Meson Dynamics in Hot Hadron Gas

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Introduction

K* interactions with light mesons

Evolution of meson multiplicities

Cooling

Kinetic freeze-out

K* / K ratio

D* / D ratio

20 years of the K^* / K ratio

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PHYSICS LETTERS B

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Strange hadron resonances as a signature of freeze-out dynamics

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Strange resonance production: probing chemical and thermal freeze-out in relativistic heavy ion collisions

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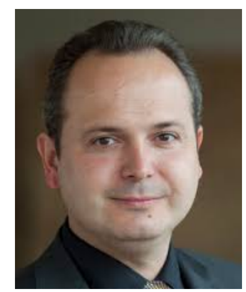
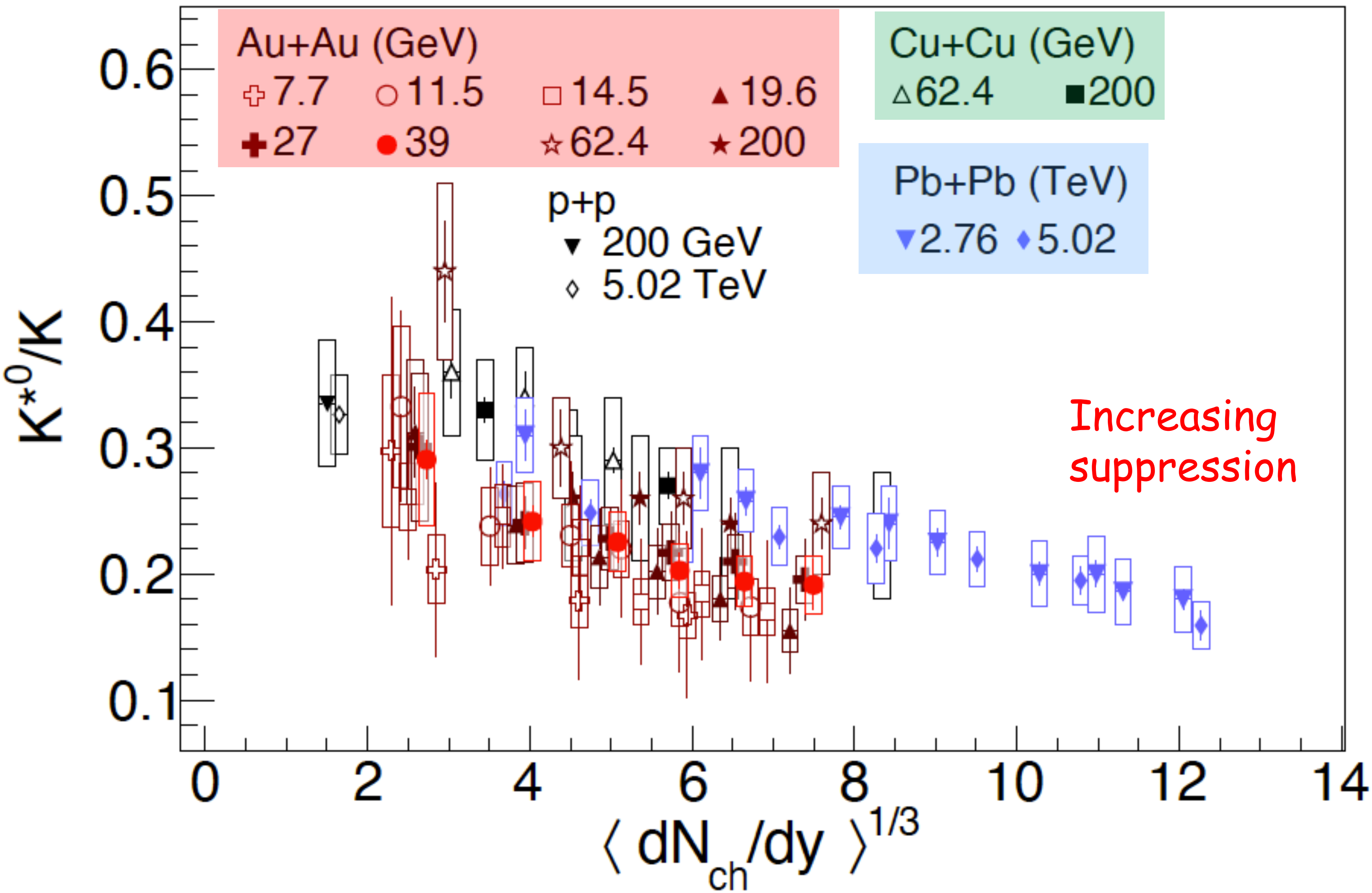


Marcus Bleicher

Jörg Aichelin

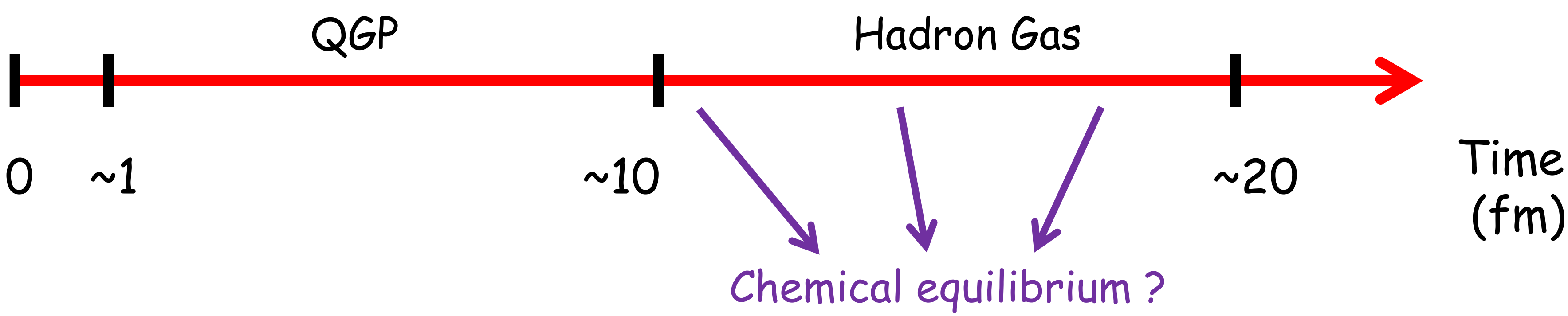
From the idea...

...to a wealthy of data



STAR, arXiv:2210.02909

How to understand the suppression



$$K^* \rightarrow K + \pi \qquad \Gamma(K^*) \simeq 50\,\mathrm{MeV} \qquad \longrightarrow \qquad \tau_{life} = \frac{1}{\Gamma(K^*)} \simeq 4\,\mathrm{fm}$$

Mesons are produced at the end of the QGP phase

They live ~ 10 fm in a hadron gas before kinetic freeze-out

K* lifetime is ~ 4fm ! It decays into K + pion and is lost !

Larger systems -> longer living fireballs -> stronger suppression

What can we learn from this ratio?

Interactions of K and K* in a hot hadron gas

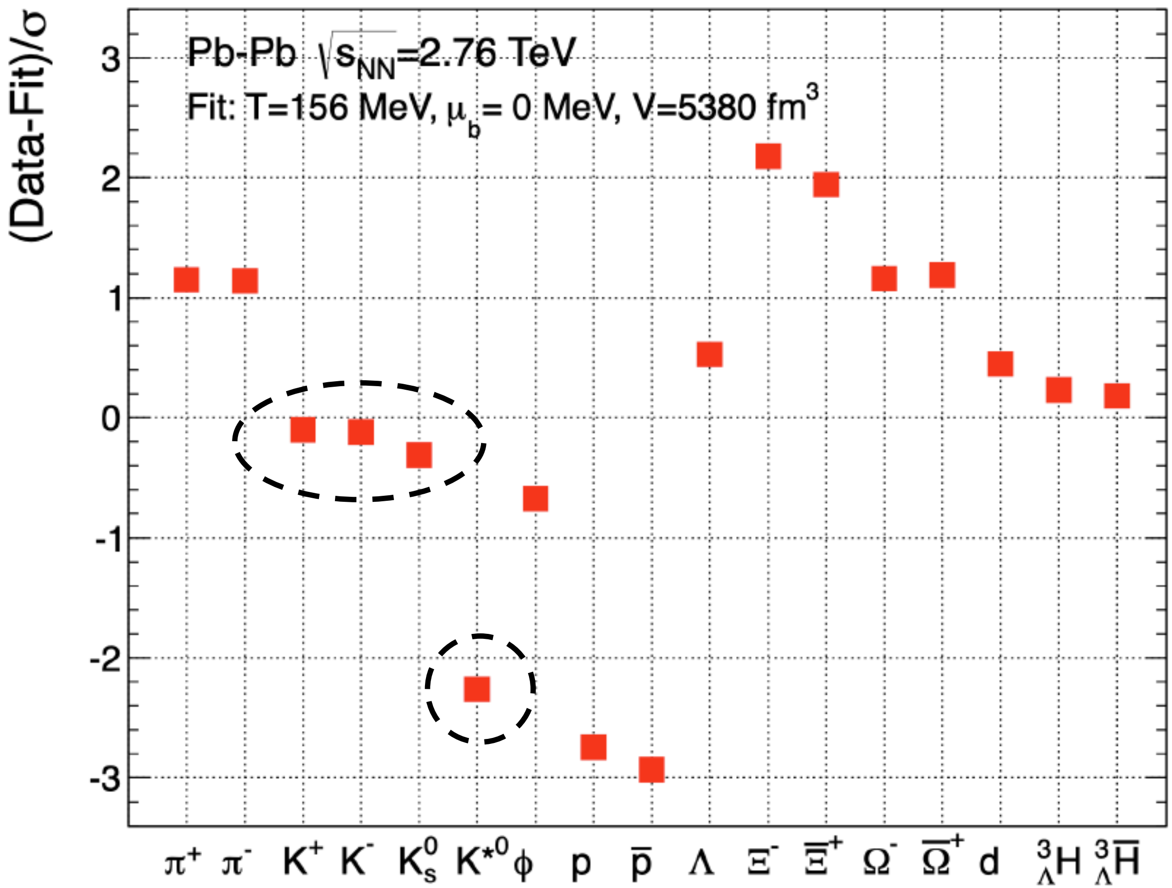
Emergence of chemical equilibrium (freeze-out)

Kinetic freeze-out: lifetime of the hadron gas phase

Confirm the existence of a hot hadron gas

Do we have a good theory ?

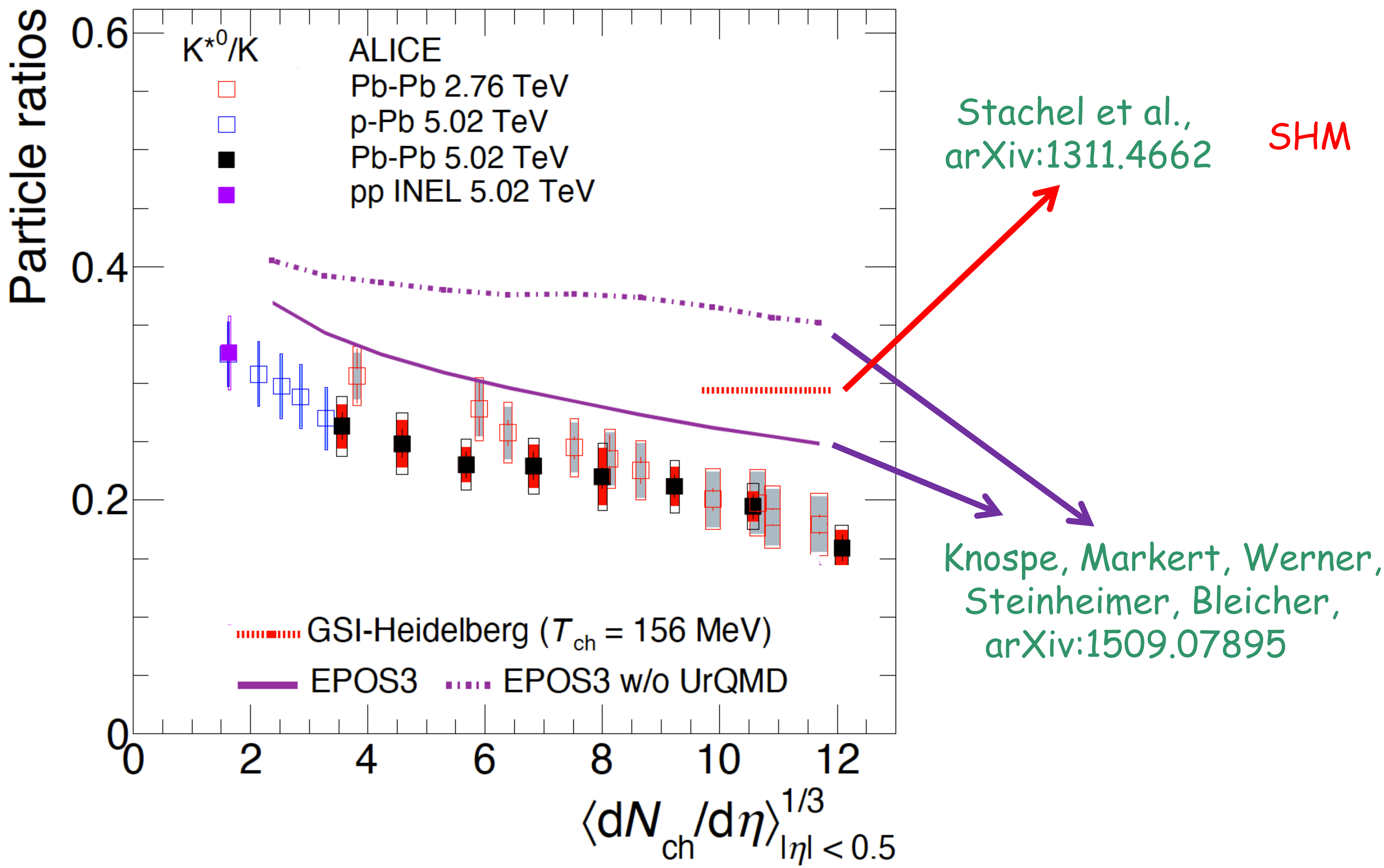
Statistical Hadronization Model fails...



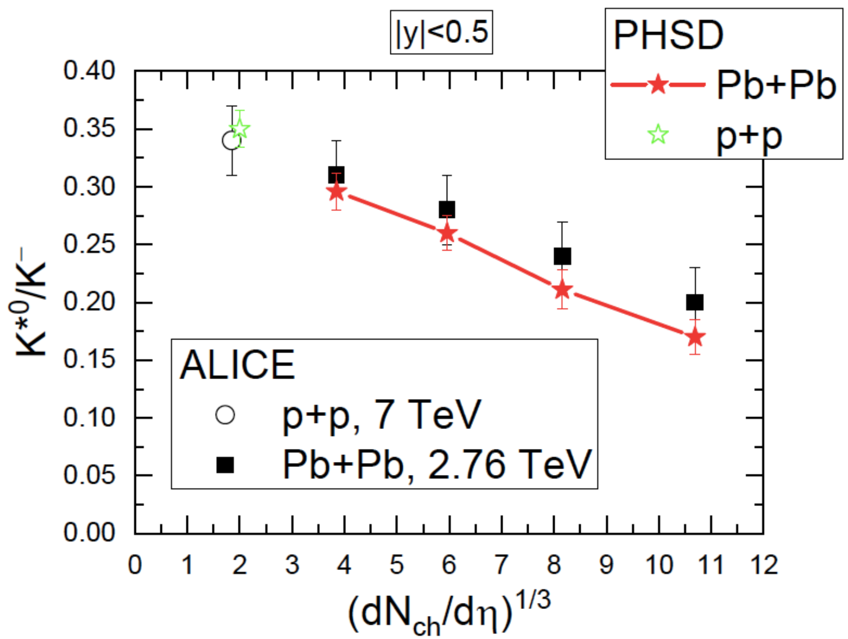
Stachel et al., arXiv:1311.4662

We must include rescattering and/or decay !

ALICE, arXiv:1910.14419

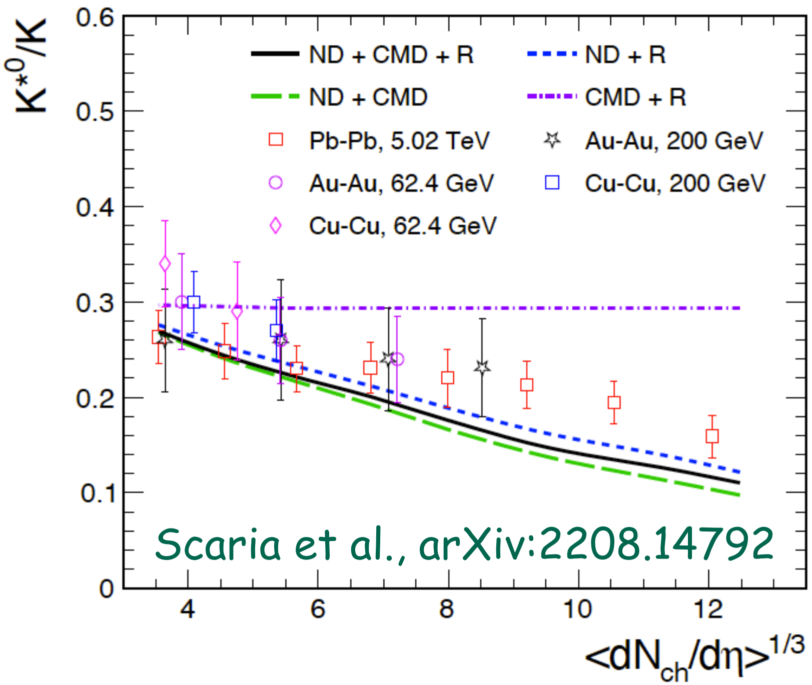


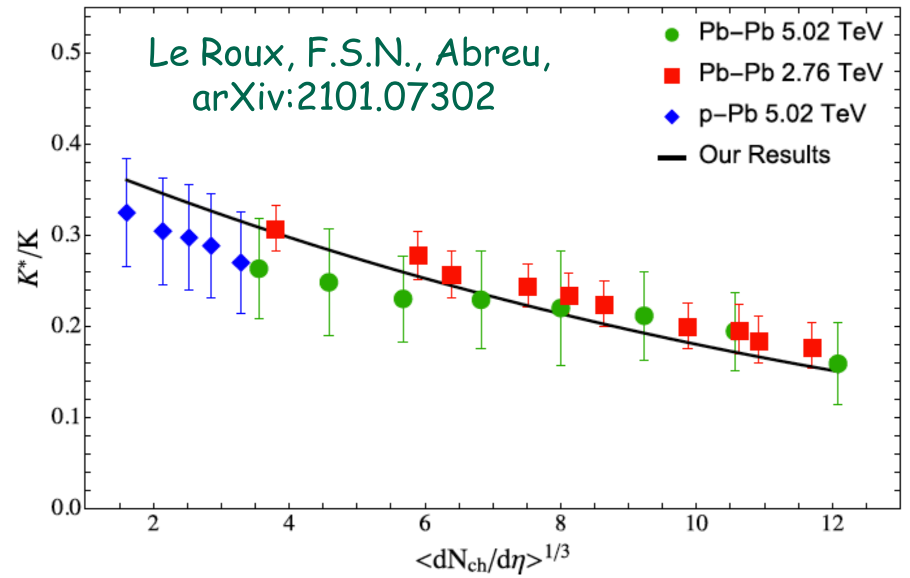
Interactions with the hadron gas improve agreement with data !



Ilner, Cabrera, Markert, Bratkovskaya, arXiv:1609.02778

Ilner, Blair,Cabrera, Markert, Bratkovskaya, arXiv:1707.00060





The hadron gas contribution

Start with the multiplicities at the hadronization

Study the changes produced by interactions in the hadron gas

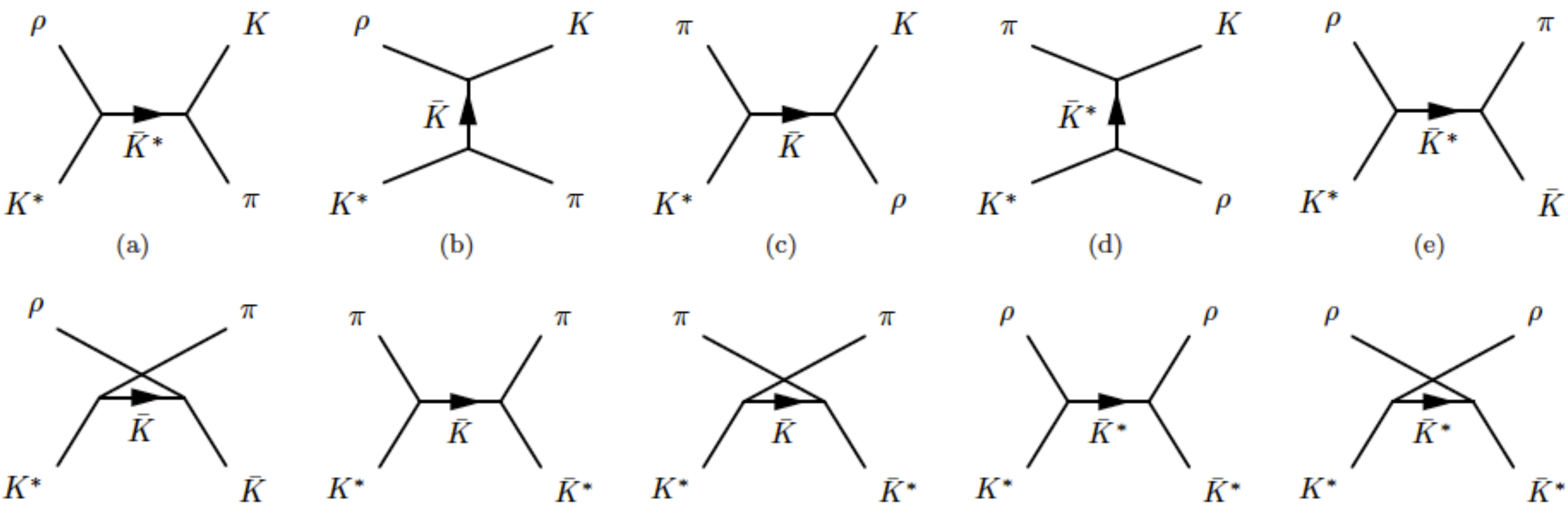
Lagrangians -> Amplitudes -> Cross Sections -> Thermal Cross Sections

Evolution equations -> Expansion and cooling -> Freeze-out

K and K* interactions with light hadrons

$$
\begin{aligned}
\mathcal{L}_{\pi K K^*} &= i g_{\pi K^* K} K^{*\mu} \vec{\tau} \cdot (\bar{K} \partial_\mu \vec{\pi} - \partial_\mu \bar{K} \vec{\pi}) \\
\mathcal{L}_{\rho K K} &= i g_{\rho K K} (K \vec{\tau} \partial_\mu \bar{K} - \partial_\mu K \vec{\tau} \bar{K}) \cdot \vec{\rho}^{\mu}, \\
\mathcal{L}_{\rho K^* K^*} &= i g_{\rho K^* K^*} \left[(\partial_\mu K^{*\nu} \vec{\tau} \bar{K}^*_\nu - K^{*\nu} \vec{\tau} \partial_\mu \bar{K}^*_\nu) \cdot \vec{\rho}^{\mu} \right. \\
&+ (K^{*\nu} \vec{\tau} \cdot \partial_\mu \vec{\rho}_\nu - \partial_\mu K^{*\nu} \vec{\tau} \cdot \vec{\rho}_\nu) \bar{K}^{*\mu} \\
&+ \left. K^{*\mu} (\vec{\tau} \cdot \vec{\rho}^{\nu} \partial_\mu \bar{K}^*_\nu - \vec{\tau} \cdot \partial_\mu \vec{\rho}^{\nu} \bar{K}^*_\nu) \right],
\end{aligned}
$$

S. Cho and S.H. Lee, arXiv:1509.04092



Cross Sections :
$$\sigma = \frac{1}{64\pi^2 s g_1 g_2} \frac{|\vec{p}_f|}{|\vec{p}_i|} \int d\Omega \overline{|\mathcal{M}|^2} F^4$$

Form Factors :
$$F_{u,t}(\vec{q}) = \frac{\Lambda^2 - m_{ex}^2}{\Lambda^2 + \vec{q}^2} \qquad \Lambda = 1.8\,\mathrm{GeV}$$

Thermal Cross Sections :
$$\langle \sigma_{ab\to cd}\, v_{ab} \rangle = \frac{\int d^3\mathbf{p}_a d^3\mathbf{p}_b\, f_a(\mathbf{p}_a)\, f_b(\mathbf{p}_b)\, \sigma_{ab\to cd}\ v_{ab}}{\int d^3\mathbf{p}_a d^3\mathbf{p}_b\, f_a(\mathbf{p}_a)\, f_b(\mathbf{p}_b)}$$

$$f_i(\vec{p}_i) = \frac{1}{e^{\sqrt{\vec{p}_i{}^2 + m_i^2}/T} - 1} \qquad v_{ab} = \sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}/(E_a E_b)$$

Inverse processes with detailed balance:
$$g_a g_b |\vec{p}_{ab}|^2 \sigma_{ab\to cd}(s) = g_c g_d |\vec{p}_{cd}|^2 \sigma_{cd\to ab}(s)$$

Rate equations

$$\frac{dN_{K^*}}{d\tau} = \langle\sigma_{K\rho\to K^*\pi} v_{K\rho}\rangle n_\rho(\tau)N_K(\tau) - \langle\sigma_{K^*\pi\to K\rho} v_{K^*\pi}\rangle n_\pi(\tau)N_{K^*}(\tau) + \langle\sigma_{K\pi\to K^*\rho} v_{K\pi}\rangle n_\pi(\tau)N_K(\tau)$$
$$- \langle\sigma_{K^*\rho\to K\pi} v_{K^*\rho}\rangle n_\rho(\tau)N_{K^*}(\tau) + \langle\sigma_{\pi\rho\to K^*\bar{K}} v_{\pi\rho}\rangle n_\pi(\tau)N_\rho(\tau) - \langle\sigma_{K^*\bar{K}\to\rho\pi} v_{K^*\bar{K}}\rangle n_{\bar{K}}(\tau)N_{K^*}(\tau)$$
$$+ \langle\sigma_{\pi\pi\to K^*\bar{K}^*} v_{\pi\pi}\rangle n_\pi(\tau)N_\pi(\tau) - \langle\sigma_{K^*\bar{K}^*\to\pi\pi} v_{K^*\bar{K}^*}\rangle n_{\bar{K}^*}(\tau)N_{K^*}(\tau) + \langle\sigma_{\rho\rho\to K^*\bar{K}^*} v_{\rho\rho}\rangle n_\rho(\tau)N_\rho(\tau)$$
$$- \langle\sigma_{K^*\bar{K}^*\to\rho\rho} v_{K^*\bar{K}^*}\rangle n_{\bar{K}^*}(\tau)N_{K^*}(\tau) + \langle\sigma_{K\pi\to K^*} v_{K\pi}\rangle n_\pi(\tau)N_K(\tau) - \langle\Gamma_{K^*}\rangle N_{K^*}(\tau),$$
$$\frac{dN_K}{d\tau} = \langle\sigma_{\pi\pi\to K\bar{K}} v_{\pi\pi}\rangle n_\pi(\tau)N_\pi(\tau) - \langle\sigma_{K\bar{K}\to\pi\pi} v_{K\bar{K}}\rangle n_{\bar{K}}(\tau)N_K(\tau) + \langle\sigma_{\rho\rho\to K\bar{K}} v_{\rho\rho}\rangle n_\rho(\tau)N_\rho(\tau)$$
$$- \langle\sigma_{K\bar{K}\to\rho\rho} v_{K\bar{K}}\rangle n_{\bar{K}}(\tau)N_K(\tau) + \langle\sigma_{K^*\pi\to K\rho} v_{K^*\pi}\rangle n_\pi(\tau)N_{K^*}(\tau) - \langle\sigma_{K\rho\to K^*\pi} v_{K\rho}\rangle n_\rho(\tau)N_K(\tau)$$
$$+ \langle\sigma_{K^*\rho\to K\pi} v_{K^*\rho}\rangle n_\rho(\tau)N_{K^*}(\tau) - \langle\sigma_{K\pi\to K^*\rho} v_{K\pi}\rangle n_\pi(\tau)N_K(\tau) + \langle\sigma_{\pi\rho\to K^*\bar{K}} v_{\pi\rho}\rangle n_\pi(\tau)N_\rho(\tau)$$
$$- \langle\sigma_{K^*\bar{K}\to\rho\pi} v_{K^*\bar{K}}\rangle n_{\bar{K}}(\tau)N_{K^*}(\tau) + \langle\Gamma_{K^*}\rangle N_{K^*}(\tau) - \langle\sigma_{K\pi\to K^*} v_{K\pi}\rangle n_\pi(\tau)N_K(\tau).$$

$$n_i(\tau) = \frac{g_i}{2\pi^2}\int_0^\infty \frac{p^2dp}{e^{\sqrt{p_i^2+m_i^2}/T(\tau)}-1} \simeq \frac{g_i}{2\pi^2} m_i^2 T(\tau) K_2\left(\frac{m_i}{T(\tau)}\right) \qquad N_i = n_i V$$

Expansion and Cooling :

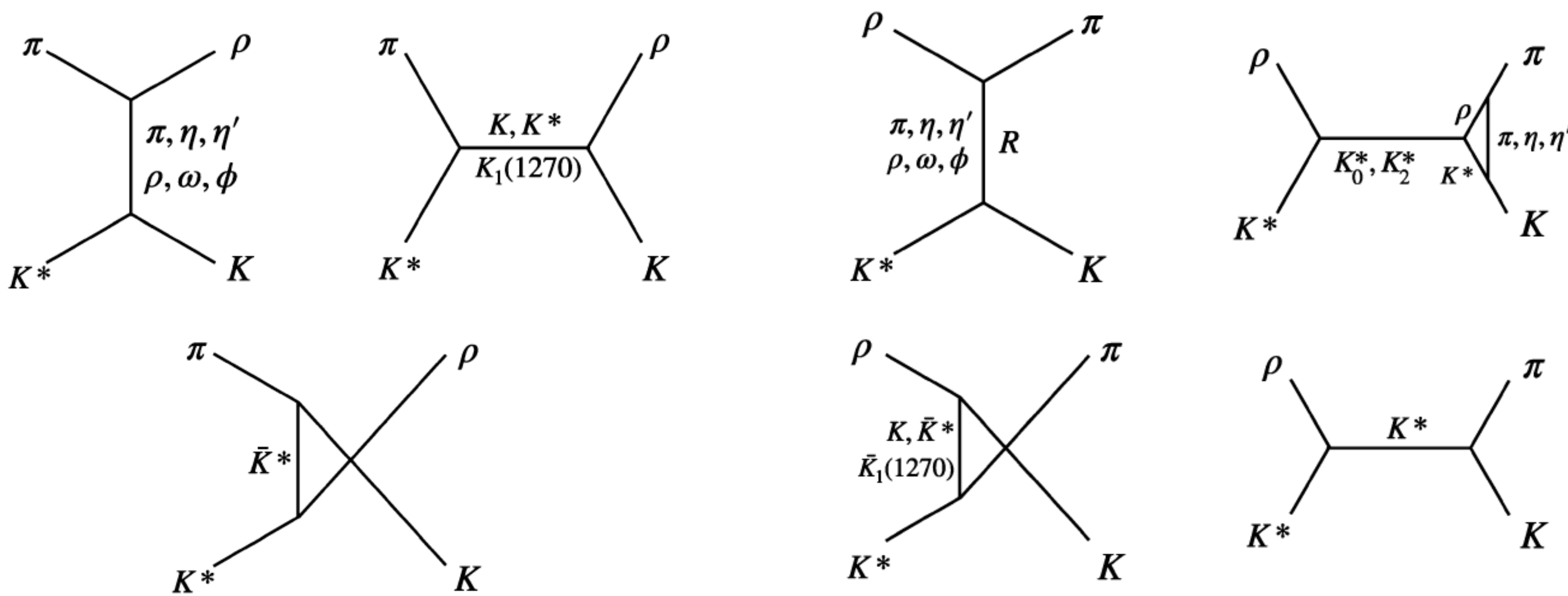
$$\begin{cases} V(\tau) = \pi[R_c + v_c(\tau-\tau_c) + a_c/2(\tau-\tau_c)^2]^2 \tau c, \\ T(\tau) = T_c - (T_h - T_f)\left(\dfrac{\tau-\tau_h}{\tau_f-\tau_h}\right)^{4/5}, \end{cases}$$

K and K* interactions with light hadrons improved

Martinez Torres, Khemchandani, Abreu, F.S.N., Nielsen, arXiv:1708.05784

Inclusion of anomalous parity VVP interactions

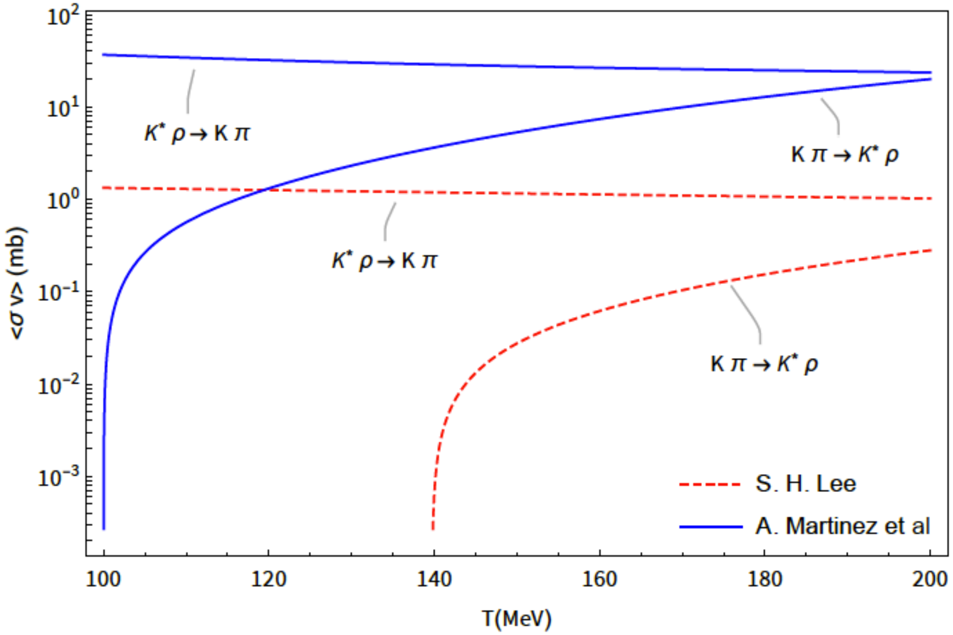
Exchange of axial resonances



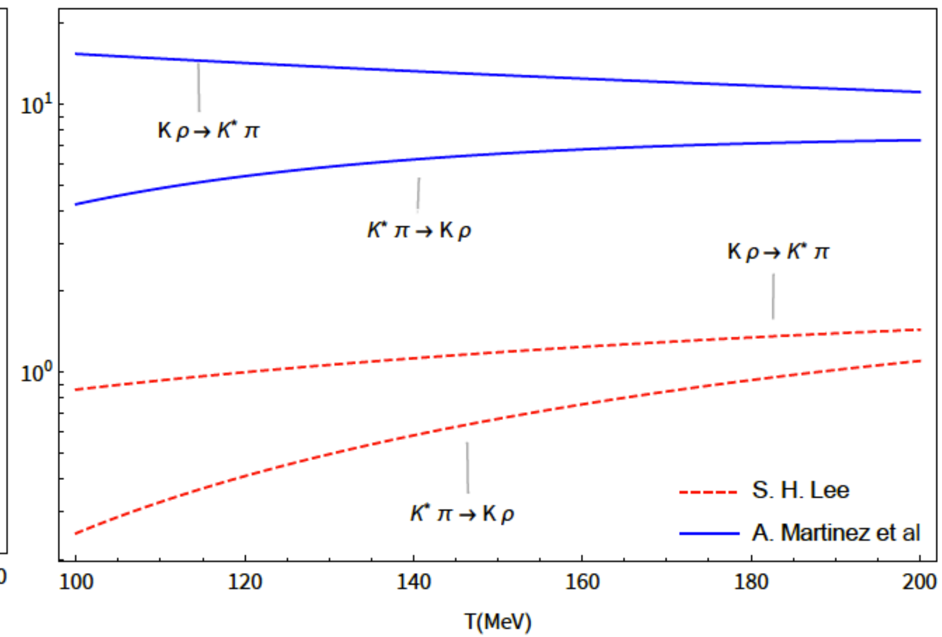
Many processes but only 3 are really important:

$$\begin{cases} K^* \rho \leftrightarrow K \pi \\ K^* \pi \leftrightarrow K \rho \\ K^* \leftrightarrow K + \pi \end{cases}$$

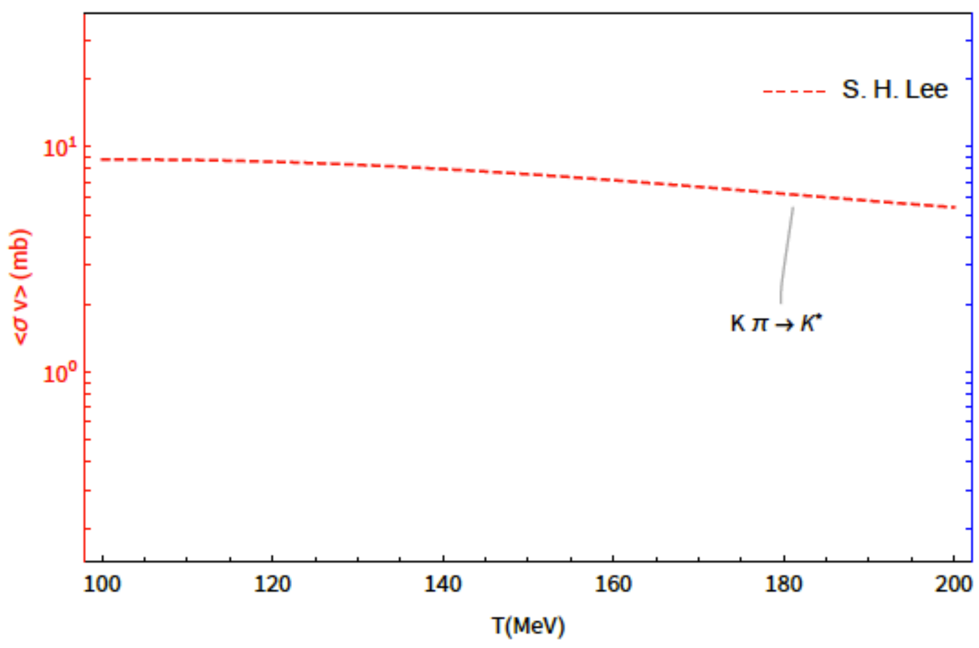
$K^* \rho \leftrightarrow K \pi$



$K^* \pi \leftrightarrow K \rho$



$K^* \leftrightarrow K + \pi$



Simplified evolution equations :

$$\begin{cases} \dfrac{dN_{K^*}(\tau)}{d\tau} = \gamma_K N_K(\tau) - \gamma_{K^*} N_{K^*}(\tau), \\ \dfrac{dN_K(\tau)}{d\tau} = -\gamma_K N_K(\tau) + \gamma_{K^*} N_{K^*}(\tau), \end{cases}$$

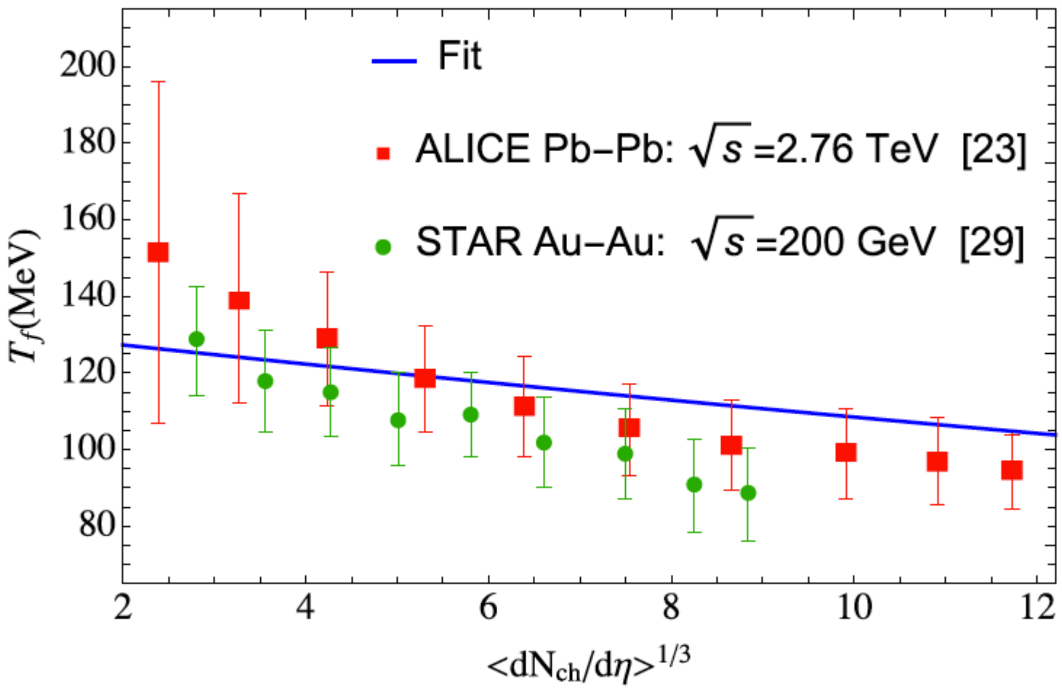
$$\gamma_K = \langle \sigma_{K\pi \to K^*\rho} v_{K\pi} \rangle n_\pi + \langle \sigma_{K\rho \to K^*\pi} v_{K\rho} \rangle n_\rho + \langle \sigma_{K\pi \to K^*} v_{K\pi} \rangle n_\pi ,$$

$$\gamma_{K^*} = \langle \sigma_{K^*\rho \to K\pi} v_{K^*\rho} \rangle n_\rho + \langle \sigma_{K^*\pi \to K\rho} v_{K^*\pi} \rangle n_\pi + \langle \Gamma_{K^*} \rangle .$$

Bjorken cooling : $T = T_h \left(\dfrac{\tau_h}{\tau}\right)^{1/3} \rightarrow \tau_f = \tau_h \left(\dfrac{T_h}{T_f}\right)^3$

T_f depends on the system size : $T_f = T_f\left(\dfrac{dN}{d\eta}(\eta = 0)\right)$

System size dependent freeze-out temperature

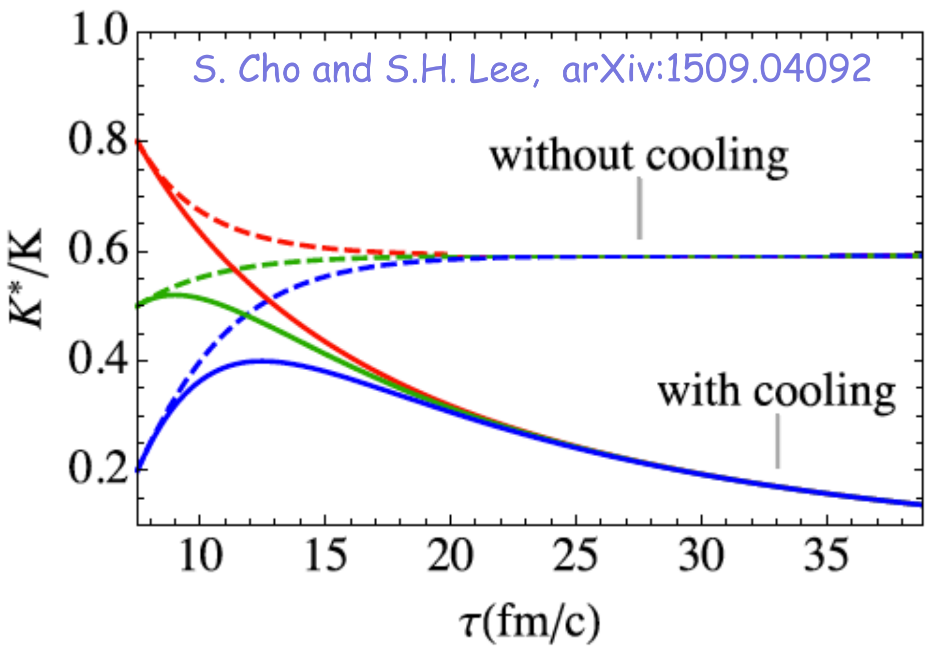


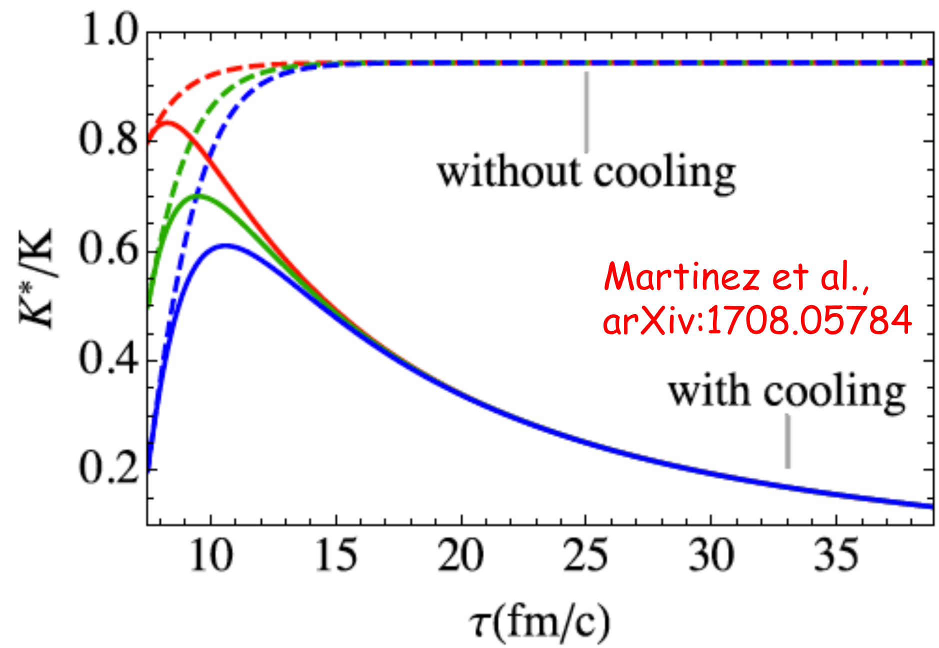
ALICE, arXiv:1303.0737

$$T_f = T_{f0}\, e^{-b\mathcal{N}}$$

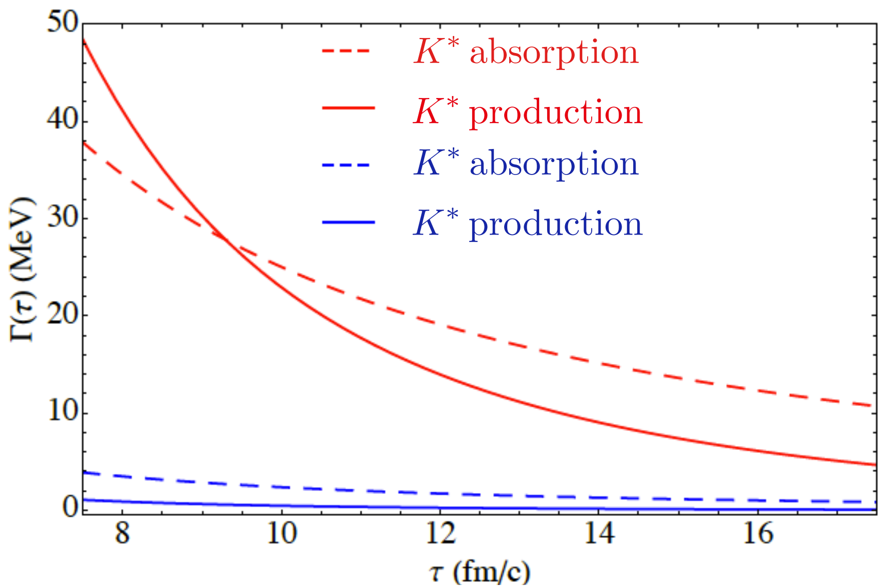
$$\mathcal{N} = \left[\left(\frac{dN}{d\eta} \right)_{\eta=0} \right]^{1/3}$$

Results





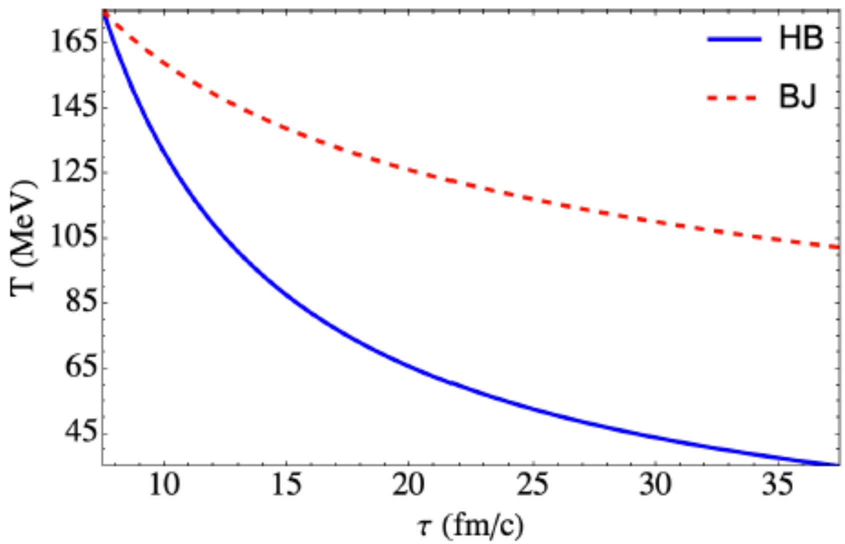
Reaction
Rate



Martinez et al.,
arXiv:1708.05784

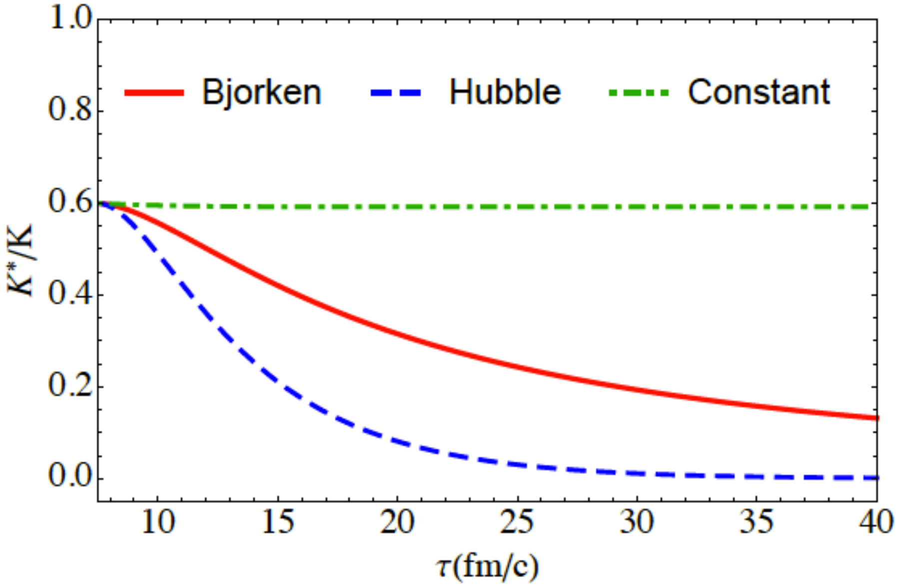
S. Cho,S.H. Lee,
arXiv:1509.04092

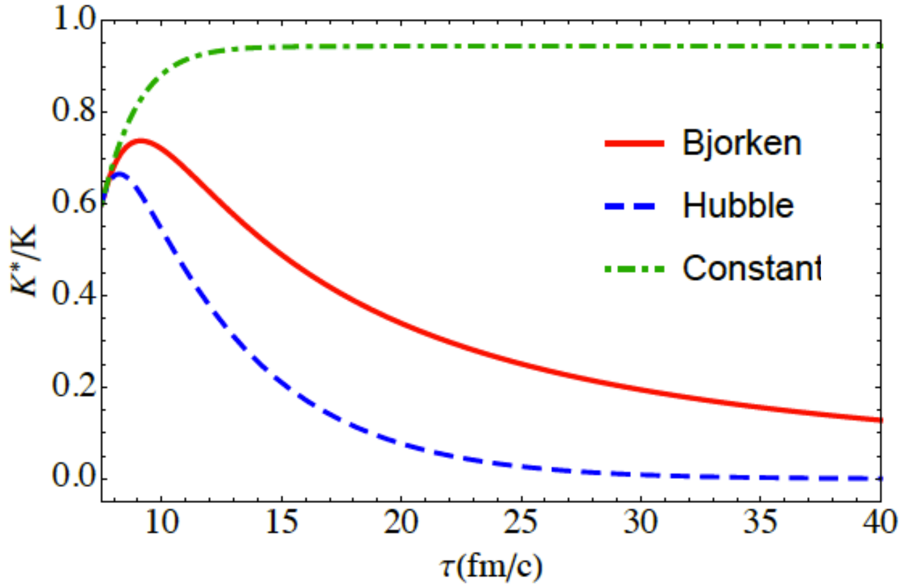
Effect of cooling



$$T(\tau) = T_h \left(\frac{\tau_h}{\tau}\right)^{\frac{1}{3}}$$ Bjorken

$$T(\tau) = T_h \left(\frac{\tau_h}{\tau}\right)$$ Hubble



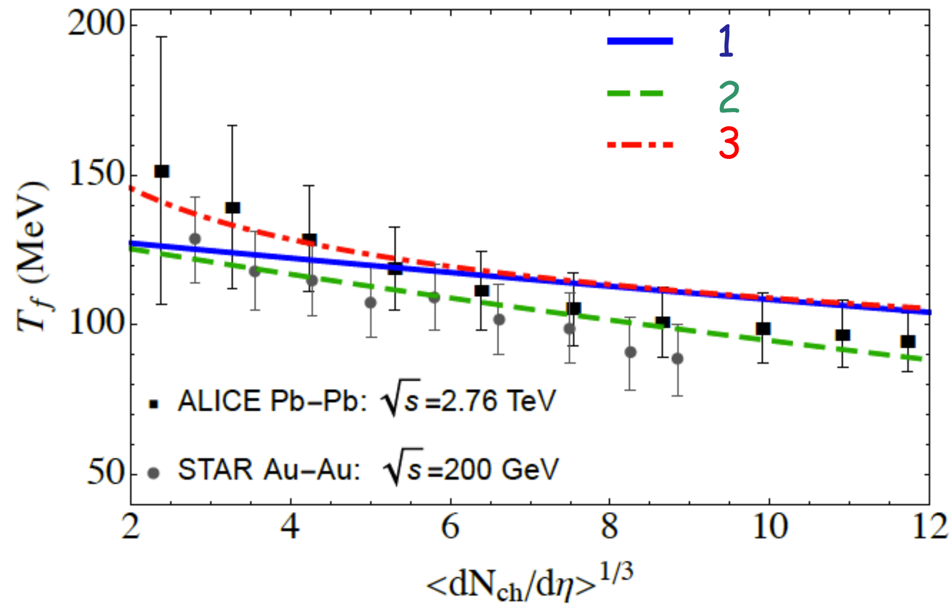


Effect of the parametrization of T_f

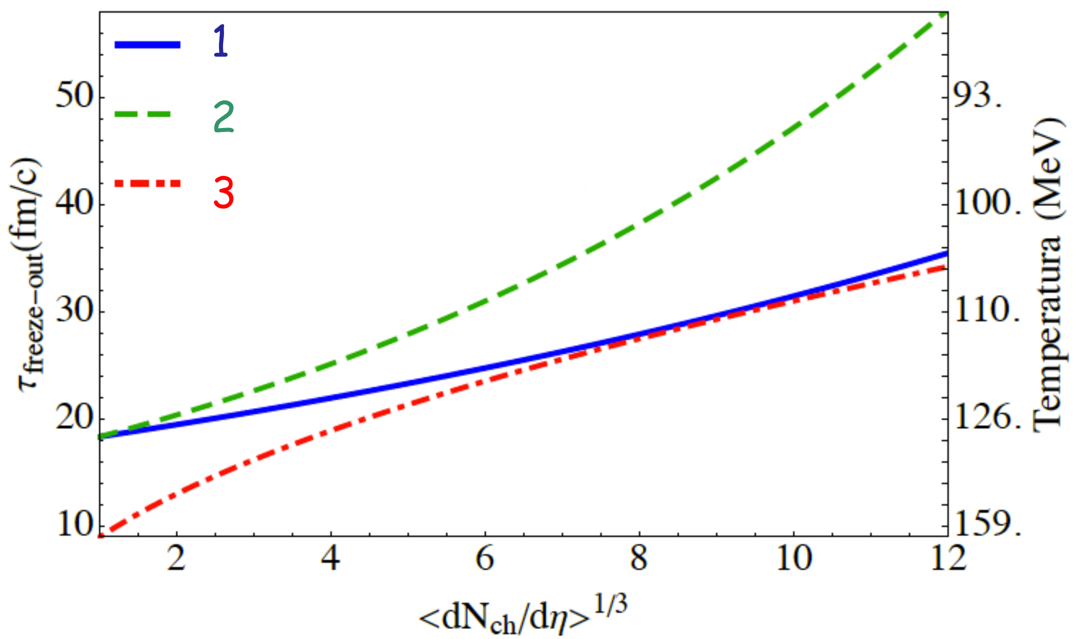
1 $T_f = 132\, e^{-0.02\,\mathcal{N}}$

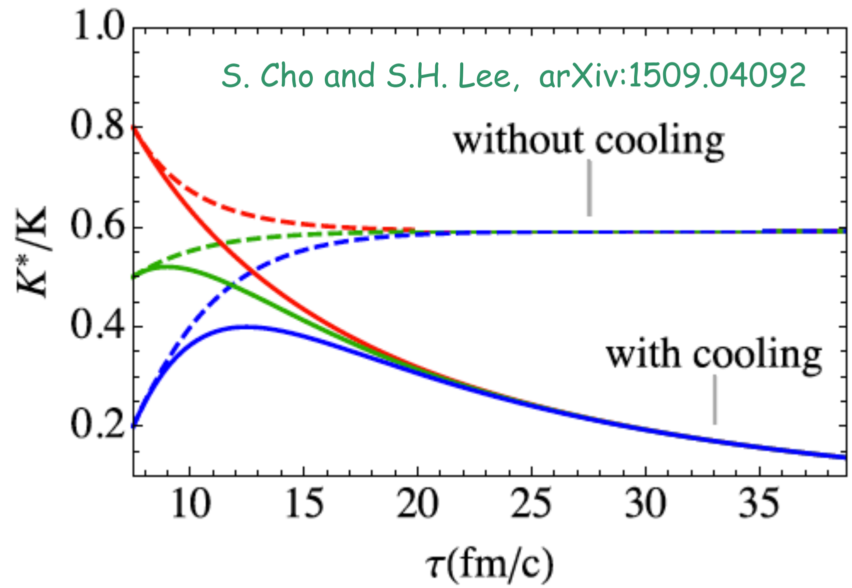
2 $T_f = 134\, e^{-0.035\,\mathcal{N}}$

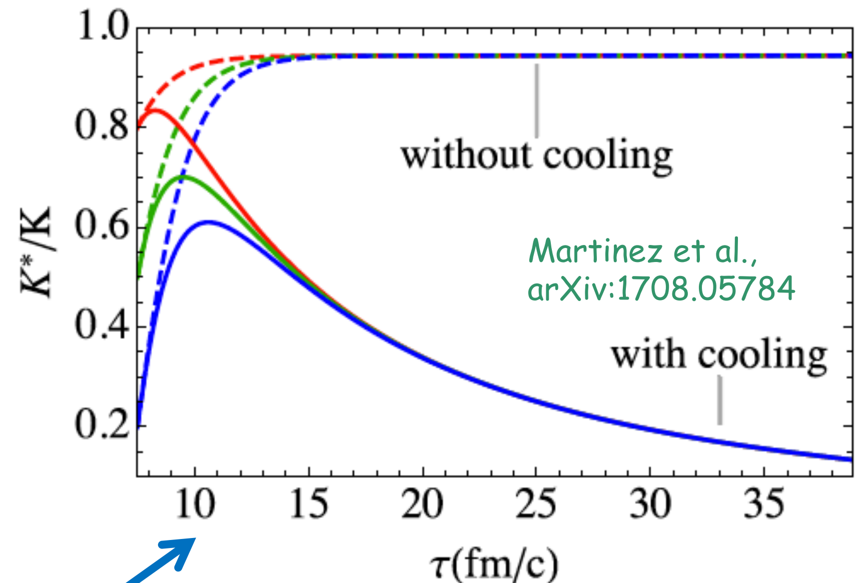
3 $T_f = 165\, e^{-0.18\,\mathcal{N}}$



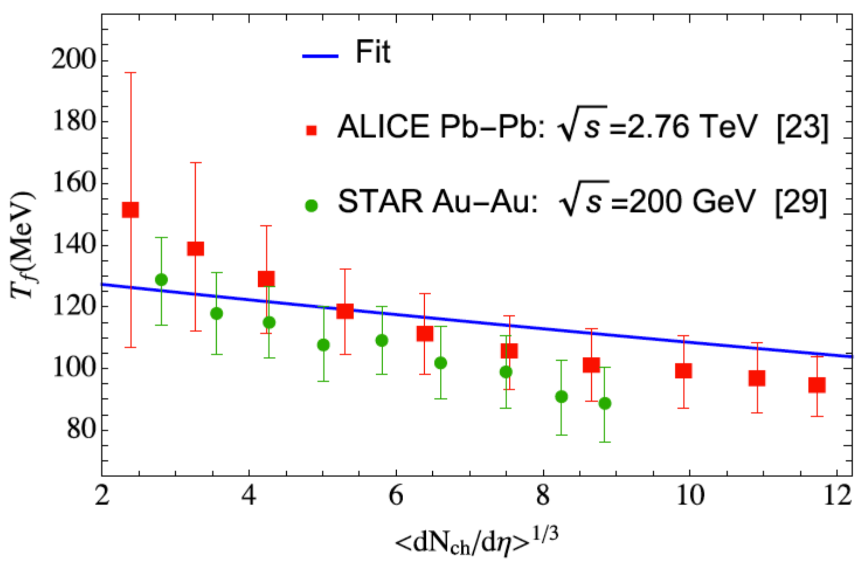
$$\tau_f = \tau_h \left(\frac{T_h}{T_f}\right)^3$$

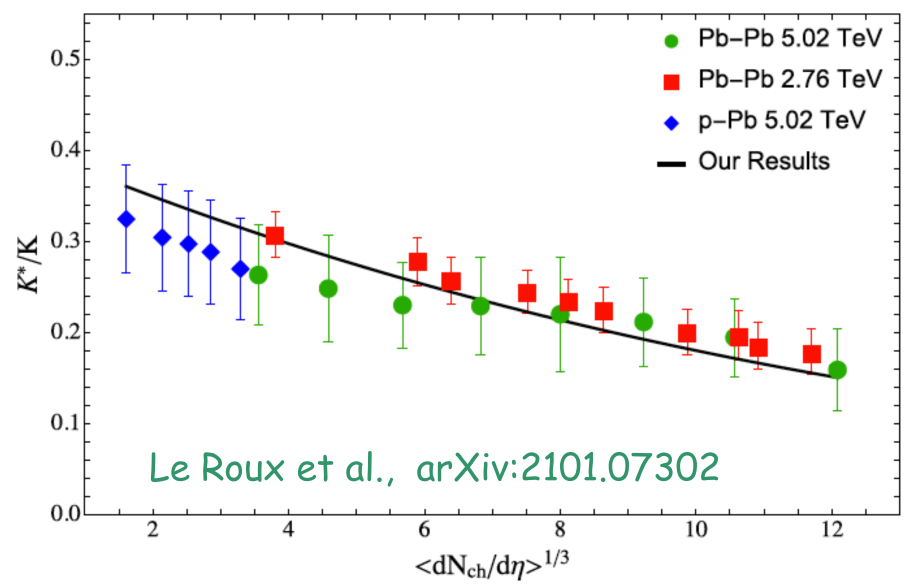






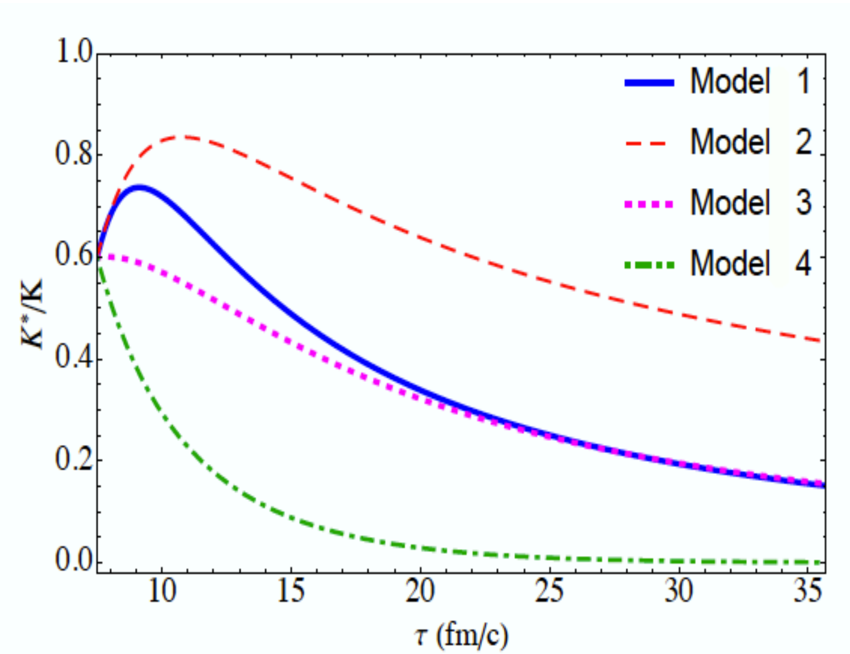
$$\tau_f = \tau_h \left(\frac{T_h}{T_f} \right)^3$$

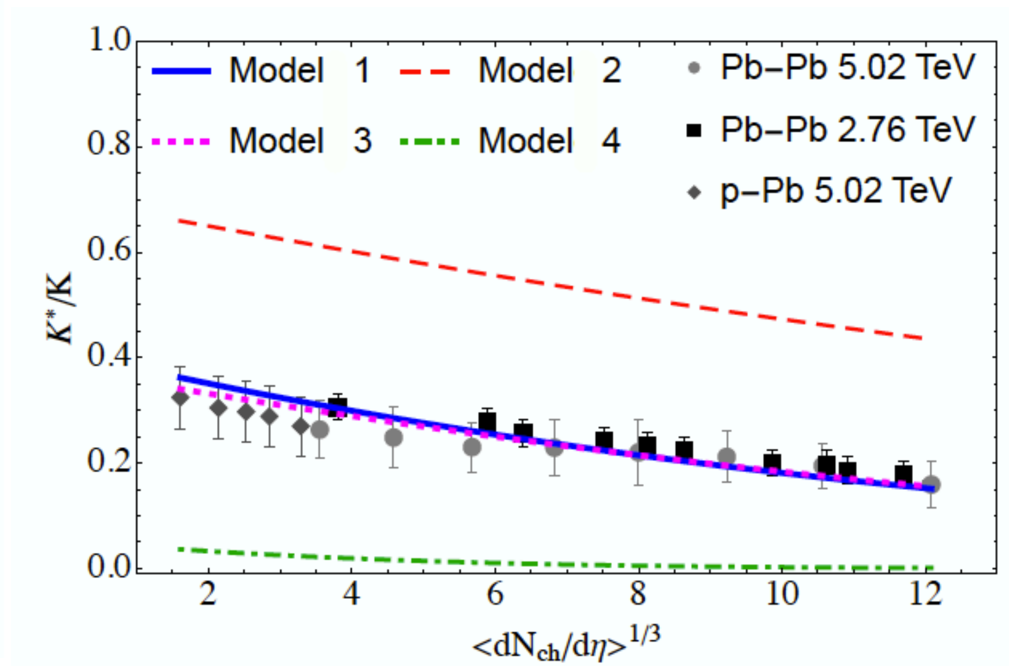




Which reaction is more important?

	$K^*\pi \leftrightarrow K\rho$	$K^*\rho \leftrightarrow K\pi$	$K^* \rightarrow K\pi$	$K\pi \rightarrow K^*$
Model 1	✓	✓	✓	✓
Model 2	✓	✓		
Model 3			✓	✓
Model 4			✓	





C.~Le Roux and F.~S.~Navarra, J. Phys. Conf. Ser. {2340}, 012010 (2022)

Summary

To describe the data we need:

- K* decay and formation
- Cooling
- System dependent freeze-out

D* / D Ratio

Lagrangians -> Amplitudes -> Cross Sections -> Thermal Cross Sections

Evolution equations -> Expansion and cooling -> Freeze-out

Abreu, FSN and Vieira, arXiv:2209.03814

Decay: $D^* \to D + \pi$ $\Gamma(D^*) \simeq 1\,\mathrm{MeV}$ $\tau_{life} = \frac{1}{\Gamma(D^*)} \simeq 200\,\mathrm{fm}$

Not relevant !

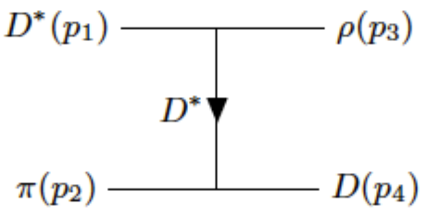
Interactions with rhos and pions

$$\begin{aligned}
\mathcal{L}_{\pi DD^*} &= i g_{\pi DD^*} D^{*\mu} \vec{\tau} \cdot (\bar{D}\partial_\mu \vec{\pi} - \partial_\mu \bar{D} \vec{\pi}) \\
\mathcal{L}_{\rho DD} &= i g_{\rho DD} (D \vec{\tau} \partial_\mu \bar{D} - \partial_\mu D \vec{\tau} \bar{D}) \cdot \vec{\rho}^{\mu}, \\
\mathcal{L}_{\rho D^* D^*} &= i g_{\rho D^* D^*} \left[(\partial_\mu D^{*\nu} \vec{\tau} \bar{D}^*_\nu - D^{*\nu} \vec{\tau} \partial_\mu \bar{D}^*_\nu) \cdot \vec{\rho}^{\mu} \right. \\
&\quad + (D^{*\nu} \vec{\tau} \cdot \partial_\mu \vec{\rho}_\nu - \partial_\mu D^{*\nu} \vec{\tau} \cdot \vec{\rho}_\nu) \bar{D}^{*\mu} \\
&\quad \left. + D^{*\mu} (\vec{\tau} \cdot \vec{\rho}^{\nu} \partial_\mu \bar{D}^*_\nu - \vec{\tau} \cdot \partial_\mu \vec{\rho}^{\nu} \bar{D}^*_\nu) \right], \\
\mathcal{L}_{\pi D^* D^*} &= -g_{\pi D^* D^*} \epsilon^{\mu\nu\alpha\beta} \partial_\mu D^*_\nu \pi \partial_\alpha \bar{D}^*_\beta, \\
\mathcal{L}_{\rho DD^*} &= -g_{\rho DD^*} \epsilon^{\mu\nu\alpha\beta} (D \partial_\mu \rho_\nu \partial_\alpha \bar{D}^*_\beta + \partial_\mu D^*_\nu \partial_\alpha \rho_\beta \bar{D})
\end{aligned}$$

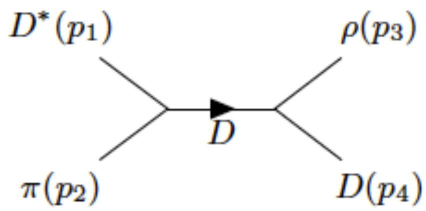
All couplings and form factors calculated with QCD sum rules!

M.~E.~Bracco, M.~Chiapparini, F.~S.~Navarra and M.~Nielsen, arXiv:1104.2864

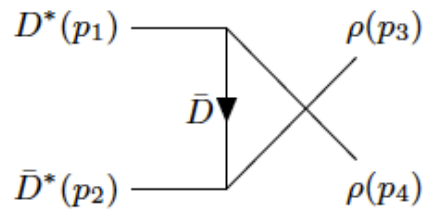
Amplitudes



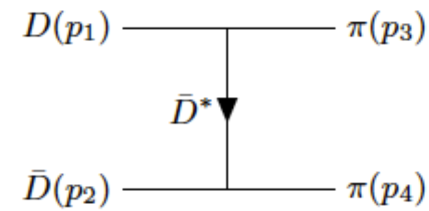
(1.a)



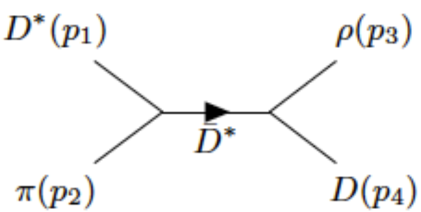
(1.b)



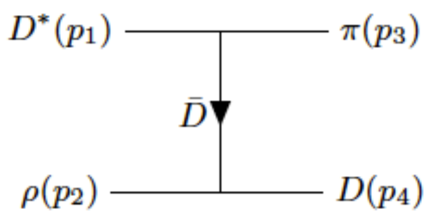
(5.d)



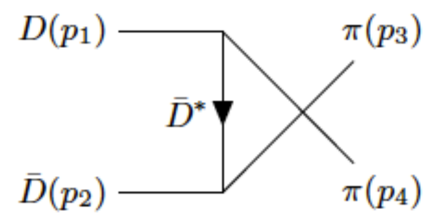
(6.a)



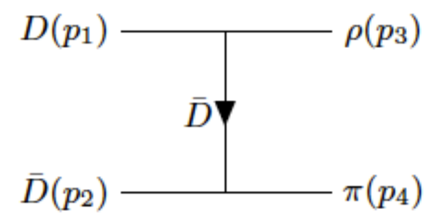
(1.c)



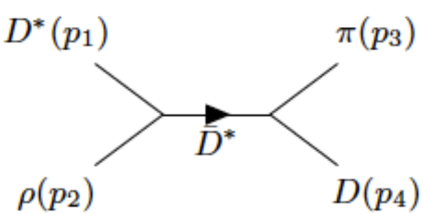
(2.a)

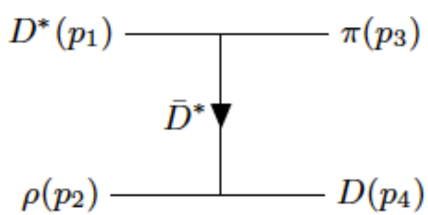


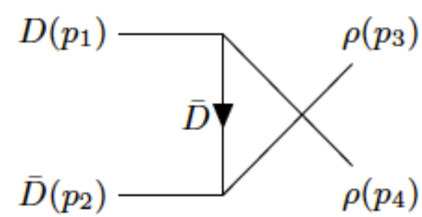
(6.b)

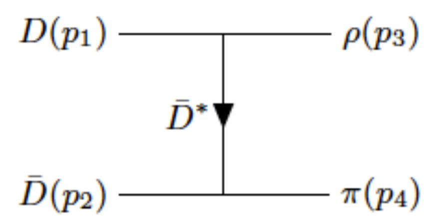


(7.a)









Expansion, cooling and initial conditions

$$T(\tau) = T_C - (T_H - T_F)\left(\frac{\tau - \tau_H}{\tau_F - \tau_H}\right)^{\frac{4}{5}},$$

$$V(\tau) = \pi\left[R_C + v_C(\tau - \tau_C) + \frac{a_C}{2}(\tau - \tau_C)^2\right]^2 \tau c,$$

TABLE II. Parameters used in Eq. (12) for central $Pb - Pb$ collisions at $\sqrt{s_{NN}} = 5$ TeV [25].

v_C (c)	a_C (c^2/fm)	R_C (fm)
0.5	0.09	11
τ_C (fm/c)	τ_H (fm/c)	τ_F (fm/c)
7.1	10.2	21.5
T_C(MeV)	T_H(MeV)	T_F(MeV)
156	156	115
N_c	$N_\pi(\tau_F)$	$N_\rho(\tau_F)$
14	2410	179
$N_D(\tau_H)$	$N_{D^*}(\tau_H)$	
4.7	6.3	

EXHIC, arXiv:1702.00486

Time evolution and multiplicities

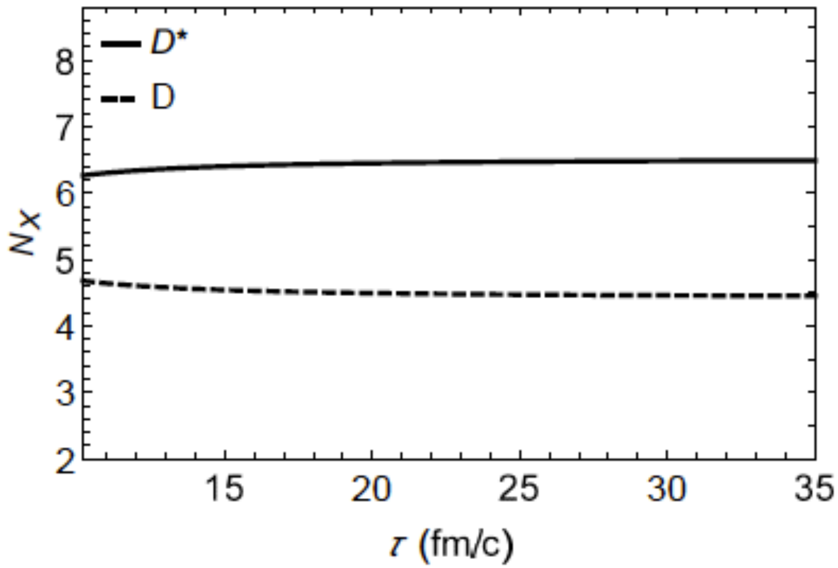
$$\frac{dN_{D^*}}{d\tau} = \langle\sigma_{D\rho\to D^*\pi}v_{D\rho}\rangle n_\rho(\tau)N_D(\tau) - \langle\sigma_{D^*\pi\to D\rho}v_{D^*\pi}\rangle n_\pi(\tau)N_{D^*}(\tau) + \langle\sigma_{D\pi\to D^*\rho}v_{D\pi}\rangle n_\pi(\tau)N_D(\tau)$$
$$- \langle\sigma_{D^*\rho\to D\pi}v_{D^*\rho}\rangle n_\rho(\tau)N_{D^*}(\tau) + \langle\sigma_{\pi\rho\to D^*\bar{D}}v_{\pi\rho}\rangle n_\pi(\tau)N_\rho(\tau) - \langle\sigma_{D^*\bar{D}\to\rho\pi}v_{D^*\bar{D}}\rangle n_{\bar{D}}(\tau)N_{D^*}(\tau)$$
$$+ \langle\sigma_{\pi\pi\to D^*\bar{D}^*}v_{\pi\pi}\rangle n_\pi(\tau)N_\pi(\tau) - \langle\sigma_{D^*\bar{D}^*\to\pi\pi}v_{D^*\bar{D}^*}\rangle n_{\bar{D}^*}(\tau)N_{D^*}(\tau) + \langle\sigma_{\rho\rho\to D^*\bar{D}^*}v_{\rho\rho}\rangle n_\rho(\tau)N_\rho(\tau)$$
$$- \langle\sigma_{D^*\bar{D}^*\to\rho\rho}v_{D^*\bar{D}^*}\rangle n_{\bar{D}^*}(\tau)N_{D^*}(\tau) + \langle\sigma_{D\pi\to D^*}v_{D\pi}\rangle n_\pi(\tau)N_D(\tau) - \langle\Gamma_{D^*}\rangle N_{D^*}(\tau),$$

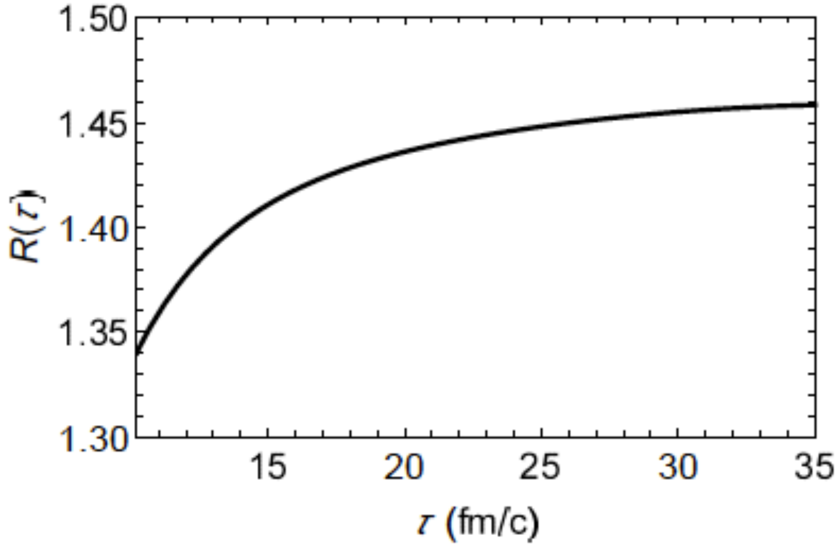
$$\frac{dN_D}{d\tau} = \langle\sigma_{\pi\pi\to D\bar{D}}v_{\pi\pi}\rangle n_\pi(\tau)N_\pi(\tau) - \langle\sigma_{D\bar{D}\to\pi\pi}v_{D\bar{D}}\rangle n_{\bar{D}}(\tau)N_D(\tau) + \langle\sigma_{\rho\rho\to D\bar{D}}v_{\rho\rho}\rangle n_\rho(\tau)N_\rho(\tau)$$
$$- \langle\sigma_{D\bar{D}\to\rho\rho}v_{D\bar{D}}\rangle n_{\bar{D}}(\tau)N_D(\tau) + \langle\sigma_{D^*\pi\to D\rho}v_{D^*\pi}\rangle n_\pi(\tau)N_{D^*}(\tau) - \langle\sigma_{D\rho\to D^*\pi}v_{D\rho}\rangle n_\rho(\tau)N_D(\tau)$$
$$+ \langle\sigma_{D^*\rho\to D\pi}v_{D^*\rho}\rangle n_\rho(\tau)N_{D^*}(\tau) - \langle\sigma_{D\pi\to D^*\rho}v_{D\pi}\rangle n_\pi(\tau)N_D(\tau) + \langle\sigma_{\pi\rho\to D^*\bar{D}}v_{\pi\rho}\rangle n_\pi(\tau)N_\rho(\tau)$$
$$- \langle\sigma_{D^*\bar{D}\to\rho\pi}v_{D^*\bar{D}}\rangle n_{\bar{D}}(\tau)N_{D^*}(\tau) + \langle\Gamma_{D^*}\rangle N_{D^*}(\tau) - \langle\sigma_{D\pi\to D^*}v_{D\pi}\rangle n_\pi(\tau)N_D(\tau),$$

$$n_i(\tau) \approx \frac{1}{2\pi^2}\gamma_i g_i m_i^2 T(\tau) K_2\left(\frac{m_i}{T(\tau)}\right) \qquad N_i = n_i V$$

$$\tau_f = \tau_h \left(\frac{T_H}{T_F}\right)^3 \qquad T_F = T_{F0}e^{-b\mathcal{N}} \quad \longrightarrow \quad \tau_F \propto e^{3b\mathcal{N}}$$

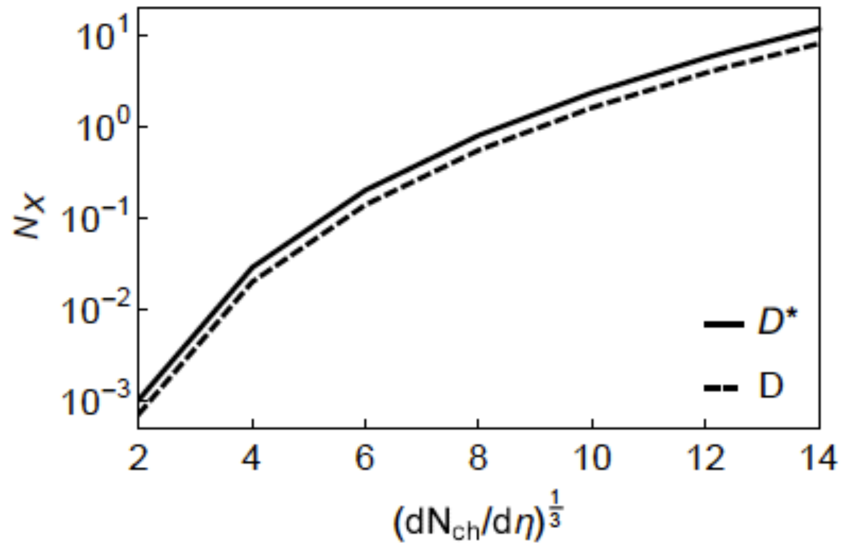
Time evolution and multiplicities

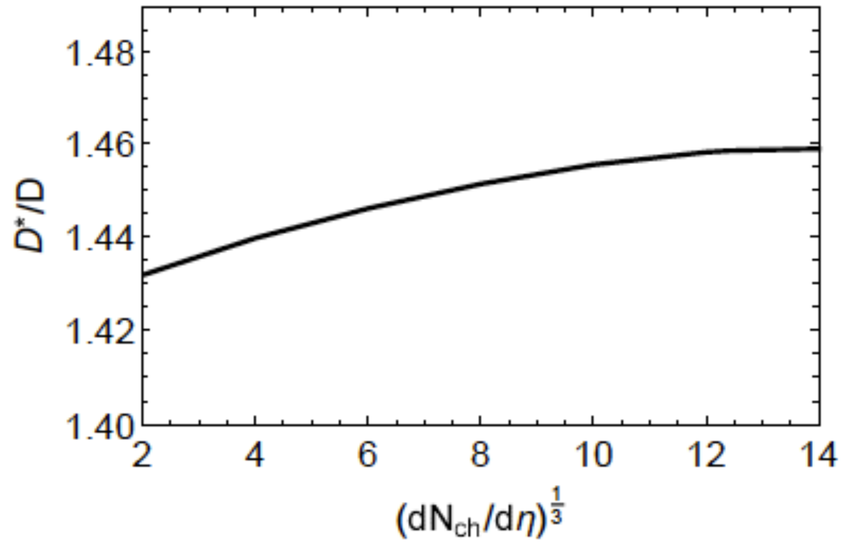




Abreu, FSN and Vieira,
arXiv:2209.03814

System size dependence





Abreu, FSN and Vieira,
arXiv:2209.03814

Summary

K* / K ratio can be well understood with a hadron gas phase

K* decay and formation are the dominant reactions

Cooling and system size dependence of the freeze-out are crucial

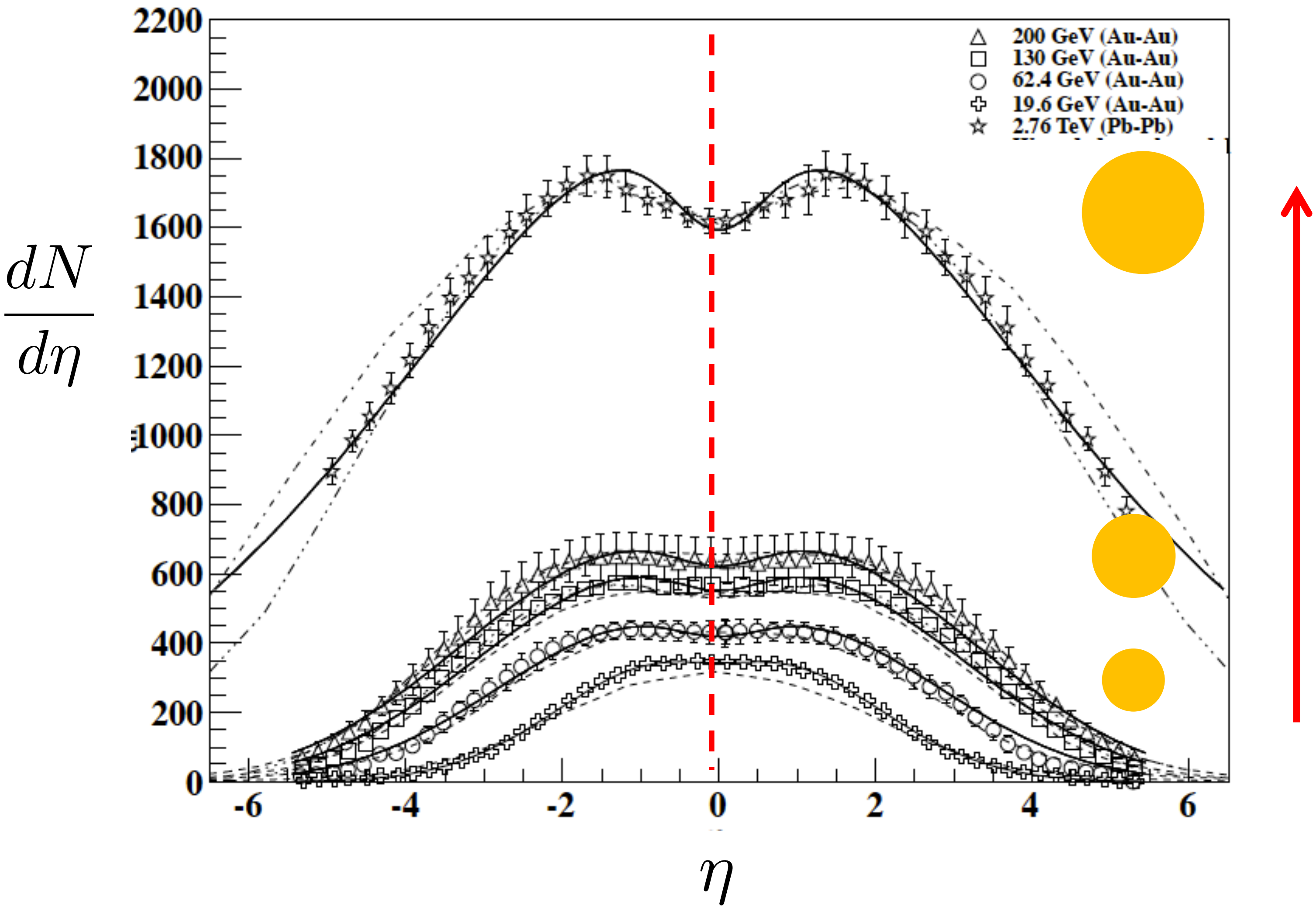
Predictions for the D* / D ratio

Thank you very much !!!

Back-ups

System Size

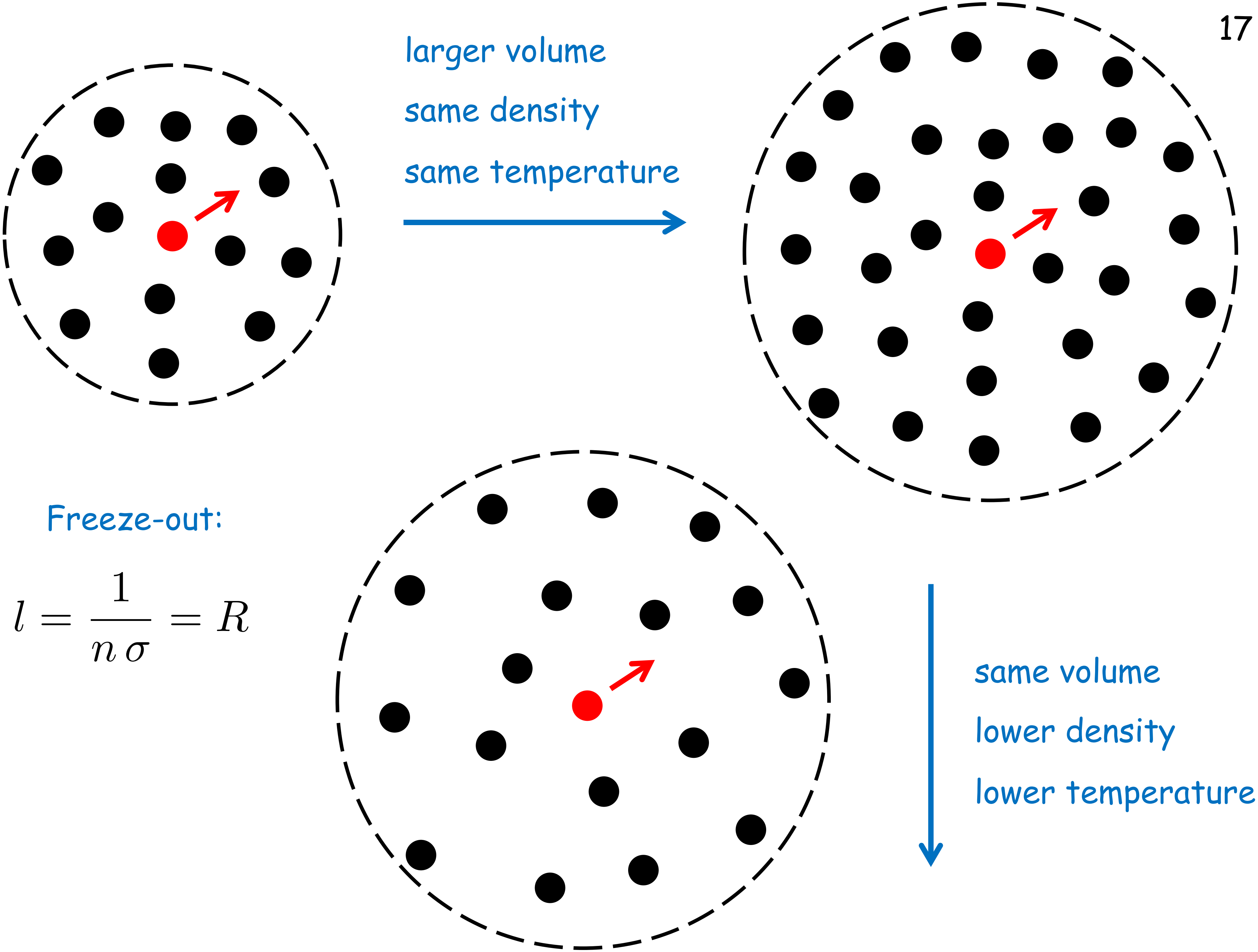
$$\left(\frac{dN}{d\eta}\right)_{\eta=0} \qquad \mathcal{N} = \left[\left(\frac{dN}{d\eta}\right)_{\eta=0}\right]^{1/3}$$



More mesons produced

Larger volume

Lives longer



Freeze-out:

$$l = \frac{1}{n\,\sigma} = R$$

Back to Giorgio

$$\Gamma(D^*) \simeq 1\,\mathrm{MeV} \quad \tau_{life} = \frac{1}{\Gamma(D^*)} \simeq 200\,\mathrm{fm}$$

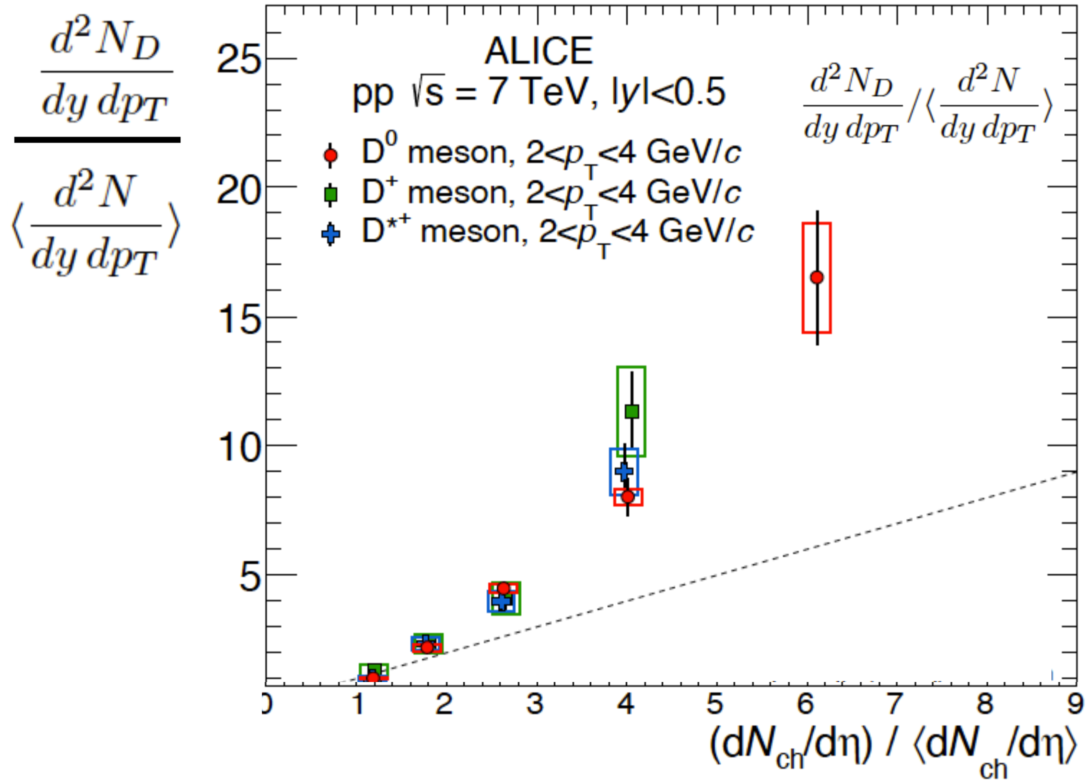
Rapidity and pt dependence of R

Freeze-out e tamanho

SU(4)

Gamma térmico = loops

System size and number of charm quarks



ALICE, JHEP (2015), arXiv:1505.00664

$$\frac{d^2 N_D}{dy\,dp_T} / \langle \frac{d^2 N}{dy\,dp_T} \rangle = \alpha' \left(\frac{dN_{ch}}{d\eta} / \langle \frac{dN_{ch}}{d\eta} \rangle \right)^{\beta}$$

$$\beta = 1.6$$

Assume that:

$$N_D \propto (\mathcal{N}^3)^{\beta}$$

$$N_c \propto (\mathcal{N}^3)^{\beta}$$

Fix the constant using EXHIC estimates:

$$N_c = 7.9 \times 10^{-5}\,\mathcal{N}^{4.8}$$

Lifetime as a function of the size