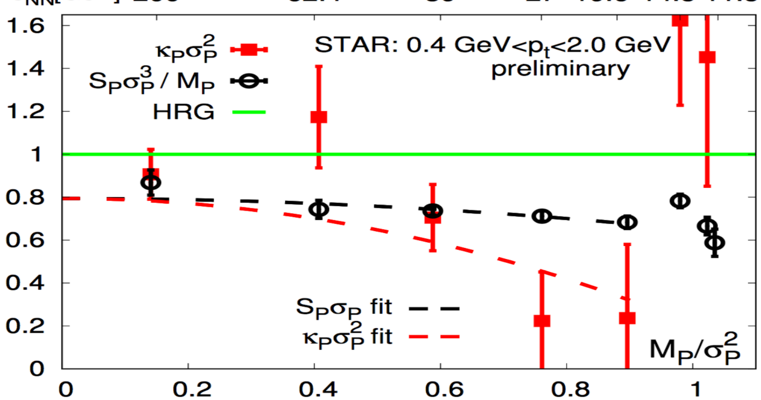
See talk by Patrick Steinbrecher on Wednesday

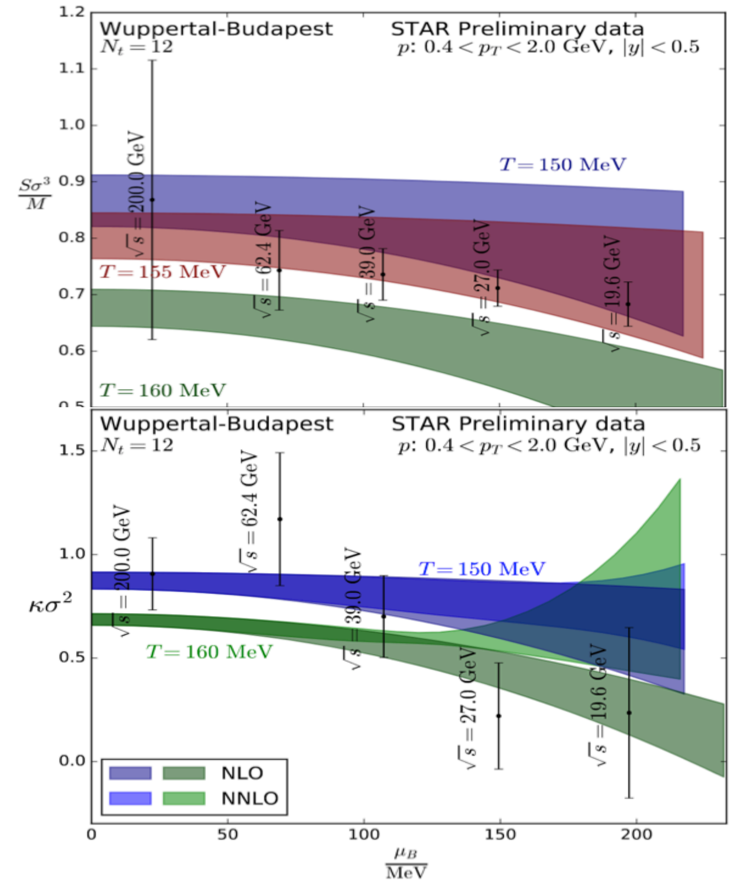
Higher order fluctuations

HotQCD, PRD (2017)

WB, 1805.04445 (2018)







$$\frac{S_B\sigma_B^3}{M_B} = \frac{\chi_3^B(T,\mu_B)}{\chi_1^B(T,\mu_B)} = \frac{\chi_4^B + s_1\chi_{31}^{BS} + q_1\chi_{31}^{BQ}}{\chi_2^B + s_1\chi_{11}^{BS} + q_1\chi_{11}^{BQ}} + \mathcal{O}(\mu_B^2) \equiv r_{31}^{B,0} + r_{31}^{B,2}\hat{\mu}_B^2 + \mathcal{O}(\mu_B^4)$$

$$\kappa_B\sigma_B^2 = \frac{\chi_4^B(T,\mu_B)}{\chi_2^B(T,\mu_B)} = \frac{\chi_4^B}{\chi_2^B} + \mathcal{O}(\mu_B^2) \equiv r_{42}^{B,0} + r_{42}^{B,2}\hat{\mu}_B^2 + \mathcal{O}(\mu_B^4)\ ,$$

A. Rustamov @QM2018

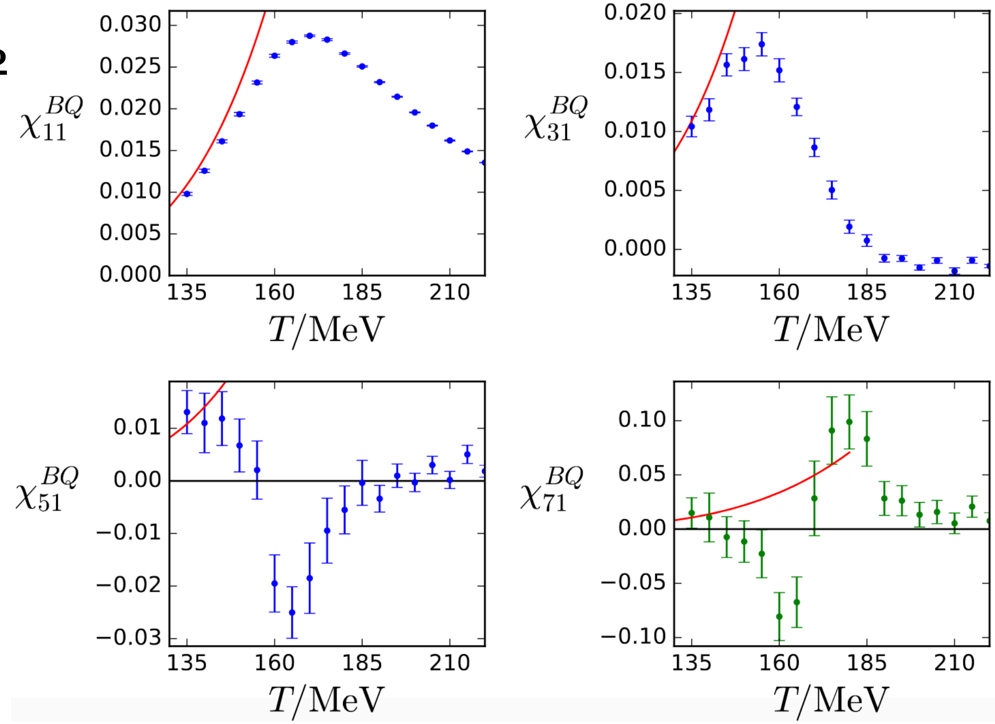
Alternative explanation: canonical suppression

Off-diagonal correlators

WB, 1805.04445 (2018)

- Simulation of the lower order correlators at imaginary μ_B
- Fit to extract higher order terms
- Results exist also for BS, QS and BQS correlators

Nt=12



$$\chi_{11}^{BS}(\hat{\mu}_B) = \chi_{11}^{BS} + \frac{1}{2!}\chi_{31}^{BS}\hat{\mu}_B^2 + \frac{1}{4!}\chi_{51}^{BS}\hat{\mu}_B^4 + \frac{1}{6!}\chi_{71}^{BS}\hat{\mu}_B^6 + \frac{1}{8!}\chi_{91}^{BS}\hat{\mu}_B^8$$

$$\chi_{21}^{BS}(\hat{\mu}_B) = \chi_{31}^{BS}\hat{\mu}_B + \frac{1}{3!}\chi_{51}^{BS}\hat{\mu}_B^3 + \frac{1}{5!}\chi_{71}^{BS}\hat{\mu}_B^5 + \frac{1}{7!}\chi_{91}^{BS}\hat{\mu}_B^7$$

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Forthcoming experimental data at RHIC

See talk by Jana Guenther on Wednesday

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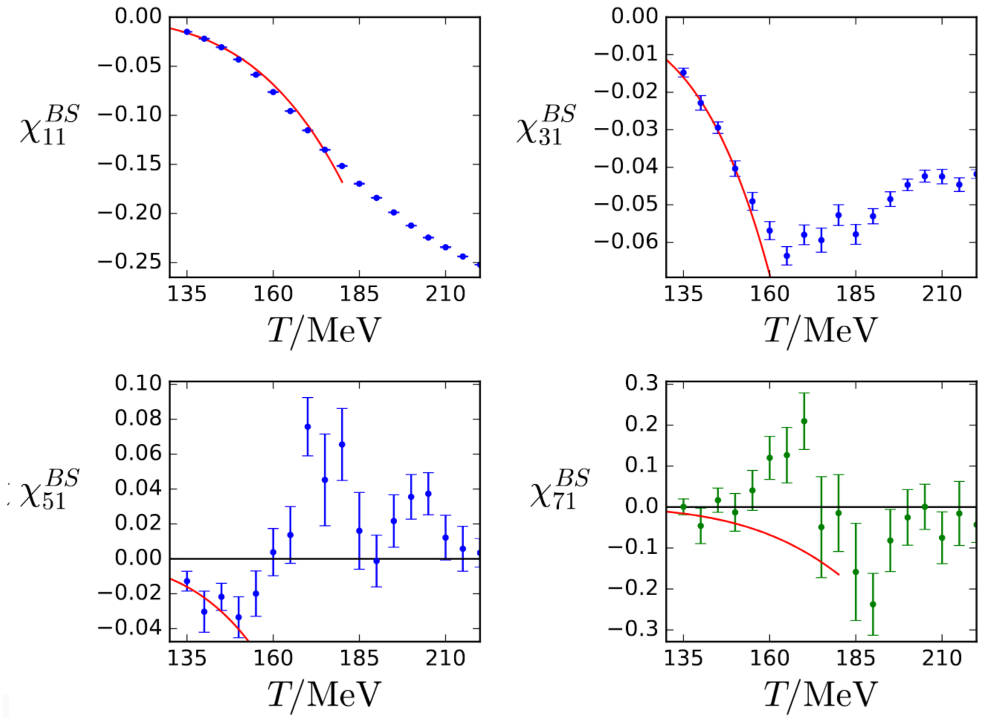
- Results exist also for BS, QS and BQS correlators

WB, 1805.04445 (2018)

Nt=12



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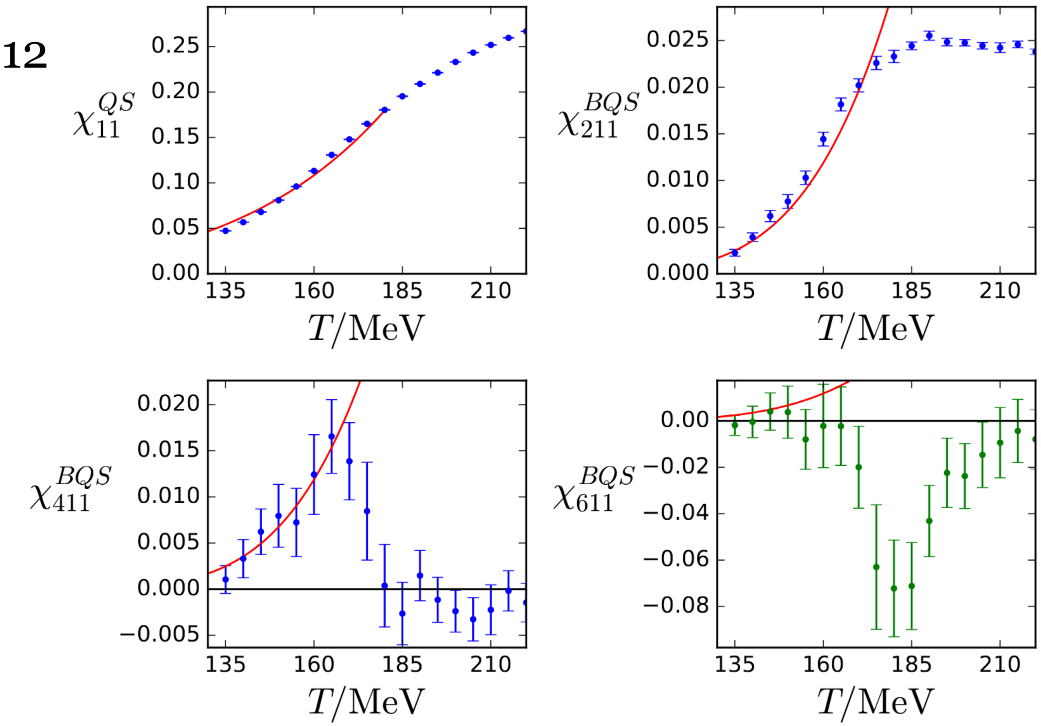
Nt=12



$$\chi_{11}^{BS}(\hat{\mu}_B) = \chi_{11}^{BS} + \frac{1}{2!}\chi_{31}^{BS}\hat{\mu}_B^2 + \frac{1}{4!}\chi_{51}^{BS}\hat{\mu}_B^4 + \frac{1}{6!}\chi_{71}^{BS}\hat{\mu}_B^6 + \frac{1}{8!}\chi_{91}^{BS}\hat{\mu}_B^8$$

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Forthcoming experimental data at RHIC

See talk by Jana Guenther on Wednesday

Other approaches I did not have time to address

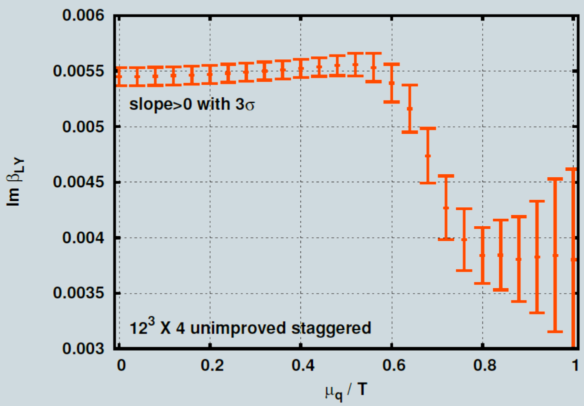
- Reweighting techniques (Fodor & Katz)
- Canonical ensemble (Alexandru et al., Kratochvila, de Forcrand, Ejiri, Bornyakov, Goy, Lombardo, Nakamura)
- Density of state methods (Fodor, Katz & Schmidt, Alexandru et al.)
- Two-color QCD (ITEP Moscow lattice group, Kogut et al., S. Hands et al., von Smekal et al.)
- Scalar field theories with complex actions (See talk by M. Ogilvie on Tuesday)
- Complex Langevin (see talks by D. Sinclair, S. Tsutsui, F. Attanasio, Y. Ito, A. Joseph on Monday)
- Lefshetz Thimble (see talks by K. Zambello, S. Lawrence, N. Warrington, H. Lamm on Monday)
- Phase unwrapping (see talks by G. Kanwar and M. Wagman on Friday)

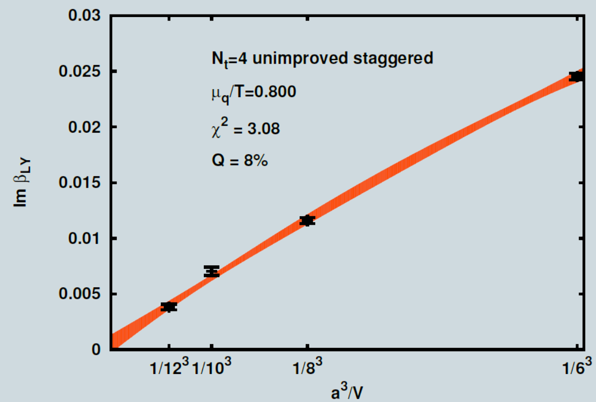
Radius of convergence of Taylor series

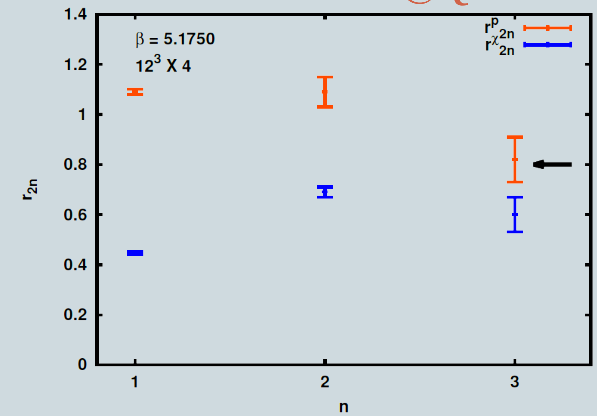
Plenary talk by Sayantan Sharma on Tuesday

- For a genuine phase transition, we expect the ∞-volume limit of the Lee-Yang zero to be real

A. Pasztor for WB @QM2018

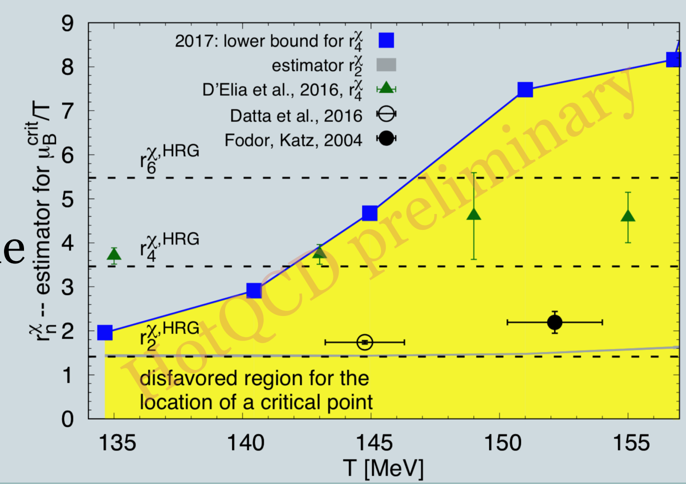


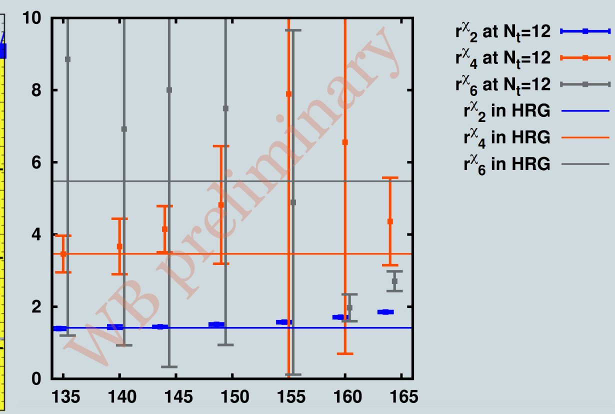




$$r^{\chi}_{2n} = \left| \frac{2n(2n-1)\chi^B_{2n}}{\chi^B_{2n+2}} \right|^{1/2}$$

- It grows as ~n in the HRG model





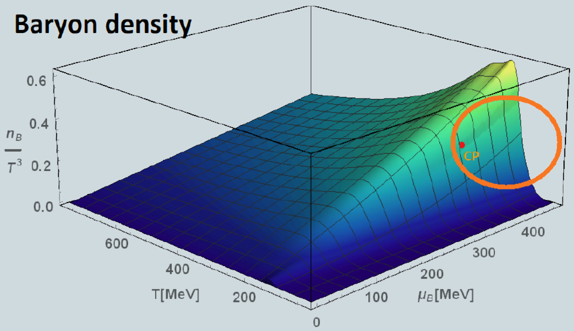
Alternative EoS at large densities

EoS for QCD with a 3D-Ising critical point

$T^4 c_n^{LAT}(T) = T^4 c_n^{Non-Ising}(T) + T_c^4 c_n^{Ising}(T)$

P. Parotto et al., 1805.05249 (2018)

- Implement scaling behavior of 3D-Ising model EoS
- Define map from 3D-Ising model to QCD
- Estimate contribution to Taylor coefficients from 3D-Ising model critical point
- Reconstruct full pressure



- Density discontinuous at $\mu_B > \mu_{Bc}$

Cluster expansion model

Vovchenko, Steinheimer, Philipsen, Stoecker, 1711.01261

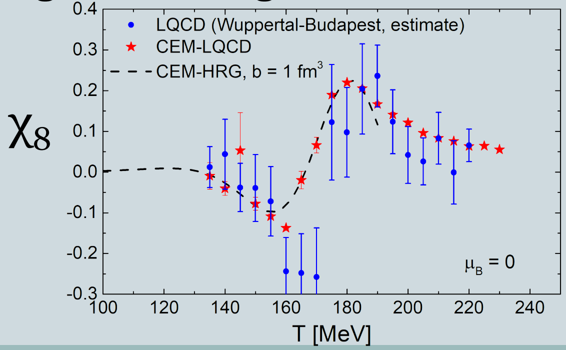
- HRG-motivated fugacity expansion for ρ_B

$$\frac{\rho_B(T,\mu_B)}{T^3} = \chi_1^B(T,\mu_B) = \sum_{k=1}^{\infty} b_k(T)\, \sinh(k\mu_B/T)$$

- b1(T) and b2(T) are model inputs
- All higher order coefficients predicted:

$$b_k(T) = \alpha_k^{SB} \frac{[b_2(T)]^{k-1}}{[b_1(T)]^{k-2}}$$

- Physical picture: HRG with repulsion at moderate T, "weakly" interacting quarks and gluons at high T, no CP



- Plan: integrate ρ_B and get $p(T,\mu_B)$

How can lattice QCD support the experiments?

- Equation of state
  - Needed for hydrodynamic description of the QGP
- QCD phase diagram
  - Transition line at finite density
  - Constraints on the location of the critical point
- Fluctuations of conserved charges
  - Can be simulated on the lattice and measured in experiments
  - Can give information on the evolution of heavy-ion collisions
  - Can give information on the critical point

Hadron Resonance Gas model

Dashen, Ma, Bernstein; Prakash, Venugopalan; Karsch, Tawfik, Redlich

- Interacting hadronic matter in the ground state can be well approximated by a non-interacting resonance gas
- The pressure can be written as:

$$p^{HRG}/T^4 = \frac{1}{VT^3} \sum_{i \in\ mesons} \ln \mathcal{Z}^M_{m_i}(T, V, \mu_{X^a}) + \frac{1}{VT^3} \sum_{i \in\ baryons} \ln \mathcal{Z}^B_{m_i}(T, V, \mu_{X^a})$$

where

$$\ln \mathcal{Z}^{M/B}_{m_i} = \mp \frac{V d_i}{2\pi^2} \int_0^\infty dk k^2 \ln(1 \mp z_i e^{-\varepsilon_i/T}) \quad ,$$

with energies $\varepsilon_i = \sqrt{k^2 + m_i^2}$, degeneracy factors d_i and fugacities

$$z_i = \exp\left((\sum_a X_i^a \mu_{X^a})/T \right) \ .$$

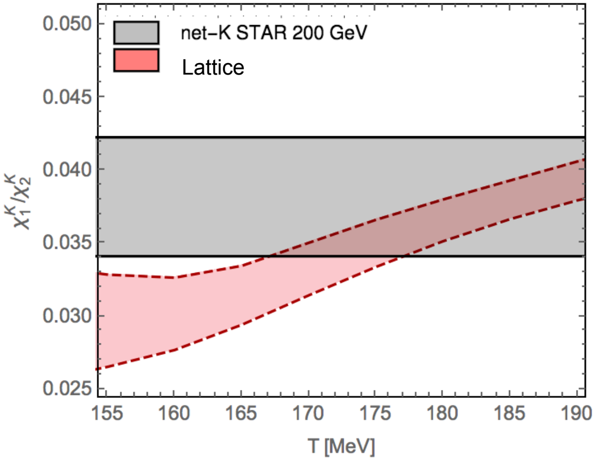
X^a: all possible conserved charges, including the baryon number B, electric charge Q, strangeness S.

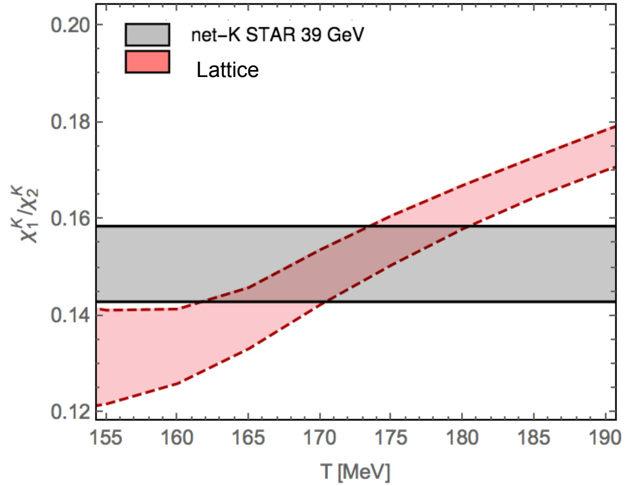
- Fugacity expansion for $\mu_S = \mu_Q = 0$: $\frac{p_B}{T^4} = \sum_{i \in B} \frac{d_i}{\pi^2} \left(\frac{m_i}{T} \right)^2 \sum_{N=1}^{\infty} (-1)^{N+1} N^{-2} K_2(N \frac{m_i}{T}) \cosh \left[N \frac{\mu_B}{T} \right]$

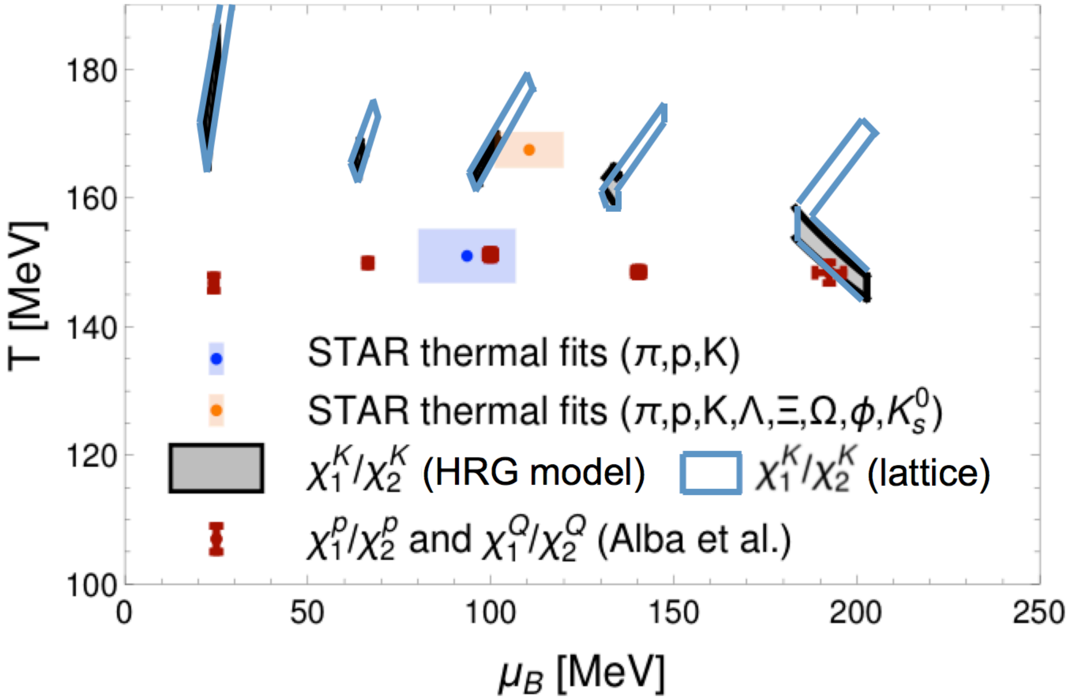
Boltzmann approximation: N=1

Kaon fluctuations on the lattice

J. Noronha-Hostler, C.R. et al. forthcoming





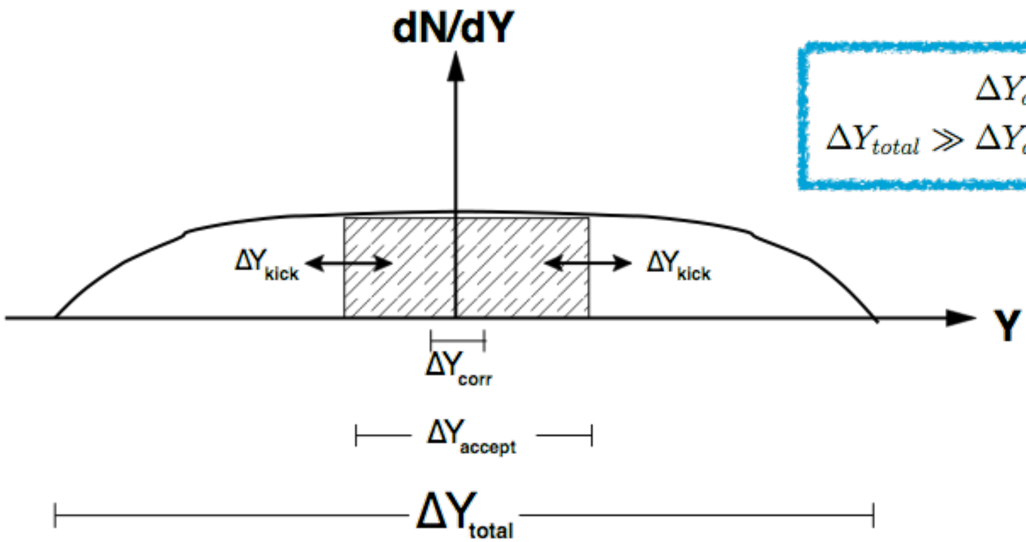


- Lattice QCD temperatures have a large uncertainty but they are above the light flavor ones

Fluctuations of conserved charges?

* If we look at the entire system, none of the conserved charges will fluctuate

* By studying a sufficiently small subsystem, the fluctuations of conserved quantities become meaningful



$$\Delta Y_{accept} \gg \Delta Y_{corr}$$
$$\Delta Y_{total} \gg \Delta Y_{accept} \gg \Delta Y_{kick}$$

- ❑ ΔYtotal: range for total charge multiplicity distribution
- ❑ ΔYaccept: interval for the accepted charged particles
- ❑ ΔYkick: rapidity shift that charges receive during and after hadronization

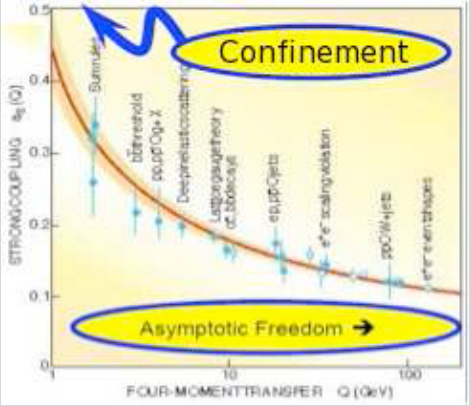
QCD matter under extreme conditions

To address these questions we need fundamental theory and experiment

Theory: Quantum Chromodynamics

- QCD is the fundamental theory of strong interactions
- It describes interactions among quarks and gluons

$$L_{QCD} = \sum_{i=1}^{n_f} \overline{\psi}_i \gamma_\mu \left(i\partial^\mu - g A_a^\mu \frac{\lambda_a}{2} \right) \psi_i - m_i \overline{\psi}_i \psi_i - \frac{1}{4} \sum_a F_a^{\mu\upsilon} F_a^{\mu\upsilon}$$



Experiment: heavy-ion collisions

- Quark-gluon plasma (QGP) discovery at RHIC and the LHC
- QGP is a strongly interacting (almost) perfect fluid

Cumulants of multiplicity distribution

- Deviation of N_Q from its mean in a single event: $\delta N_Q = N_Q - \langle N_Q \rangle$
- The cumulants of the event-by-event distribution of NQ are:

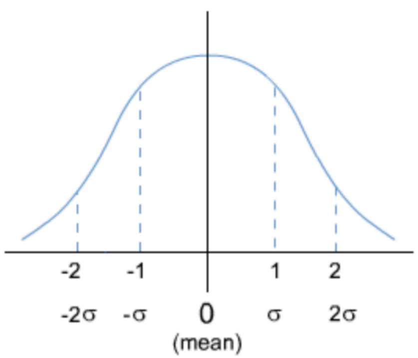
$$\chi_2 = \langle (\delta NQ)^2 \rangle \qquad \chi_3 = \langle (\delta NQ)^3 \rangle \qquad \chi_4 = \langle (\delta NQ)^4 \rangle - 3\langle (\delta NQ)^2 \rangle^2$$

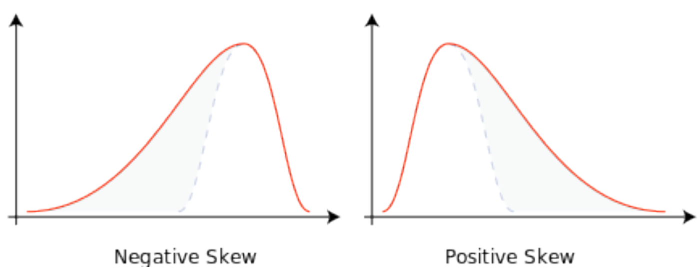
- The cumulants are related to the central moments of the distribution by:

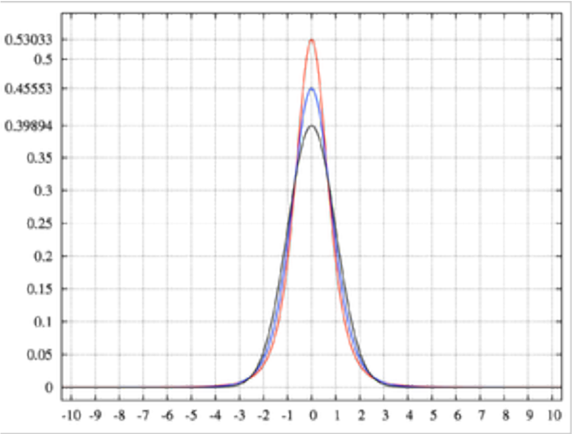
variance: $\sigma^2 = \chi_2$

Skewness: $S = \chi_3/(\chi_2)^{3/2}$

Kurtosis: $\kappa = \chi_4/(\chi_2)^2$





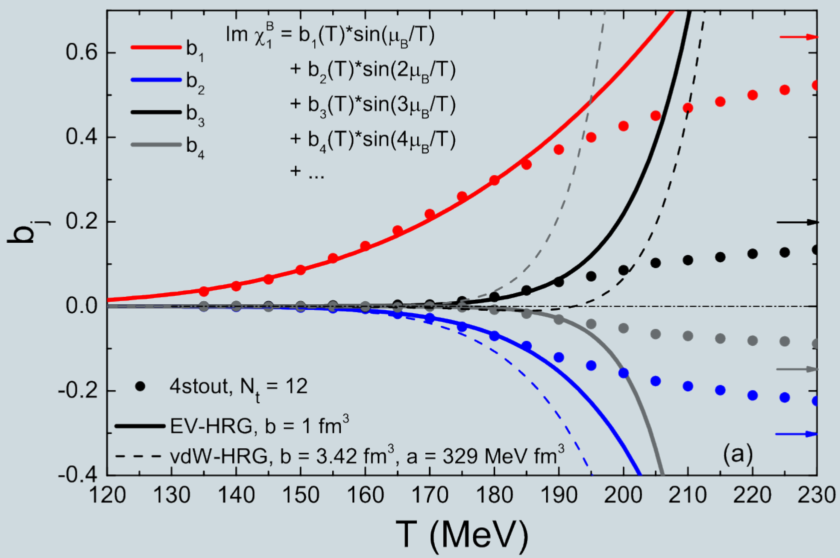


Fluctuations and hadrochemistry

$$\chi_1^B(T,\mu_B) = \frac{\rho_B(T,\mu_B)}{T^3} = \sum_{k=1}^{\infty} b_k(T)\sinh(k\,\mu_B/T)$$

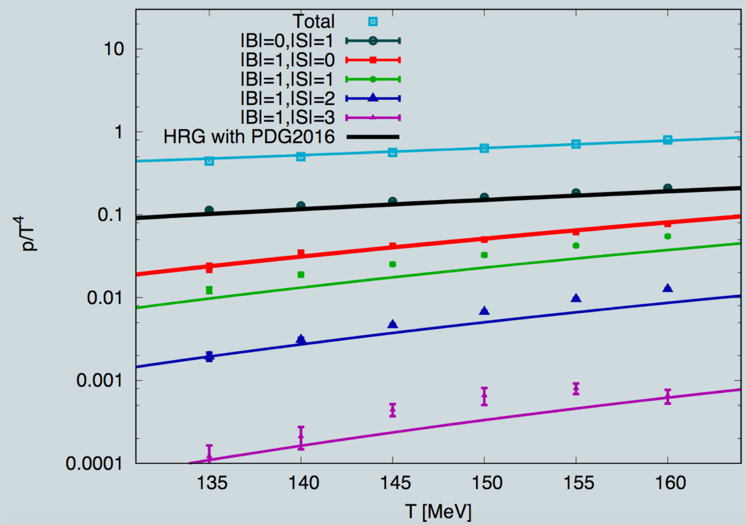
$$P(\hat{\mu}_B,\hat{\mu}_S) = P_{00}^{BS} + P_{10}^{BS}\cosh(\hat{\mu}_B) + P_{01}^{BS}\cosh(\hat{\mu}_S) + P_{11}^{BS}\cosh(\hat{\mu}_B-\hat{\mu}_S) + P_{12}^{BS}\cosh(\hat{\mu}_B-2\hat{\mu}_S) + P_{13}^{BS}\cosh(\hat{\mu}_B-3\hat{\mu}_S)$$

V. Vovchenko et al., PLB (2017)



- Consistent with HRG at low temperatures
- Consistent with approach to ideal gas limit
- b_2 departs from zero at T~160 MeV
- Deviation from ideal HRG

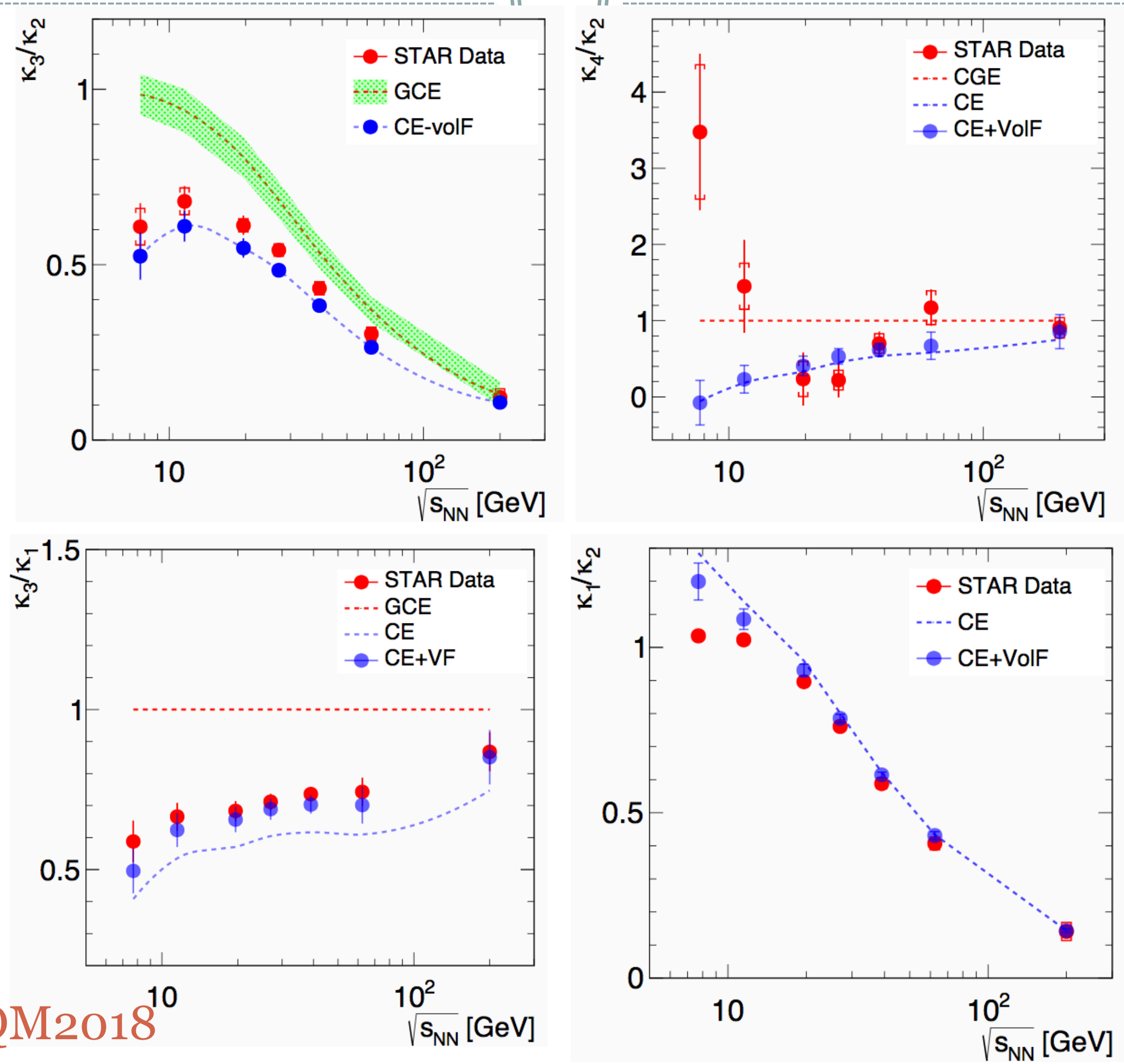
P. Alba et al., PRD (2017)



- Need of additional strange hadrons, predicted by the Quark Model but not yet detected
- First pointed out in Bazavov et al., PRL(2014)

(see talk by J. Glesaaen on Friday)

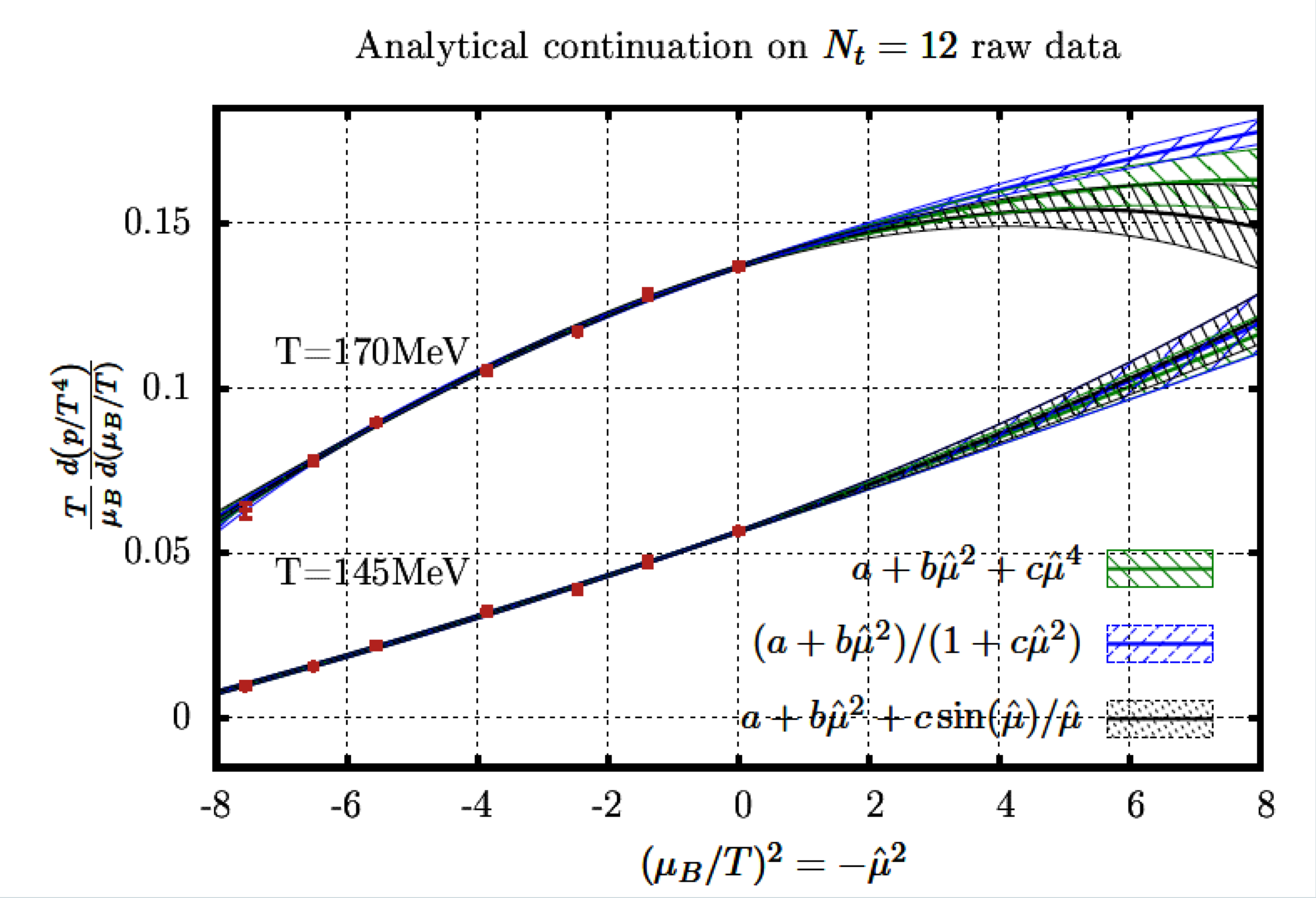
Canonical suppression



A. Rustamov @QM2018

above 11.5 GeV CE suppression accounts for measured deviations from GCE

Analytical continuation – illustration of systematics



Analytical continuation – illustration of systematics

Condition: $\chi_8 \lesssim \chi_4 \longrightarrow f(\hat{\mu}_B) = a + b\hat{\mu}_B^2 + c\hat{\mu}_B^4 + \frac{b\epsilon}{840}\hat{\mu}_B^6$