# The hidden gauge symmetry of relativistic dissipative hydrodynamics

# or... Hydrodynamics with 50 particles. What does it mean and how to think about it?

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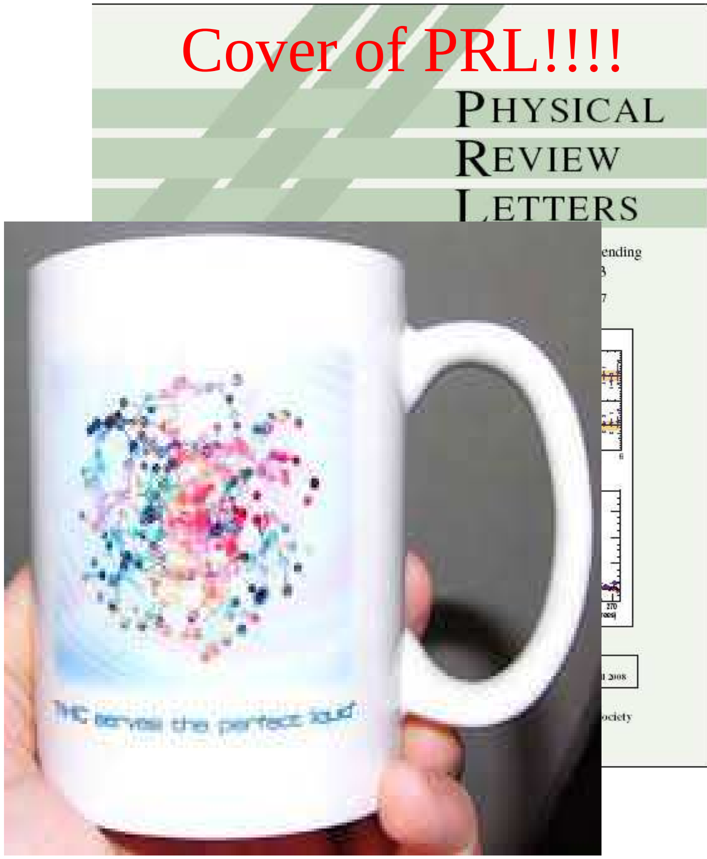
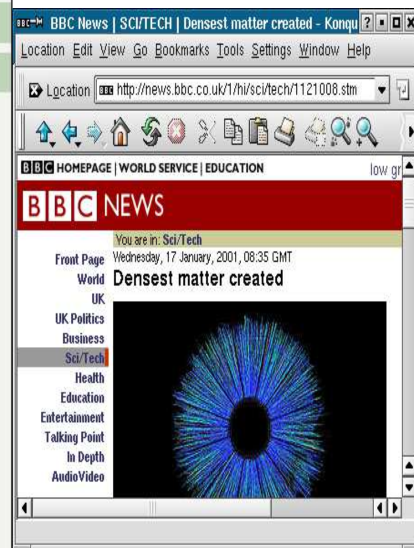
2007.09224 (JHEP),
2109.06389 (Annals of Physics, With T.Dore,M.Shokri,L.Gavassino,D.Montenegro)
Answers somewhat speculative... but I think I am asking good questions!

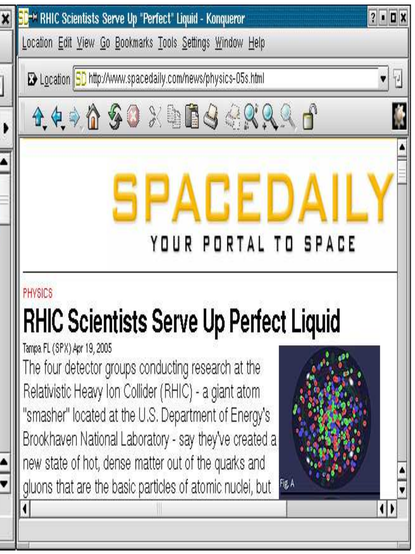
Cover of PRL!!!!




BBC!




SPACE DAILY!




We all know we created the perfect fluid... what does this mean?

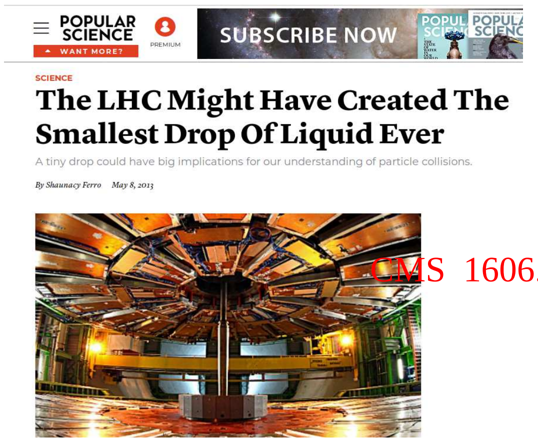
Conventional widsom: hydro EFT of gradients of conserved currents

$$\partial_\mu T^{\mu\nu}=0; T^{\mu\nu} = \underbrace{T^{\mu\nu}_{eq}}_{Thermal} + \underbrace{\Pi^{\mu\nu}}_{Relax} \equiv T^{\mu\nu}=T_0^{\mu\nu}(e,u)+\eta\mathcal{O}\left(\partial u\right)+\tau\mathcal{O}\left(\partial^2 u\right)+...$$

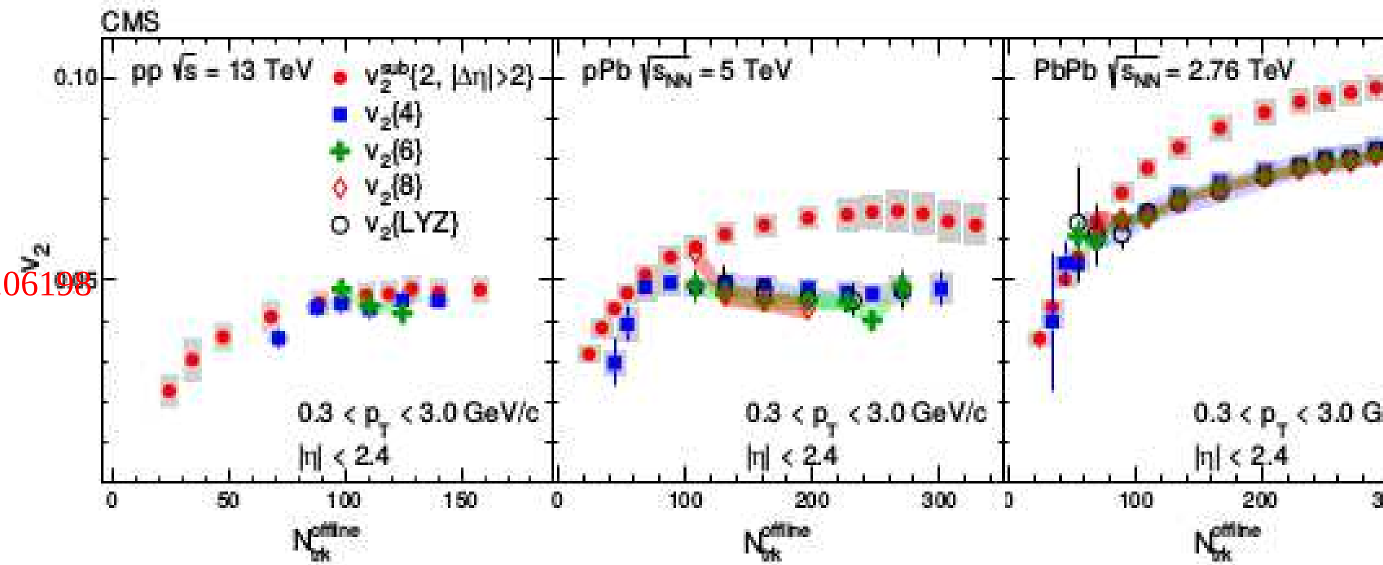
$$\eta=\lim_{k\to 0}\frac{1}{k}\mathrm{Im}\int dx\left\langle \hat{T}_{xy}(x)\hat{T}_{xy}(y)\right\rangle \exp\left[ik(x-y)\right] \quad , \quad \tau\sim\frac{\partial^2}{\partial k^2}\int e^{ikx}\left\langle TT\right\rangle ,$$

**This is a <u>classical</u> theory** , $\hat{T}_{\mu\nu}\to\langle T_{\mu\nu}\rangle$ Correlators $\langle T_{\mu\nu}(x)...T_{\mu\nu}\rangle$ play role in coefficients, <u>not</u> in EoM (if you know initial conditions, you know the whole evolution!) Kubo formula $w\to 0$ cuts out thermal fluctuations. <u>Implicitly assumed$\ll$</u> mean free path

**Both top-down** ultimately derived from "microscopic" theories (Boltzmann equation,AdS/CFT), not "bottom up" statistical mechanics ("universality", independent from microscopic physics)!

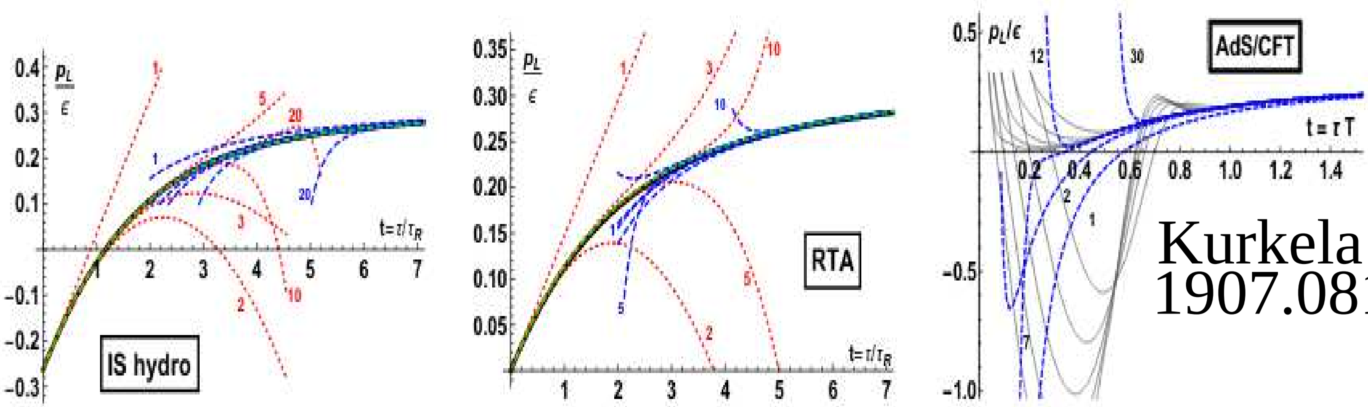



CMS 1606.06198




1606.06198 (CMS) : When you consider geometry differences and multi-particle cumulants (remove momentum conservation), hydro with $\mathcal{O}(20)$ particles "just as collective" as for 1000. Also cold atom fluids with 10,000 particles $\sim 1mm^3$. Controversial (A.Bilandzic et al) but AFAIK no evidence collectivity goes down with $A, N$! Little understanding of this in "conventional widsom". What si the smallest possible fluid?

Hydrodynamics in small systems: “hydrodynamization”/”fake equilibrium”
A lot more work in both AdS/CFT and transport theory about ”hydrodynamization”/”Hydrodynamic attractors”




Kurkela et al 1907.08101.

Fluid-like systems far from equilibrium (large gradients )! Usually from 1D solution of Boltzmann and AdS/CFT EoMs! “hydrodynamics converges even at large gradients with no thermal equilibrium”

But I have a basic question: ensemble averaging!

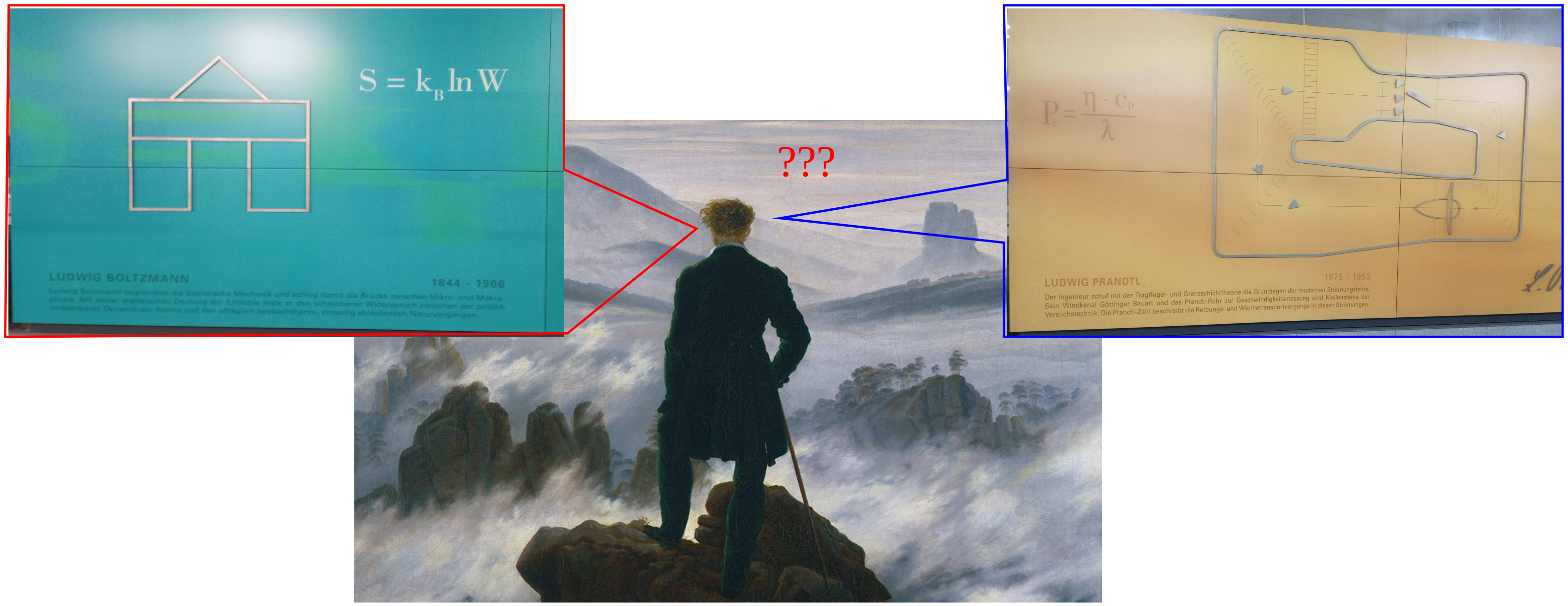
**Ensemble averaging** , $\langle F(\{x_i\},t)\rangle \neq F(\{\langle x_i\rangle\},t)$ suspect for any non-linear theory. molecular chaos in Boltzmann, Large $N_c$ in AdS/CFT, all assumed . But for $\mathcal{O}(50)$ particles?!?! For water, a cube of length $\eta/(sT)$ has $\mathcal{O}\left(10^9\right)$ molecules,

$$P(N \neq \langle N\rangle) \sim \exp\left[-\langle N\rangle^{-1}(N-\langle N\rangle)^2\right] \ll 1$$

. EoS is given by $p = T\ln Z$ but $\partial^2 \ln Z/\partial T^2, dP/dV$?? NB: nothing to do with equilibration timescale . Even "things born in equilibrium" locally via Eigenstate thermalization have fluctuations!

**How** does dissipation work in such a "semi-microscopic system"? If $T_{\mu\nu} \to \hat{T}_{\mu\nu}$ what is $\hat{\Pi}_{\mu\nu}$ Second law fluctuations? Sometimes because of a fluctuation entropy decreases! What is the role of microstates?

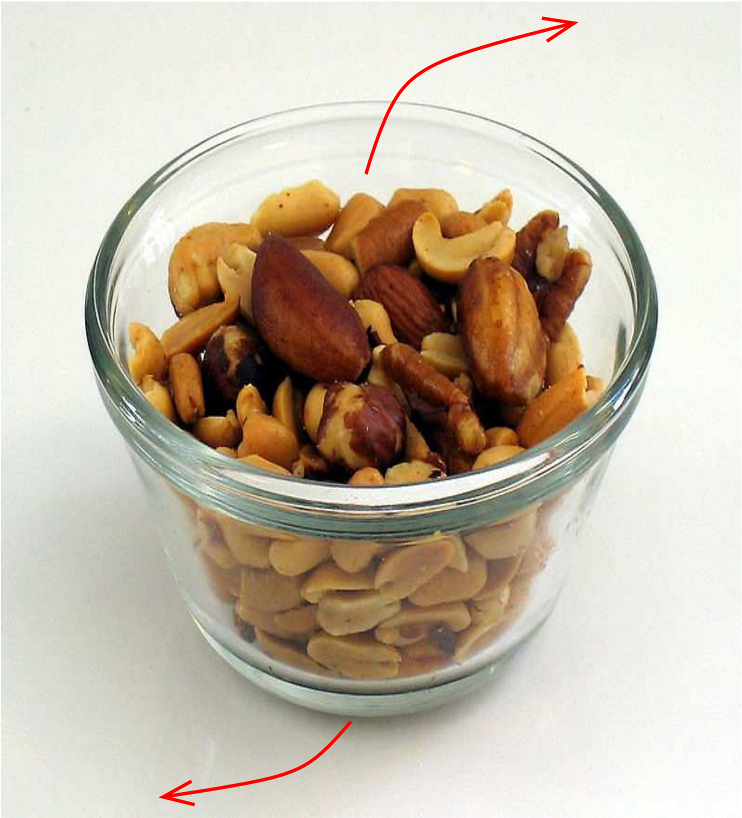
**The obvious conclusion** is Fluctuations only help dissipation, they are random . Can $l_{mfp} \geq \mathcal{O}(1)\left(V/N_{dof}\right)^{1/3}$ be wrong?




Bottom line: Either hydrodynamics is not the right explanation for these observables (possible! But small/big systems similar! ) or we are not understanding something basic about what's behind the hydrodynamics! What do fluctuations do? Just a lower limit to dissipation?
more fundamentally , the relationship between hydrodynamics and statistical mechanics is not as understood as one might think!

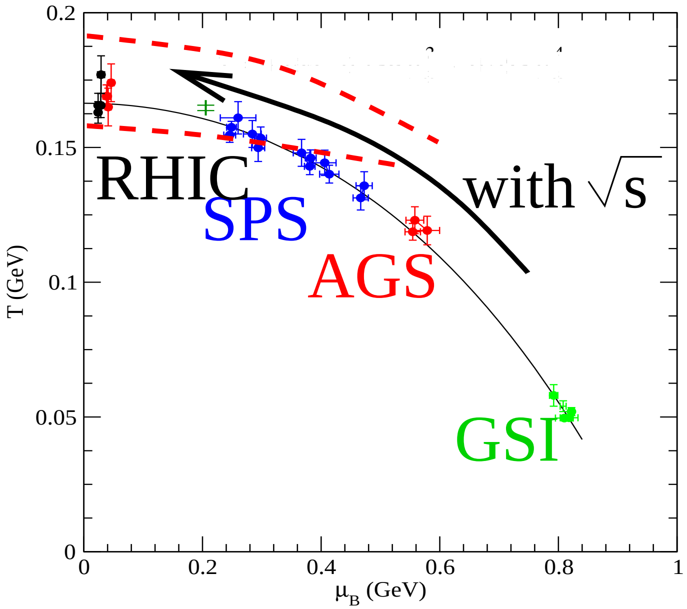
Perhaps even related to everyday physics?

The
Brazil
nut effect

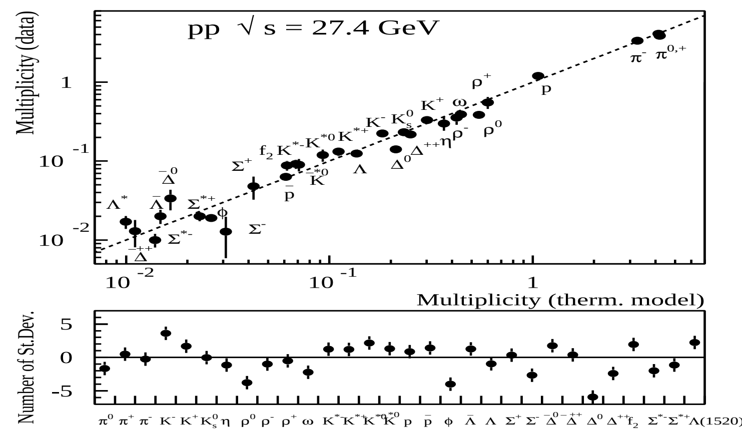


# Statistical mechanics in small systems

Smaller energies
Cleymans
Oeschler
Redlich

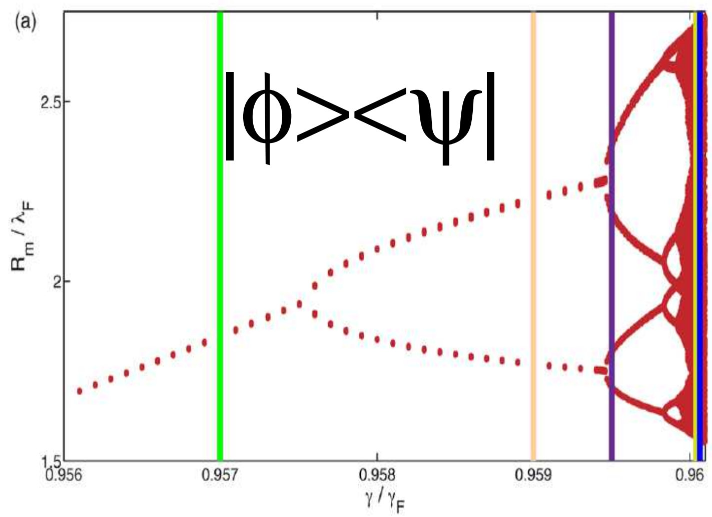



Smaller sizes (p−p,e−e)
Becattini



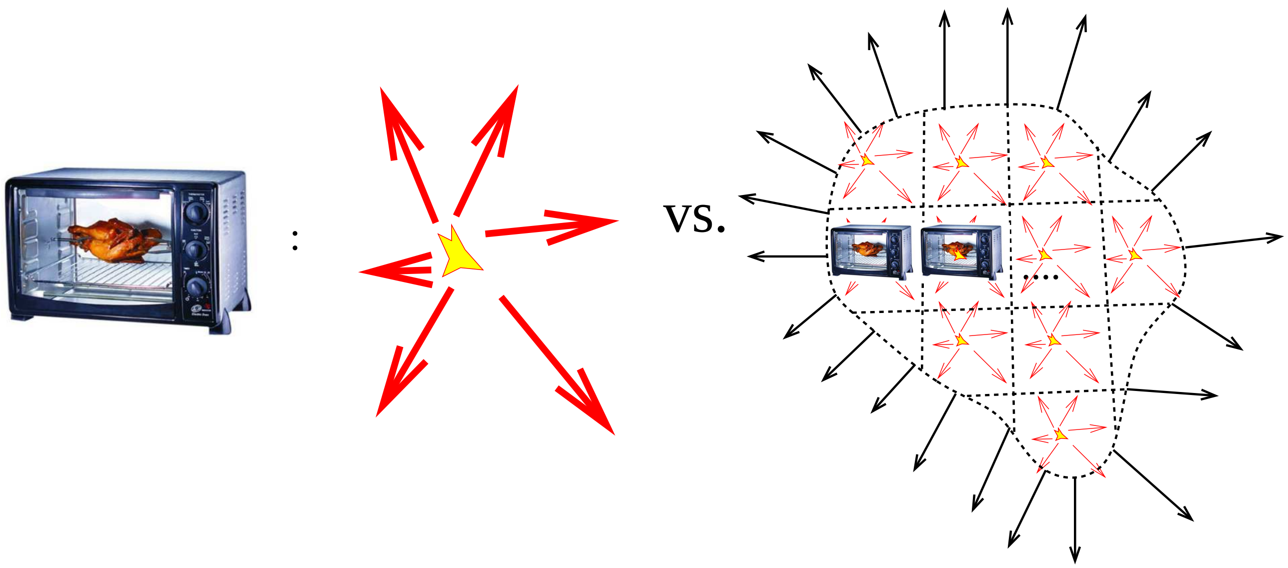

If you consider chemistry (particle ratios), statistical mechanics does seem to work reasonably well to smallest systems! What is the relationship between this,hydrodynamics and microscopic theory?

Statistical behavior is actually not surprising




Berry/Bohigas/Eigenstate thermalization hypothesys: $E_{n>>1}$ of quantum systems whose classical correspondent is chaotic have density matrices that look like pseudo-random. If off-diagonal elements oscillate fast or observables simple, indistinguishable from Micro-canonical ensemble!

What we lack...




We need to build a hydrodynamics from such a picture away from the many particle limit So fluctuations are included. Boltzmann,AdS/CFT both assume $s^{-1/3} \ll \eta/(sT)$ Can intuition that fluctuations "only add dissipation" away from thermalization be wrong? actually in Gauge theory the opposite happens! Fluctuations "add to equilibrium"!

What is a gauge theory,exactly?

$$\mathcal{Z} = \int \mathcal{D}A^\mu \exp\left[S[F_{\mu\nu}]\right] \equiv \int \mathcal{D}A_1^\mu \mathcal{D}A_2^\mu \exp\left[S[A_1^\mu]\right]$$

$A^\mu_{1,2}$ can be separated since physics sensitive to derivatives of $\ln \mathcal{Z}$

$$\ln \mathcal{Z} = \Lambda + \ln \mathcal{Z}_G \quad , \quad Z_G = \int \mathcal{D}\mathcal{A}^\mu \delta\left(G(A^\mu)\right) \exp\left[S(A_\mu)\right]$$

Ghosts come from expanding $\delta(...)$ term. In KMS condition/Zubarev

$$Z = \int \mathcal{D}\phi \quad , \quad {}''S'' \rightarrow d\Sigma_\nu \beta_\mu T^{\mu\nu}$$

Multiple $T_{\mu\nu}(\phi) \rightarrow$ Gauge-like configuration . Related to Phase space fluctuations of $\phi$

## A proposal for a different point of view: Inverse ("Bayesian") attractor

**Close to local equilibrium** is not on gradient expansion but the approximate applicability of fluctuation-dissipation
These are not automatically the same!

**For smaller fluctuating systems** many equivalent definitions of $T_0^{\mu\nu}, \Pi^{\mu\nu}$

Different Boltzmannian entropy but all counted as Gibbsian entropy

**If many equivalent choices of $\Pi_{\mu\nu}$** likely in one its "small"! Ideal hydro behavior.

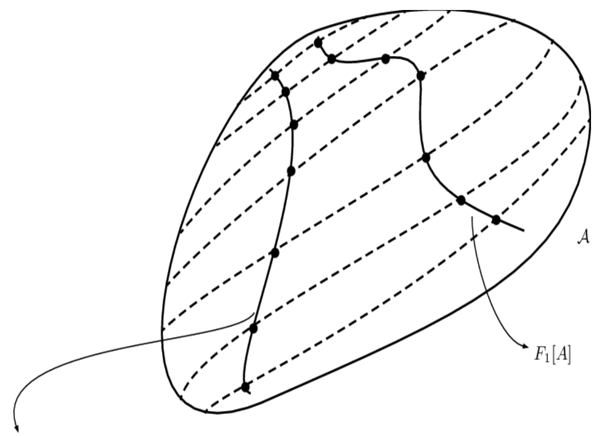
So indeed Ambiguity from fluctuations makes system look like a fluid.

So could fluctuations help thermalize? A key insight is redundances

Some qualitative developments: $T_0^{\mu\nu}, \Pi^{\mu\nu}, u^\mu$ are not actually experimental observables! Only total energy momentum tensor

$$\hat{T}^{\mu\nu} = \hat{T}_0^{\mu\nu} + \hat{\Pi}^{\mu\nu}$$

and its correlators are! Changing $d\Sigma_\mu$ in Zubarev $\equiv$ changing $\Pi^{\mu\nu}, T_0^{\mu\nu}$ !




Analogy to choosing a gauge in gauge theory?

This is relevant for current hydrodynamic research

Causal relativistic hydrodynamics still contentious, with many definitions

**Israel-Stewart** Relaxing $\Pi_{\mu\nu}$ .

Causal, but up to 9 additional DoFs (not counting conserved charges), blow-up possible (M.Disconzi, 2008.03841). $\Pi_{\mu\nu}$ "evolving" microstates!

**BDNK,earlier Hiscock,Lindblom,Geroch,...** $\Pi_{\mu\nu} \sim \partial u$ At a price

- Arbitrary (up to causality constaints) $u_\mu$ .
- Entropy "temporarily decreases" with perturbations (Gavassino et al, arXiv:2006.09843 ). Kovtun in 2112.14042 derives BDNK from a truncation of the Boltzmann equation generally violating the H-theorem

For phenomenology because of conservation laws “any” $\partial_\mu T^{\mu\nu}$ “can be integrated” but lack of link with equilibration and multiple definitions of “near-equilibrium” problematic.

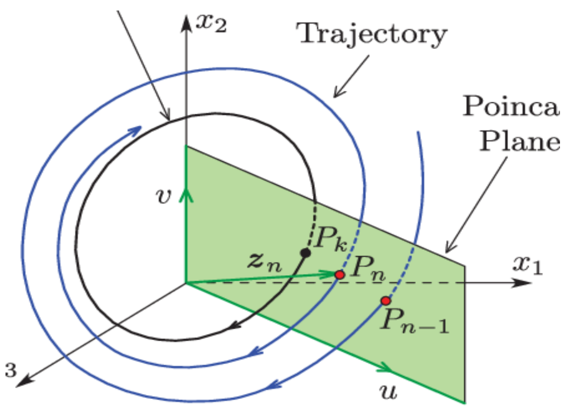
**If you care** about statistical mechanics, price is steep!
“special” time foliation from ergodic hypothesis/Poncaire cycles!

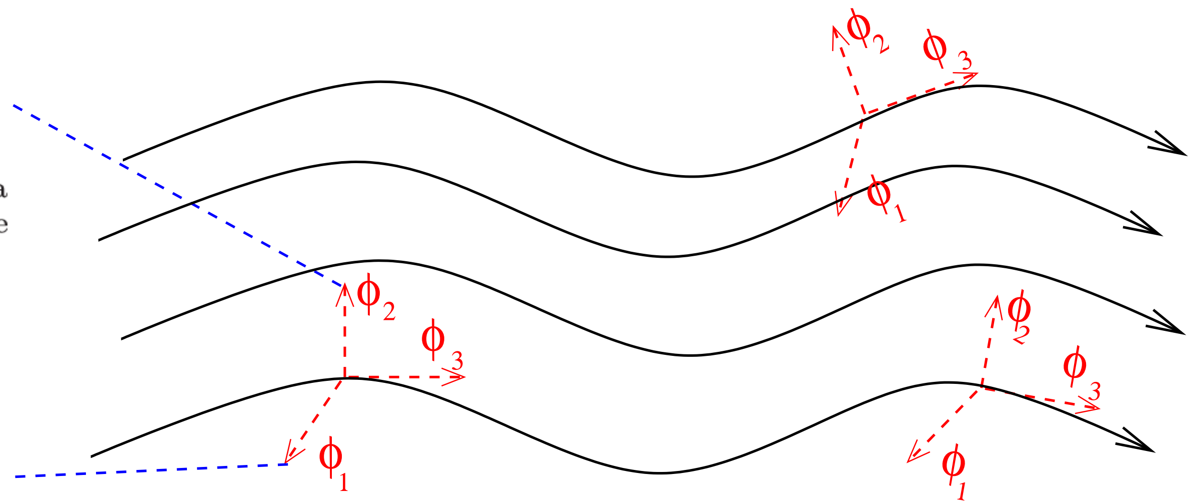
**But** entropy decrease physically reasonable from Zubarev definition. But not from H-theorem!

**Fluctuations** come with redundances in $T_0^{\mu\nu}, \Pi^{\mu\nu}$

Could these definitions of $u_\mu$ be just “Gauge” choices?

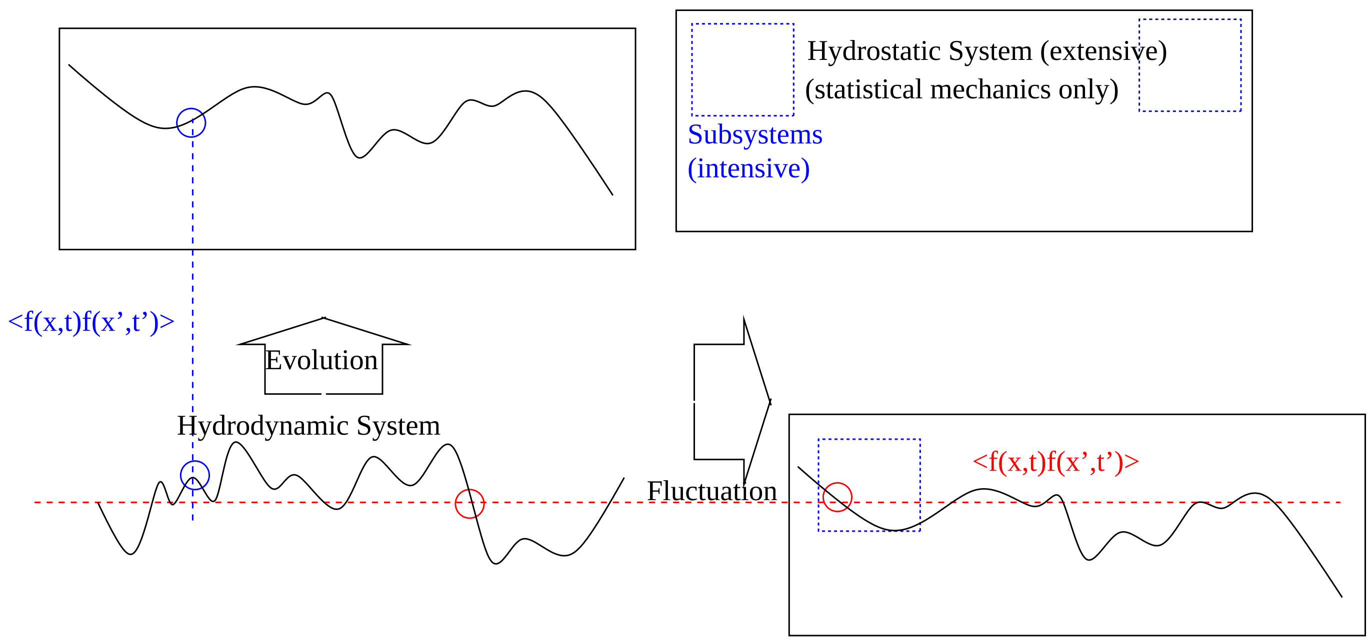
How to make physics fully "gauge"-invariant? Ergodicity/Poncaire cycles meet relativity slightly away from equilibrium!



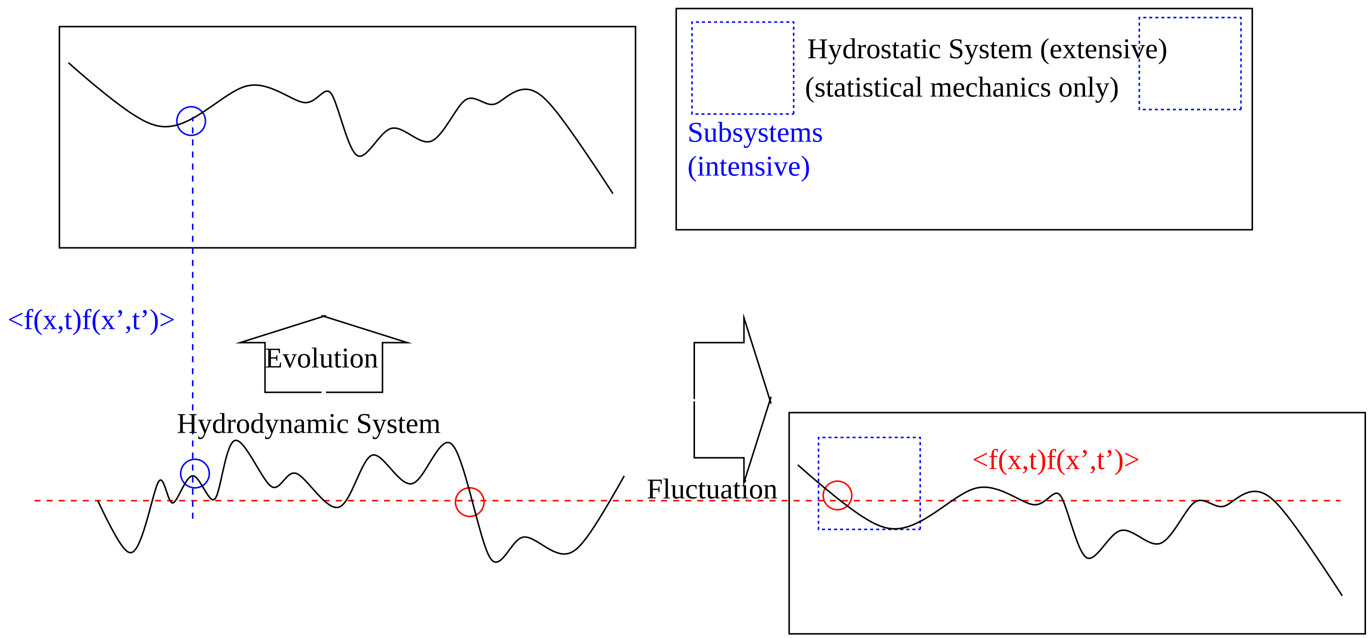




Gibbs entropy level+relativity : Lack of equilibrium is equivalent to "loss of phase" of Poncaire cycles. one can see a slightly out of equilibrium cell <u>either</u> as a "mismatched $u_\mu$" (fluctuation) or as lack of genuine equilibrium (dissipation)

# How to make physics fully "gauge"-invariant?




Fluctuation-dissipation at the cell level could do it! We don't know if a "step" is fluctuation ($T_0^{\mu\nu}$ or evolution ($\Pi_{\mu\nu}$ )-driven!




But in hydro $T_0^{\mu\nu}, \Pi_{\mu\nu}$ treated very differently! "Sound-wave" $u \sim \exp[ik_\mu x^\mu]$ or "non-hydrodynamic Israel-Stewart mode?"
$D\Pi_{\mu\nu} + \Pi_{\mu\nu} = \partial u$
Only in EFT $1/T \ll l_{mfp}$ they are truly different!

Infinitesimal transformation $dM_{\mu\nu}$ such that $dM_{\mu\nu}(x)\frac{\delta \ln \mathcal{Z}_E[\beta_\mu]}{dg^{\alpha\mu}(x)} = 0$

**Change in microscopic fluctuation** $\ln \mathcal{Z} \to \ln \mathcal{Z} + d \ln \mathcal{Z}$

$$d \ln \mathcal{Z} = \sum_{N=0}^{\infty} \int \prod_{j=1}^{N} d^4 p_j \delta \left( E_N(p_1, ...p_j) - \sum_j p_j^0 \right) \sqrt{|dM|} \exp\left( -\frac{dM_{0\mu} p^\mu}{T} \right)$$

**Change in macroscopic dissipative term**

$$\Pi_{\mu\nu} \to \Pi_{\alpha\gamma} \left( g_\mu^\alpha g_\nu^\gamma - g_\mu^\alpha dM_\nu^\gamma - g_\nu^\gamma dM_\mu^\alpha \right) \quad , \quad u_\mu \to u_\alpha \left( g_\mu^\alpha - dM_\mu^\alpha \right)$$

For $1/T \ll l_{mfp}$ probability$\to 0$, $1/T \sim l_{mfp}$ many "similar" probabilities!

## The "gauge-symmetry" in practice

Generally $dM_{\mu\nu} = \Lambda^{-1}_{\alpha\mu} dU^{\alpha\beta} \Lambda_{\beta\mu}$

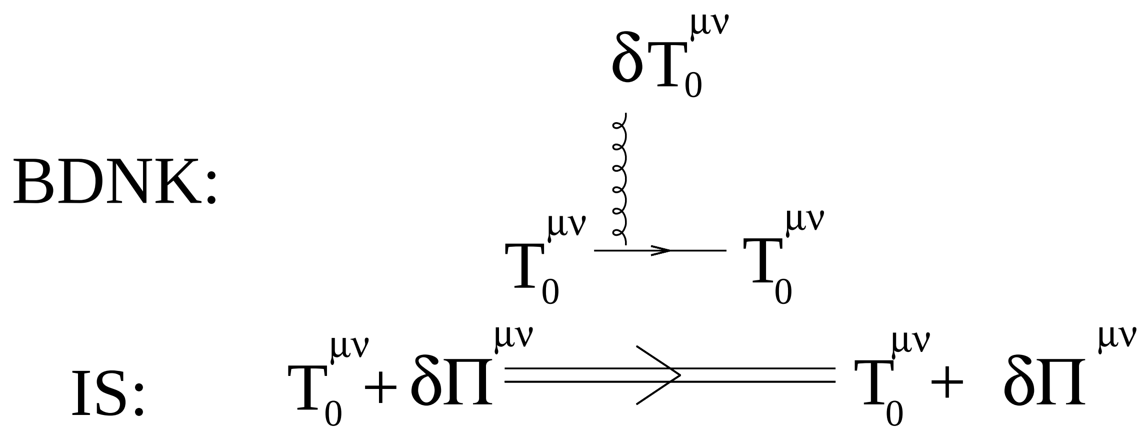
$$d\left[\ln \Pi_{\alpha\beta}\right] \Lambda^{\alpha\mu} \left(\Lambda^{\beta\nu}\right)^{-1} = \eta^{\mu\nu} d\mathcal{A} + \sum_{I=1,3} \left( d\alpha_I \hat{J}_I^{\mu\nu} + d\beta_I \hat{K}_I^{\mu\nu} \right)$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

which move components from $\Pi_{\mu\nu}$ to $Q_\mu$ as well as $K_{1,2,3}$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

## An example... bulk viscosity




$$e_{IS} \to e + (e+p)\tau \frac{\dot{e}}{e+p} + \left( (e+p)\tau + \frac{c_V}{s}\zeta \right) \partial_\mu u^\mu \quad , \quad p_{IS} \to p + \Pi$$

Considering $c_V$ controls energy fluctuations, shift from IS to BDNK equivalent to relabeling $\Pi$ dynamics as interaction with a fluctuation-generated sound wave.

Characterizing these gauge redundancies

Grossi,Floerchinger, 2102.11098 (PRD) Let us define a $J$ co-moving with $u_\mu$ and use the "exact" (before coarse-graining) partition function to build

$$\Gamma(\phi) = \mathrm{Sup_J} \left( \int J(x)\phi(x) - i \ln \mathcal{Z}[J] \right)$$

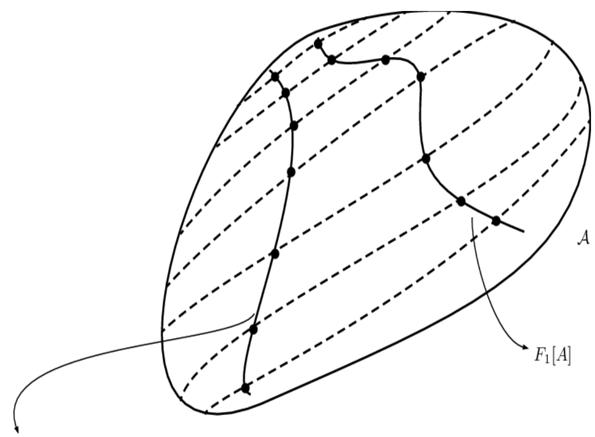
$u_\mu \to u'_\mu$ non-inertial and does not change $\langle T_{\mu\nu} \rangle$, so one can define

$$J_{\mu\nu\gamma} = \frac{1}{\sqrt{g}} \frac{\delta \ln \mathcal{Z}[J']}{\delta \Gamma^{\alpha\nu\gamma}} \quad , \quad D_\mu J^{\mu\nu\gamma} = 0$$

Setting the gauge at the level of the microscopic approximately thermalized partition function equivalent adding auxiliary field $D_\mu M_{\alpha\beta} = 0$ to

$$\mathcal{Z}[J_{\alpha\beta\gamma}] = \int \mathcal{D}\phi \mathcal{D} M_{\alpha\beta} \exp \left[ \int det[M] d^4x \mathcal{L}\left(\phi, \partial_\mu + \Gamma ...\right) + \int d\Sigma^\gamma M^{\alpha\beta} J_{\alpha\beta\gamma} \right]$$

Cool but what about thermalization in small systems?
Initial and final state described by many equivalent trajectories




One of them could be close to an ideal-looking one. "reverse" attractor Few particles with strong interaction (Eigenstate thermalization? ) correspond to many hydro like-configurations $\{u_\mu, \Pi_{\mu\nu}\}$ with fluctuations , within same Gibbs entropy class. some closer to ideal? No symmetries necessary!

Irrelevant in everyday liquids since $l_{mfp} \gg 1/T$ or AdS/CFT since $N_c \ll \infty$ but perhaps not for QGP!

Every statistical theory needs a "state space" and an "evolution dynamics"
The ingredients

**State space:Zubarev hydrodynamics** Mixes micro and macro DoFs

**Dynamics: Crooks fluctuation theorem** provides the dynamics via a definition of $\Pi_{\mu\nu}$ from fluctuations

$\hat{T}^{\mu\nu}$ is an operator, so any decomposition, such as $\hat{T}_0^{\mu\nu} + \hat{\Pi}^{\mu\nu}$ must be too!

Zubarev partition function for local equilibrium: think of Eigenstate thermalization...

Let us generalize the GC ensemble to a co-moving frame $E/T \to \beta_\mu T^\mu_\nu$

$$\hat{\rho}(T_0^{\mu\nu}(x), \Sigma_\mu, \beta_\mu) = \frac{1}{Z(\Sigma_\mu, \beta_\mu)} \exp\left[ - \int_{\Sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}_0^{\mu\nu} \right]$$

$Z$ is a partition function with a field of Lagrange multiplies $\beta_\mu$ , with microscopic and quantum fluctuations included.

Effective action from $\ln[Z]$ . Correction to Lagrangian picture?

All normalizations diverge but hey, it's QFT! (Later we resolve this! )

This is perfect global equilibrium. What about imperfect local? Two vectors, $d\Sigma_\mu u_\mu T_0^{\mu\nu}$ $d\Sigma_\mu$ foliation. We can coarse-grain and gradient expand, but Kubo already proven ,can we do better?

An operator formulation $\hat{T}^{\mu\nu} = \hat{T}_0^{\mu\nu} + \hat{\Pi}_{\mu\nu}$
and $\hat{T}_0^{\mu\nu}$ truly in equilibrium! Each microscopic particle "does not know" if it "belongs" to $\hat{T}_0^{\mu\nu}, \hat{\Pi}_{\mu\nu}$

$$\hat{\rho}(T_0^{\mu\nu}(x), \Sigma_\mu, \beta_\mu) = \frac{1}{Z(\Sigma_\mu, \beta_\mu)} \exp\left[ -\int_{\Sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}_0^{\mu\nu} \right]$$

describes all cumulants and probabilities

$$\langle T_0^{\mu\nu}(x_1) T_0^{\mu\nu}(x_2) ... T_0^{\mu\nu}(x_n) \rangle = \prod_i \frac{\delta^n}{\delta \beta_\mu(x_i)} \ln Z$$

Equilibrium at "probabilistic" level and KMS Condition obeyed by "part of density matrix" in equilibrium, "expand" around that! An operator constrained by KMS condition is still an operator! $\equiv$ time dependence in interaction picture

## Entropy/Deviations from equilibrium

$$n^\nu \partial_\nu \left( s u^\mu \right) = n^\mu \frac{\Pi^{\alpha\beta}}{T} \partial_\alpha \beta_\beta \quad , \quad \geq 0$$

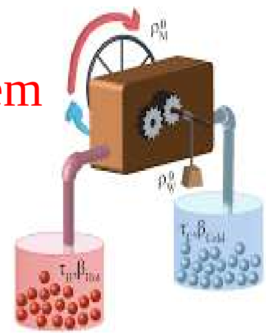
- If $n_\mu$ arbitrary cannot be true for "any" choice
- 2nd law is true for "averages" anyways, sometimes entropy can decrease

We need a fluctuating formulation!

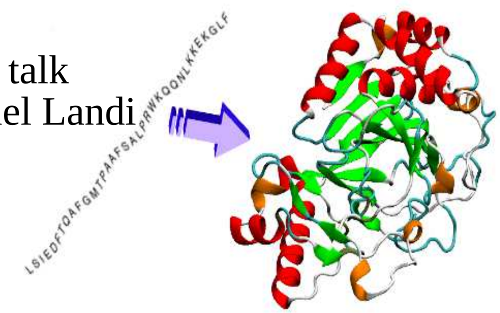
- "Statistical" (probability depends on "local microstates")
- Dynamics with fluctuations, time evolution of $\beta_\mu$ distribution

Crooks fluctuation theorem

$P(W)/P(-W)=e^{\Delta s}$



From talk
Gabriel Landi



Relates fluctuations, entropy in <u>small</u> <u>fluctuating</u> systems (Nano,proteins )

**P(W)** Probability system doing work in its usual thermal evolution

**P(-W)** Probability of the same system "running in reverse" and decreasing entropy due to a <u>thermal fluctuation</u>

$\Delta S$ Entropy produced by $P(W)$

**Valid** <u>far</u> from equilibrium, proven for non-Boltzmannian processes

How is Crooks theorem useful for what we did? Guarnieri et al, arXiv:1901.10428 (PRX) derive Thermodynamic uncertainity relations from

$$\hat{\rho}_{ness} \simeq \hat{\rho}_{les}(\lambda) e^{\hat{\Sigma}} \frac{Z_{les}}{Z_{ness}} \quad , \quad \hat{\rho}_{les} = \frac{1}{Z_{les}} \exp\left[ -\frac{\hat{H}}{T} \right]$$

$\hat{\rho}_{les}$ is Zubarev operator while $\Sigma$ is calculated with a Kubo-like formula

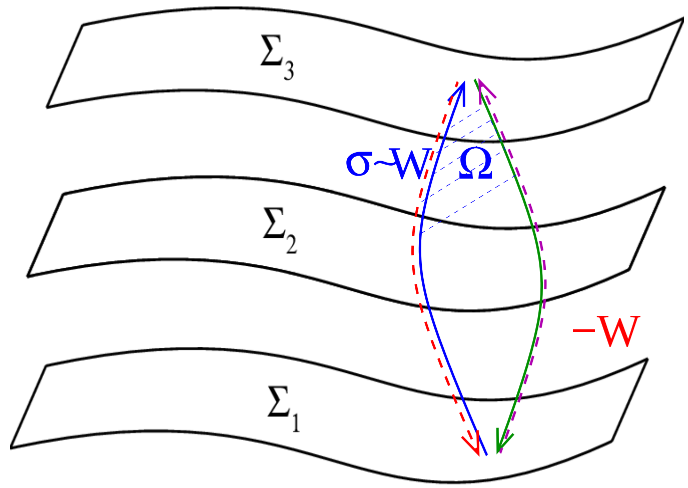
$$\hat{\Sigma} = \delta_\beta \Delta \hat{H}_+ \quad , \quad \hat{H}_+ = \lim_{\epsilon \to 0^+} \epsilon \int dt e^{\epsilon t} e^{-\hat{H} t} \Delta \hat{H} e^{\hat{H} t}$$

Relies on

$$\lim_{w \to 0} \left\langle \left[ \hat{\Sigma}, \hat{H} \right] \right\rangle \to 0 \equiv \lim_{t \to \infty} \left\langle \left[ \hat{\Sigma(t)}, \hat{H}(0) \right] \right\rangle \to 0$$

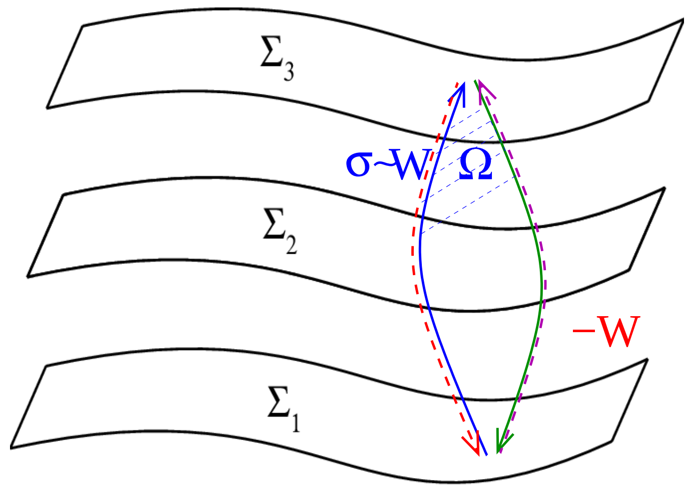
This "infinite" is "small" w.r.t. hydro gradients. $\equiv$ Markovian as in Hydro with $l_{mfp} \to \partial$ but with operators$\to$ carries all fluctuations with it!

## Applying Crooks theorem to Zubarev hydrodynamics: Stokes theorem




$$- \int_{\Sigma(\tau_0)} \mathrm{d}\Sigma_\mu \left( \widehat{T}^{\mu\nu} \beta_\nu \right) = - \int_{\Sigma(\tau')} \mathrm{d}\Sigma_\mu \left( \widehat{T}^{\mu\nu} \beta_\nu \right) + \int_\Omega \mathrm{d}\Omega \left( \widehat{T}^{\mu\nu} \nabla_\mu \beta_\nu \right),$$

true for “any” fluctuating configuration.



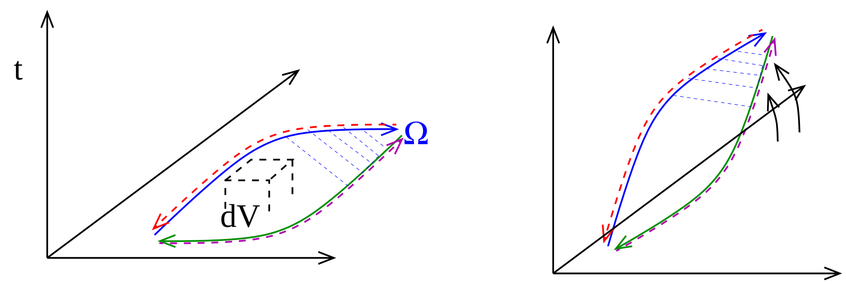

Let us now invert one foliation so it goes "backwards in time" assuming Crooks theorem means

$$\frac{\exp\left[-\int_{\sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}^{\mu\nu}\right]}{\exp\left[-\int_{-\sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}^{\mu\nu}\right]} = \exp\left[\frac{1}{2}\int_\Omega d\Omega_\mu^\mu \left[\frac{\hat{\Pi}^{\alpha\beta}}{T}\right]\partial_\beta \beta_\alpha\right]$$

Small loop limit $\left\langle \exp\left[\oint d\Sigma_\mu \omega^{\mu\nu}\beta^\alpha \hat{T}_{\alpha\nu}\right]\right\rangle = \left\langle \exp\left[\int \frac{1}{2} d\Sigma_\mu \beta^\mu \hat{\Pi}^{\alpha\beta}\partial_\alpha\beta_\beta\right]\right\rangle$

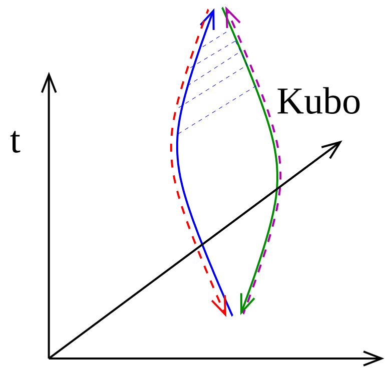
A non-perturbative operator equation,divergences cancel out...

$$\left.\frac{\hat{\Pi}^{\mu\nu}}{T}\right|_\sigma = \left(\frac{1}{\partial_\mu\beta_\nu}\right)\frac{\delta}{\delta\sigma}\left[\int_{\sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}^{\mu\nu} - \int_{-\sigma(\tau)} d\Sigma_\mu\beta_\nu\hat{T}^{\mu\nu}\right]$$




A sanity check: For a an equilibrium spacelike $d\Sigma_\mu = (dV, \vec{0})$ (left-panel) we recover Boltzmann's $\Pi^{\mu\nu} \Rightarrow \Delta S = \frac{dQ}{T} = \ln\left(\frac{N_1}{N_2}\right)$ , for an analytically continued "tilted" panel, Kubo's formula

## A sanity check




When $\eta \to 0$ *and* $s^{-1/3} \to 0$ (the first two terms in the hierarchy), Crooks fluctuation theorem gives $P(W) \to 1$ $P(-W) \to 0$ $\Delta S \to \infty$ so Crooks theorem reduces to $\delta$-functions of the entropy current

$$\delta \left( d\Sigma_\mu \left( s u^\mu \right) \right) \Rightarrow n^\mu \partial_\mu \left( s u^\mu \right) = 0$$

We therefore recover conservation equations for the entropy current, a.k.a. ideal hydro

## A numerical formulation

**Define a field** $\beta_\mu$ field and $n_\mu$

**Generate** an ensemble of

$$\ln Z|_{t+dt} = \int \mathcal{D}g_{\mu\nu}(x) T^{\mu\nu}|_{t+dt} \quad , \quad \beta_\mu|_{t+dt} = \frac{\delta \ln Z|_{t+dt}}{\delta T_{\mu\nu}} n_\nu$$

According to a Metropolis algorithm ran via Crooks theorem

**Reconstruct** the new $\beta$ and $\Pi_{\mu\nu}$ . The Ward identity will make sure $\beta_\mu \beta^\mu = -1/T^2$

**Computationally intensive** (an ensemble at every timestep), but who knows?

## Conclusions

- Linking hydrodynamics to statistical mechanics is still an open problem
  Only top-down models (Boltzmann,AdS/CFT) rather than bottom-up theory
  Is hydro universal? what are its limits of applicability? still open question

  The observation of hydro-like behavior in small systems liable to fluctuations makes this explicit!

- Crooks fluctuation theorem could provide such a link!

- redundances play crucial role in fluctuations, could mean small systems achieve "thermalization" quicker! inverse attractor!

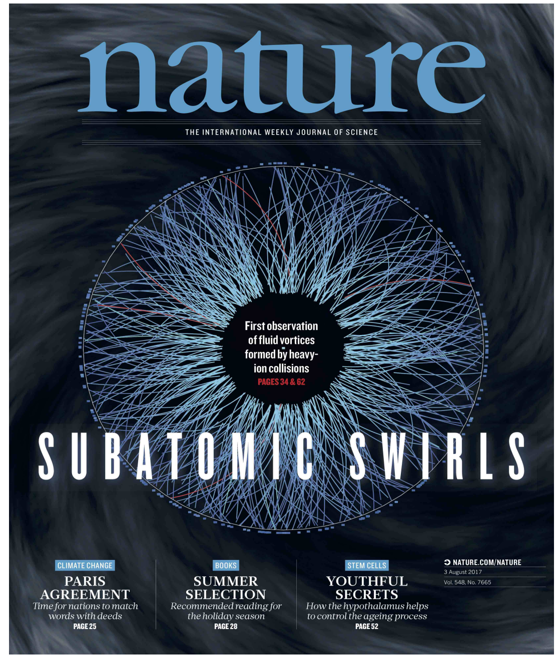
# SPARE SLIDES

## PS: transfer of micro to macro DoFs experimentally proven!

STAR collaboration
1701.06657

NATURE
August 2017

Polarization by vorticity in heavy ion collisions




Could give new talk about this, but will mention hydro with spin not developed and a lot of conceptual debates Pseudo-gauge dependence if both spin and angular momentum present in fluid? Gauge symmetry "ghosts"? GT,1810.12468 (EPJA) . redundances?

STAR collaboration

1701.06657

NATURE

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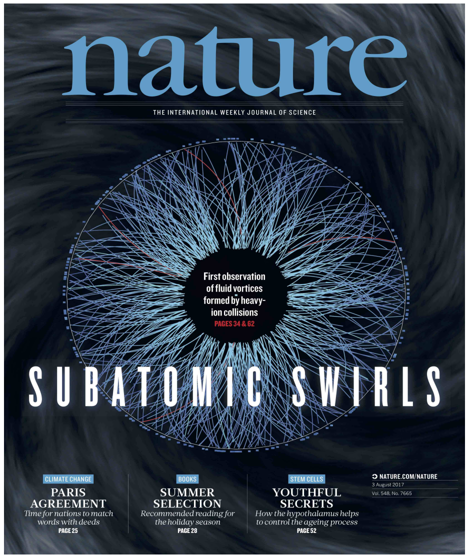
Polarization by vorticity in heavy ion collisions




Pseudo-gauge symmetries physical interpretation: T.Brauner, 1910.12224

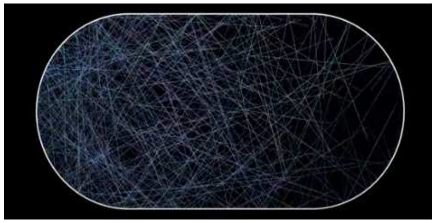
$$x^\mu \to x^\mu + \epsilon \zeta^\mu(x) \quad , \quad \psi_a \to \psi_a + \epsilon \psi'_a \to \mathcal{L} \to \mathcal{L}$$

$\ln \mathcal{Z}$ Invariant, but $\langle O \rangle$ generally is not. Spin $\leftrightarrow$ fluctuation, need equivalent of DSE equations! $D\langle O \rangle = 0 \to D\langle O \rangle = \langle O_I O_J \rangle$

**Statistical mechanics:** This is a system in global equilibrium, described by a partition function $Z(T, V, \mu)$ , whose derivatives give expectation values $\langle E \rangle$ ,fluctuations $\langle (\Delta E)^2 \rangle$ etc. in terms of conserved charges. All microstates equally likely, which leads to preferred macrostates!

**Fluid dynamics:** This is the state of a field in local equilibrium which can be perturbed in an infinity of ways. The perturbations will then interact and dissipate according to the Euler/N-S equations. many issues connecting to Stat.Mech. Wild weak solutions, millenium problem!

## The problem with general "transport thinking"

 $=$  ?

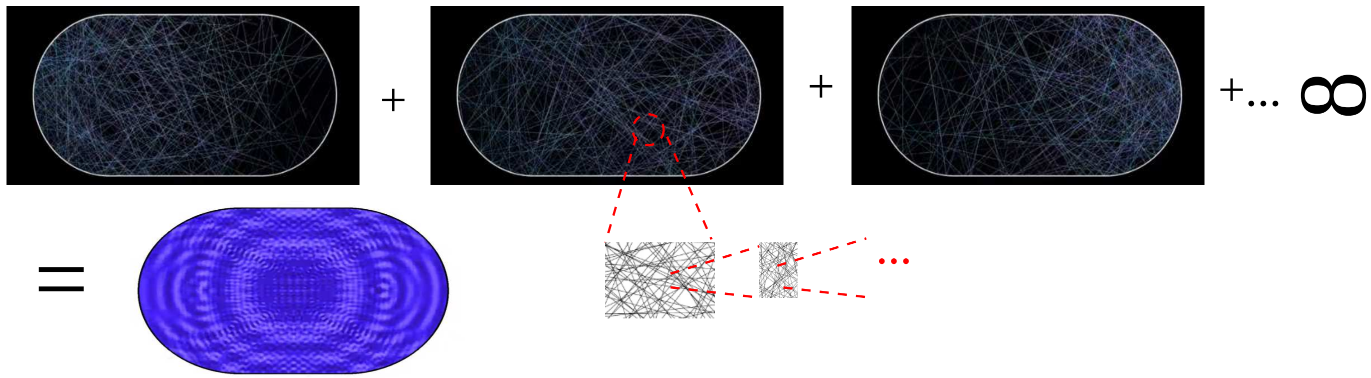
Let's solve the simplest transport equation possible: Free particles

$$\frac{p^\mu}{m}\partial_\mu f(x,p) = 0 \rightarrow f(x,p) = f\left(x_0 + \frac{p}{m}t, p\right)$$

<u>obvious</u> solution is just to propagate
What is <u>weird</u> is that "hydro-like" solution possible too (eg vortices)!

$$f(x,p) \sim \exp\left[-\beta_\mu p^\mu\right] \quad , \quad \partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$$

But obviously unphysical, no force! What's up?




This paradox is resolved by remembering that $f(x,p)$ is defined in an ensemble average limit where the number of particles is not just "large" but uncountable . curvature from continuity!

BUt this suggests Boltzmann equation disconnected from any finite number of particles!

What if $e^{-\beta_\mu p^\mu}$ used to sample strongly coupled particles in "many finite events"? Thermal fluctuations,Vlasov correlations and Boltzmann scattering "mix these words". Many ways to mix,some wrong! What is appropriate?

How "different events" correlated is crucial
Villani , `https://www.youtube.com/watch?v=ZRPT1Hzze44`

**Vlasov equation** contains all classical correlations. Relativistically numer of particles varies in each event but "evolves" deterministically. but instability-ridden, "filaments", cascade in scales.
$N_{DOF} \to \infty$ invalidates KAM theorem stability

**Boltzmann equation** "Semi-Classical UV-completion" ov Vlasov equation, first term in BBGK hyerarchy, written in terms of Wigner functions.

Infinitely unstable jerks on infinitely small scales Random scattering Statistical behavior emerges from both instabilities (chaos, Poncaire cycles) and scattering (H-theorem) but interplay non-trivial. Strong coupling away from molecular chaos not understood!

There is more to hydro than the Knudsen number

Power counting:

B.Betz,D.Henkel,D.Risc 0812.1440

3 length scales: ~~2 microscopic,~~ 1 macroscopic

- thermal wavelength $\lambda_{\rm th} \sim \beta \equiv 1/T$
- mean free path $\ell_{\rm mfp} \sim (\langle\sigma\rangle n)^{-1}$

What if these are ~?

$\langle\sigma\rangle$ averaged cross section, $n \sim T^3 = \beta^{-3} \sim \lambda_{\rm th}^{-3}$

- length scale over which macroscopic fluid fields vary $L_{\rm hydro}$ , $\partial_\mu \sim L_{\rm hydro}^{-1}$

Note: since $\eta \sim (\langle\sigma\rangle\lambda_{\rm th})^{-1}$ $\Longrightarrow$

$$\boxed{\frac{\ell_{\rm mfp}}{\lambda_{\rm th}} \sim \frac{1}{\langle\sigma\rangle n}\frac{1}{\lambda_{\rm th}} \sim \frac{\lambda_{\rm th}^3}{\langle\sigma\rangle\lambda_{\rm th}} \sim \frac{\lambda_{\rm th}^3}{\langle\sigma\rangle\lambda_{\rm th}} \sim \frac{\eta}{s}}$$

$s$ entropy density, $s \sim n \sim T^3 = \beta^{-3} \sim \lambda_{\rm th}^{-3}$

$$\underbrace{l_{micro}}_{\sim s^{-1/3}, n^{-1/3}} \ll \underbrace{l_{mfp}}_{\sim \eta/(sT)} \ll L_{macro}$$

Second inequality was developed so far, but first is suspect! experimentally

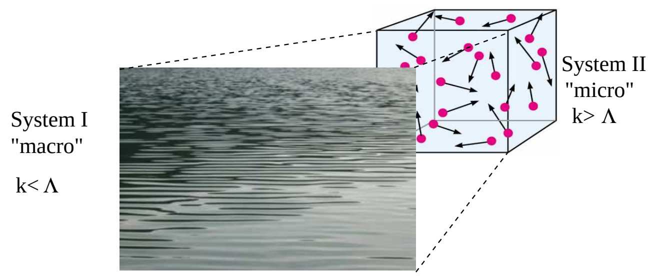
$$\underbrace{l_{micro}}_{\sim s^{-1/3}, n^{-1/3}} \ll \underbrace{l_{mfp}}_{\sim \eta/(sT)} \ll L_{macro}$$

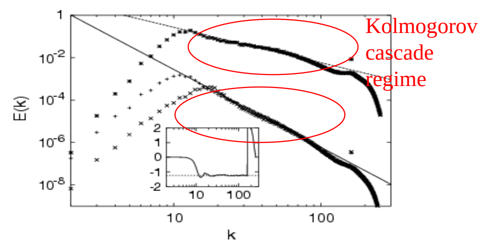
Weakly coupled: Ensemble averaging in Boltzmann equation good up to $\mathcal{O}\left((1/\rho)^{1/3}\partial_\mu f(...)\right)$
Strongly coupled: classical supergravity requires $\lambda \gg 1$ but $\lambda N_c^{-1} = g_{YM} \ll 1$ so

$$\frac{1}{T N_c^{2/3}} \ll \frac{\eta}{sT} \quad \left( \quad or \quad \frac{1}{\sqrt{\lambda} T}\right) \ll L_{macro}$$

QGP: $N_c = 3 \ll \infty$ ,so $l_{micro} \sim \frac{\eta}{sT}$ . Cold atoms: $l_{micro} \sim n^{-1/3} > \frac{\eta}{sT}$ ?

Why is $l_{micro} \ll l_{mfp}$ necessary? microscopic fluctuations (which have nothing to do with viscosity ) will drive fluid evolution. $\Delta\rho/\rho \sim C_V^{-1} \sim N_c^{-2}$



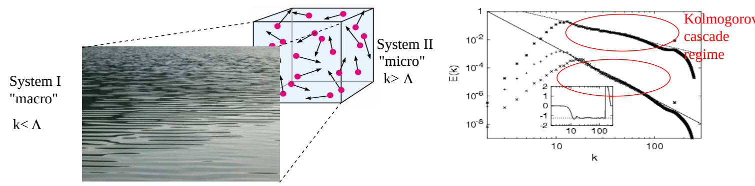




A classical low-viscosity fluid is turbulent. Typically, low-k modes cascade into higher and higher k modes In a non-relativistic incompressible fluid

$$\eta/(sT) \ll L_{eddy} \ll L_{boundary} \quad , \quad E(k) \sim \left(\frac{dE}{dt}\right)^{2/3} k^{-5/3}$$

For a classical ideal fluid, no limit! since $\lim_{\delta\rho\to 0, k\to\infty} \delta E(k) \sim \delta\rho k c_s \to 0$ but quantum $E \geq k$ so energy conservation has to cap cascade.

More fundamentally: take stationary slab of fluid at local equilibrium.

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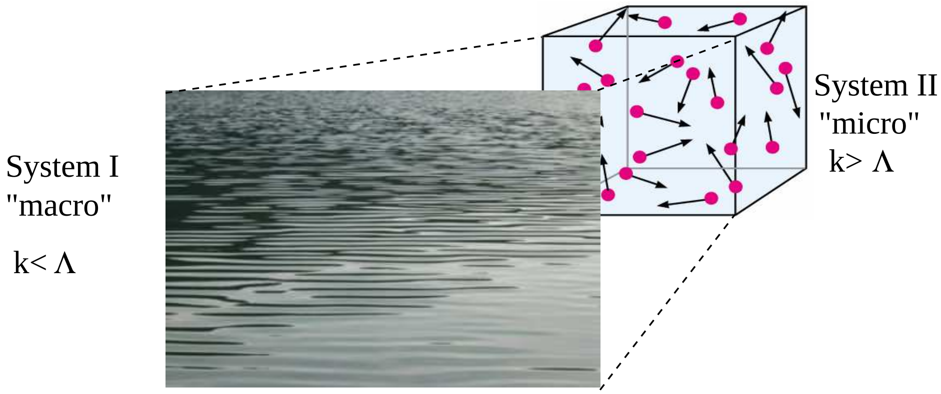
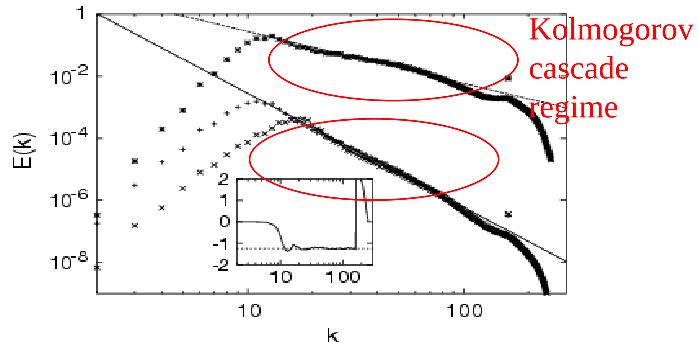
**Fluid dynamics:** This is the state of a field in local equilibrium which can be perturbed in an infinity of ways. The perturbations will then interact and dissipate according to the Euler/N-S equations. Smaller $\eta/s$ , the closer to local equilibrium (SM applies to cell) but the longer the timescale to global equilibrium (SM applies to system).




- Provided state is localized, local equilibrium is "global equilibrium in every cell", global equilibrium with spin, forces "non-local" A.Palermo et al,2007.08249,2106.08340 "global" equilibrium not necessarily stable against hydro perturbations I think "real" global equilibrium built up from local equilibria

- Dissipation scale in local equilibrium $\eta/(Ts)$ , global equilibration timescale $(Ts)/\eta$ .turbulence drastically changes this ,but "when does a small perturbation become a microstate?"

## Some insight from maths

Millenium problem: existence and smoothness of the Navier-Stokes equations

Important tool are "weak solutions" , similar to what we call "coarse-graining".

$$F\left(\frac{d}{dx}, f(x)\right) = 0 \Rightarrow F\left(\int \frac{d}{dx}\phi(x)..., f(x)\right) = 0$$

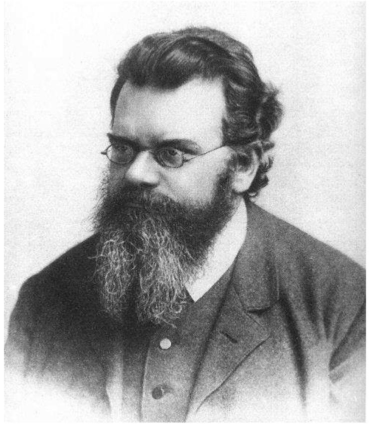
$\phi(x)$ "test function", similar to coarse-graining!

Existance of Wild/Nightmare solutions and non-uniqueness of weak solutions shows this tension is non-trivial, coarse-graining “dangerous”

I am a physicist so I care little about the ”existence of ethernal solutions” to an approximate equation, Turbulent regime and microscopic local equilibria need to be consistent

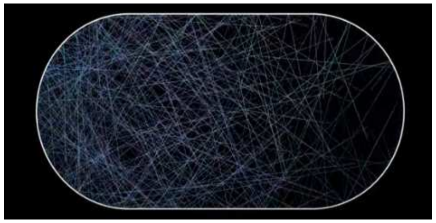
Thermal fluctuations could both ”stabilize” hydrodynamics and ”accellerate” local thermalization
But where do microstates,”local” microstates fit here?

# the battle of the entropies



Boltzmann entropy is usually a property of the "DoF", and is "kinetic" subject to the H-theorem which is really a consequence of the not-so-justified molecular chaos assumption. Gibbsian entropy is the log of the area of phase space, and is justified from coarse-graining and ergodicity , but hard to define it in non-equilibrium . The two are different even in equilibrium, with interactions! Note, Von Neumann $\langle ln\hat{\rho}\rangle$ Gibbsian

## The problem with general "transport thinking"

 =  ?

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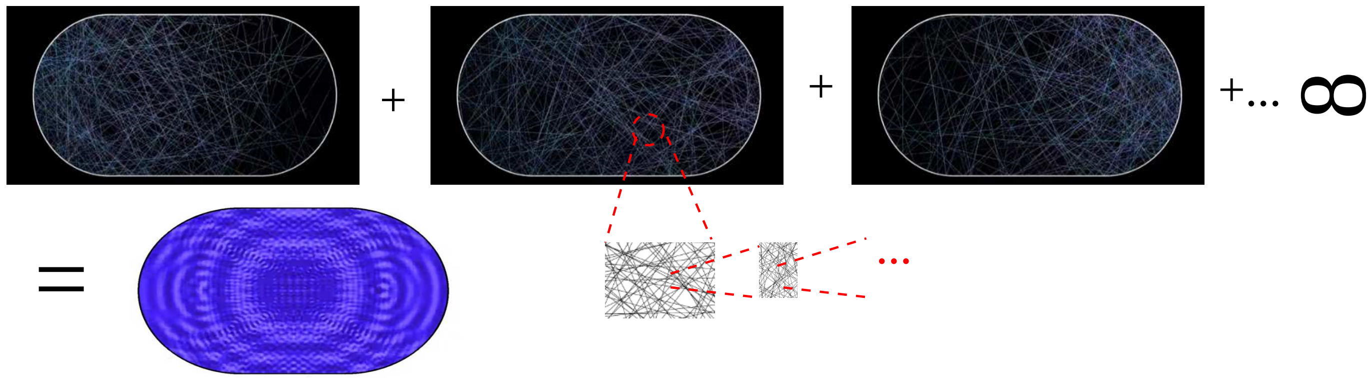


This paradox is resolved by remembering that $f(x,p)$ is defined in an ensemble average limit where the number of particles is not just "large" but uncountable . curvature from continuity!

BUt this suggests Boltzmann equation disconnected from any finite number of particles! e.g. $v_n\{M \gg 1\}$? Vorticity/polariation link?

What if $e^{-\beta_\mu p^\mu}$ used to sample strongly coupled particles in "many finite events"? Thermal fluctuations,Vlasov correlations and Boltzmann scattering "mix these words". Many ways to mix,some wrong! What is appropriate?

## Connection to transport

Villani , https://www.youtube.com/watch?v=ZRPT1Hzze44

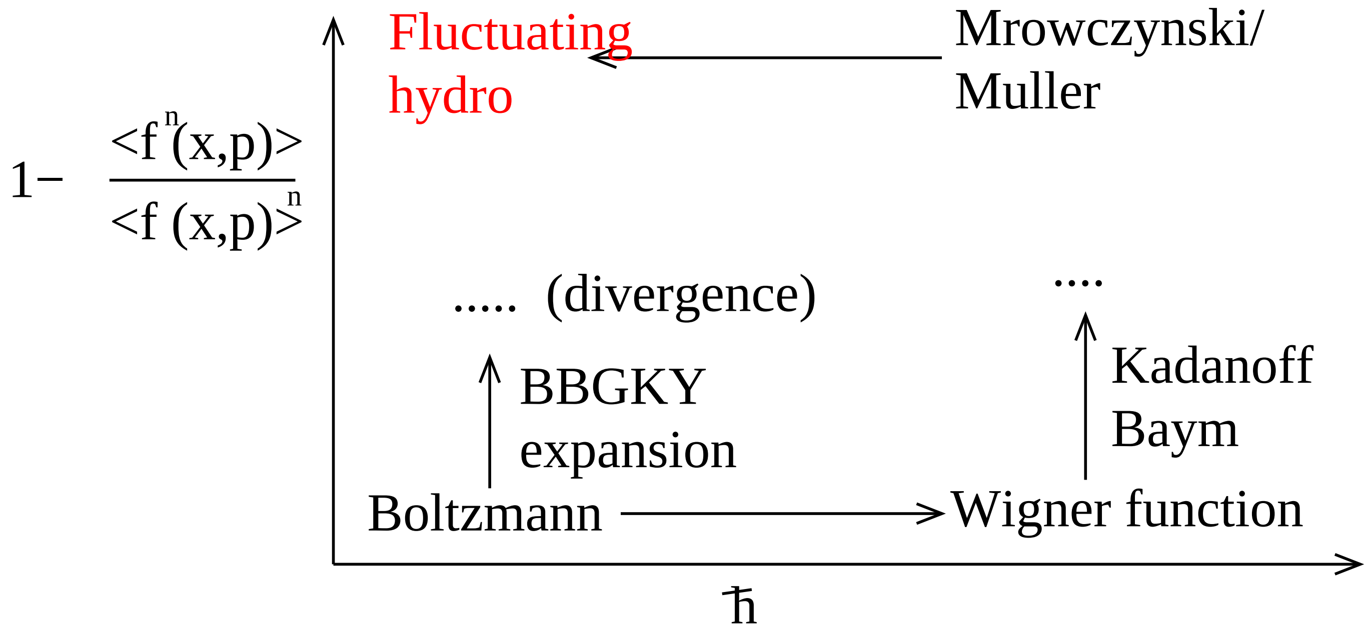
**Vlasov equation** contains all classical correlations, instability-ridden, "filaments", cascade in scales.
$N_{DOF} \to \infty$ invalidates KAM theorem stability. High T: Debye screening kills potentials

**Boltzmann equation** "Semi-Classical UV-completion" of Vlasov equation, first term in BBGK hyerarchy, written in terms of Wigner functions. pQCD@high T Boltzmann of quasi-particles,potentials "absorbed". AMY,Giglieri,.. first order, convergence unknown

Infinitely unstable jerks on infinitely small scales Random scattering .

But if number of particles $N \ll \infty$ Correlations important! .

## Boltzmann equation,BBGKY and limits



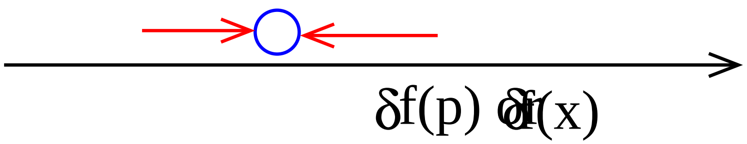

Boltzmann equation emerges as a double limit from microscopic correlations, $\hbar \to 0$ Relaxing the latter limit would destroy statistical independence CHSH relations , so probably not relevant (phases "chaotic"). But fluctuating hydro "non-perturbative" in correlations

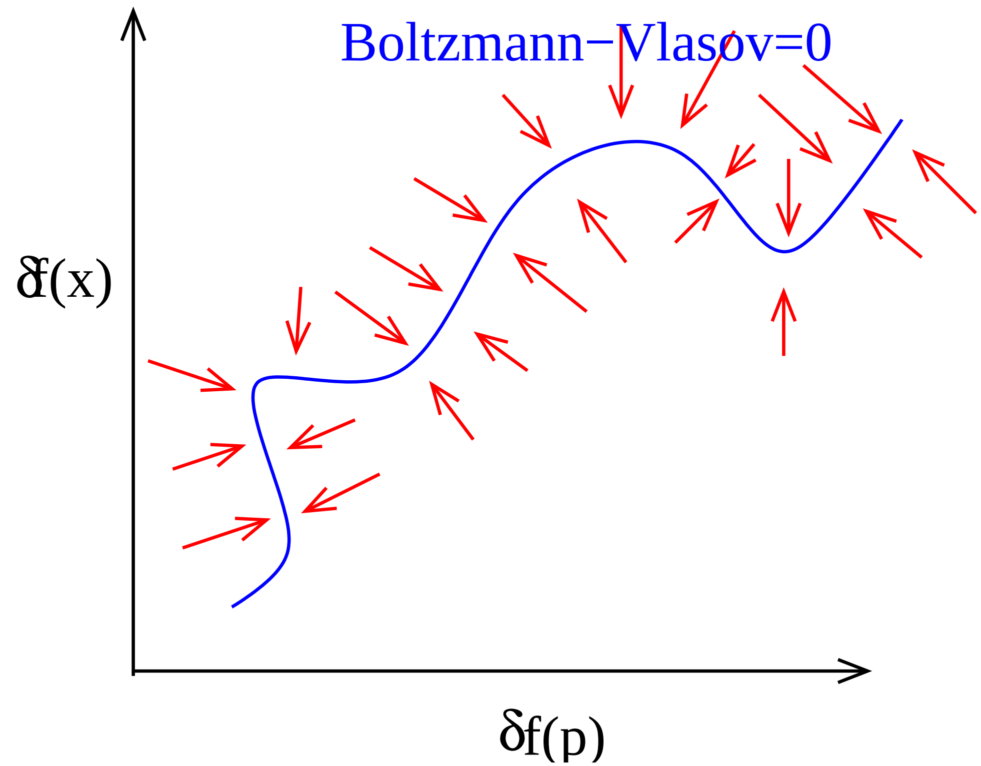
Finite number of particles: $f(x,p)$ not a function but a functional ($\mathcal{F}(f(x,p)) \underbrace{\rightarrow}_{Boltzmann} \delta\left(f' - f(x,p)\right)$ ), incorporating continuum of functions and all correlations. Perhaps solvable!

$$\frac{p^\mu}{\Lambda}\frac{\partial}{\partial x^\mu} f(x,p) = \left\langle \underbrace{\hat{C}[\tilde{W}(\tilde{f}_1,\tilde{f}_2)] - g\frac{p^\mu}{\Lambda}\hat{F}^{\mu\nu}[\tilde{f}_1,\tilde{f}_2]\frac{\delta}{\delta\tilde{f}_{1,2}}\tilde{W}\left(\tilde{f}_1,\tilde{f}_2\right)}_{How\quad many\quad A-B=0?} \right\rangle$$

Wigner functional to $\mathcal{O}\left(h^0\right)$ . What is the effect? If only Boltzmann term not much!

If Both Vlasov and Boltzmann terms, redundancy-ridden!




One can deform $f(x,p)$ by $\delta f(x)$ or $\delta f(p)$ so that $\hat{C} - \hat{W}$ cancels. In ensemble average deformation makes no sense, but away from it it does!

$$f(x,p) \to f'(x,p) \quad , \quad \underbrace{\hat{C}\left(f(x,p), f'(x,p)\right)}_{lim_{f\to f'} \sim \partial f/\partial x} = \underbrace{\hat{V}^{\mu}\left(f(x,p), f'(x,p)\right)\frac{\partial f}{\partial p_{\mu}}}_{lim_{f\to f'} \sim \partial f/\partial p}$$

Infinite number of redundancies! Close to local equilibrium limit...

$$\left\{ \begin{array}{c} f(x,p) \\ f'(x,p) \end{array} \right\} \sim \exp\left[ - \left\{ \begin{array}{c} \beta_{\mu}(x,t) \\ \beta'_{\mu}(x,t) \end{array} \right\} p^{\mu} \right] \quad , \quad \lim_{f\to f'} \left\{ \begin{array}{c} \hat{C}[f,f'] \\ \hat{V}[f,f'] \end{array} \right\} \sim \left\{ \begin{array}{c} \langle \partial\beta \rangle \\ \langle \beta^2 \rangle \end{array} \right\}$$

and these redundances look like the hydro ones