The Equation of State from Lattice QCD

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28th November 2022

BERGISCHE UNIVERSITÄT WUPPERTAL

WB collaboration

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The QCD Lagrangian

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + \overline{\psi} \left(\mathrm{i}\gamma_\mu D^\mu - m \right) \psi$$

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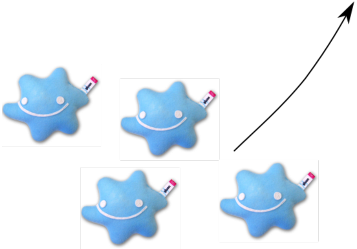

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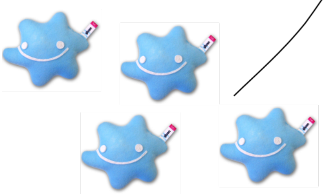

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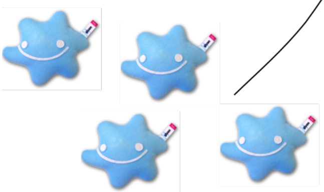
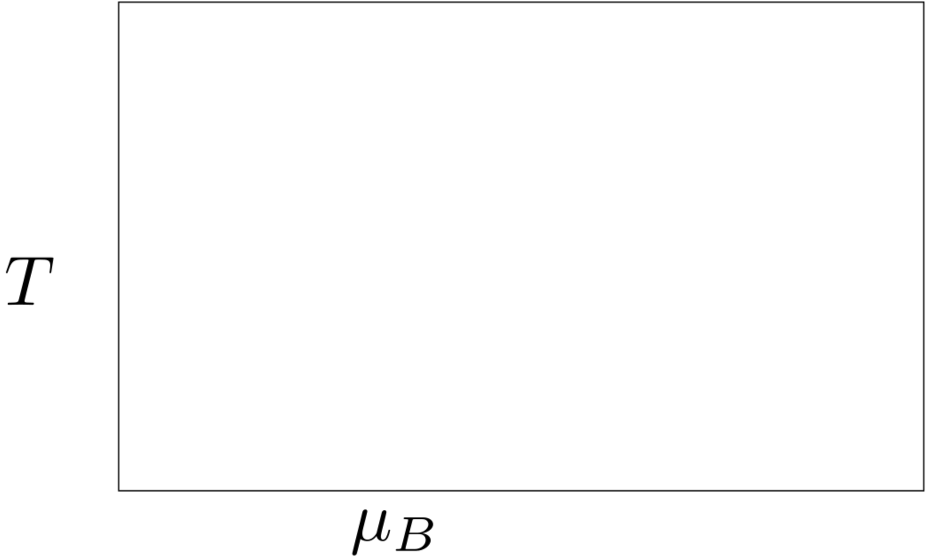


The (T, μ_B)-phase diagram of QCD



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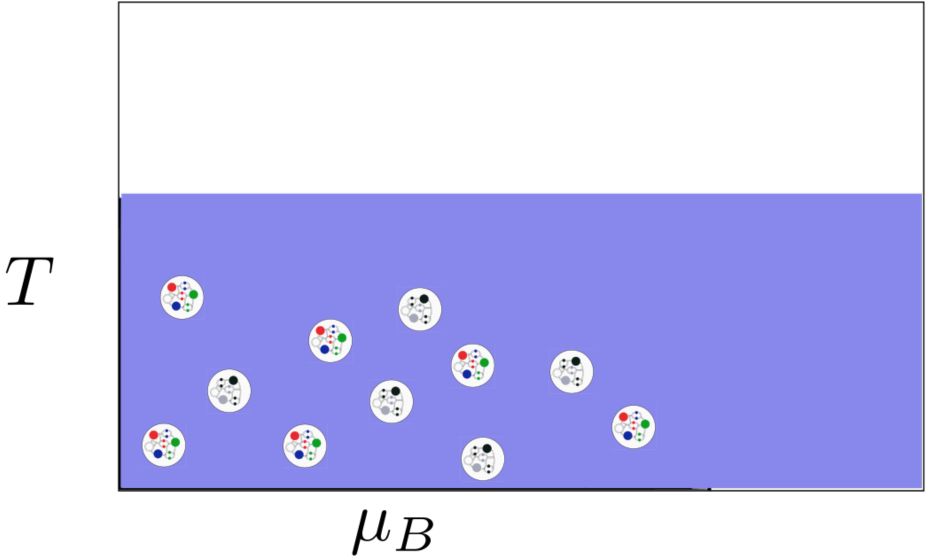

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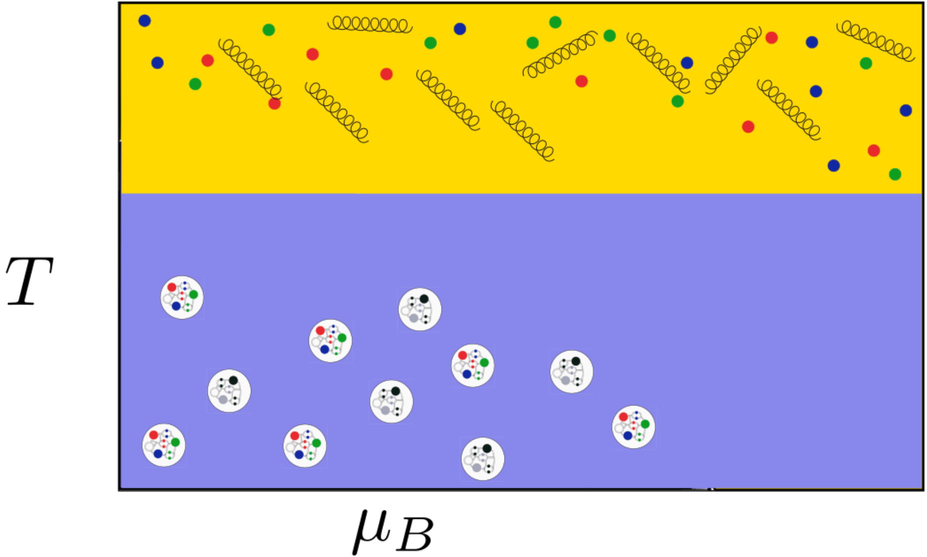

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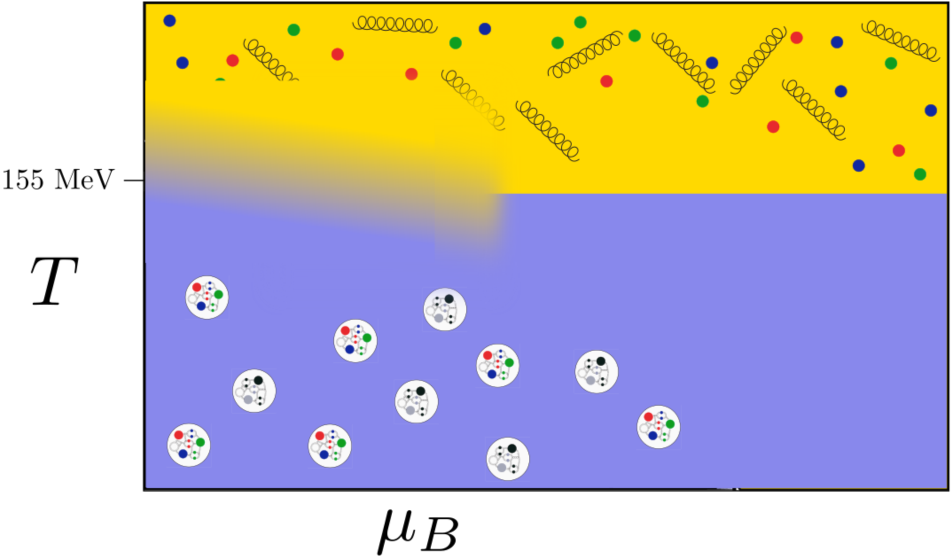

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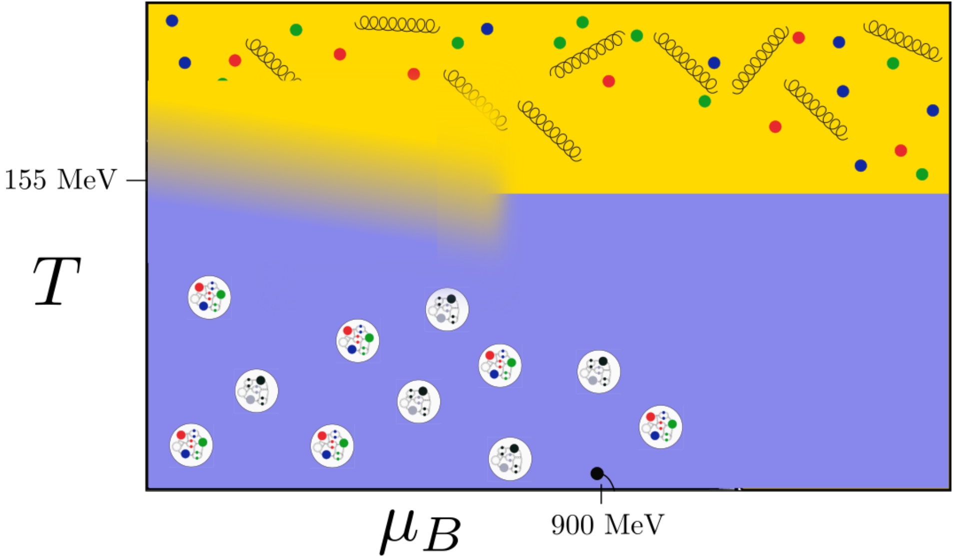

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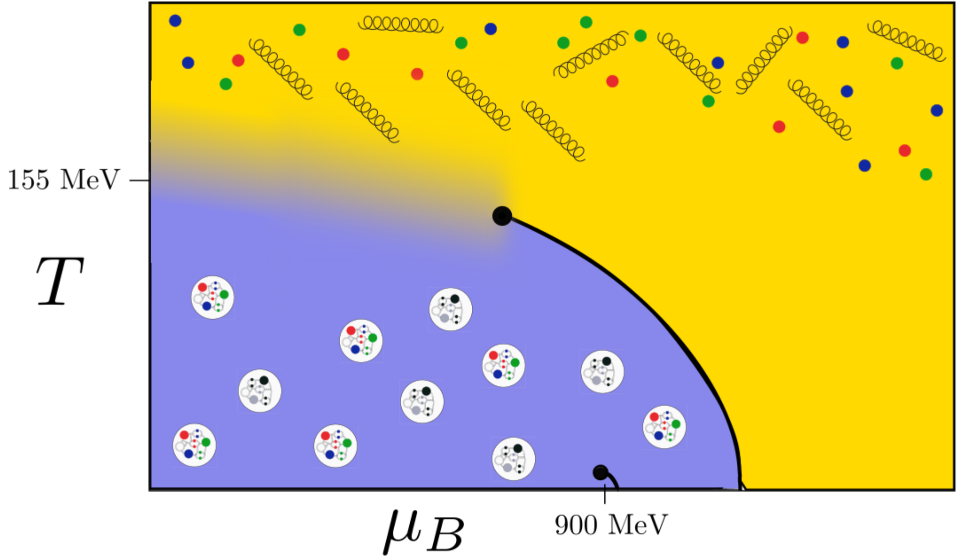

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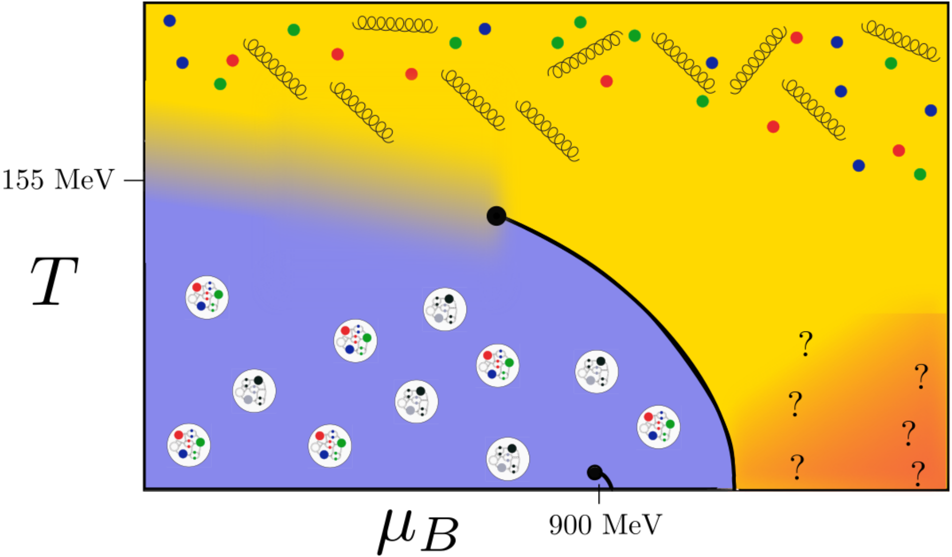
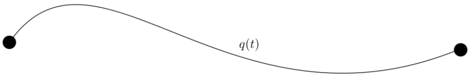


1. Lattice QCD

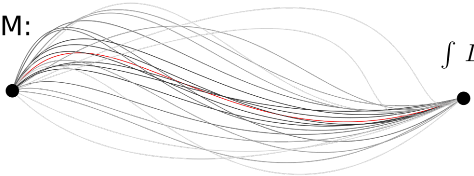
2. Equation of state
   - Rescaling and expansion - the analysis in [Borsanyi:2022qlh]
   - Results at $n_S = 0$ and $\mu_Q = 0$
   - Beyond strangeness neutrality

The path integral quantization: from M to QM to QFT

Mechanics:



QM:



$$\int Dq(t) e^{\mathrm{i}\int \mathrm{d}t\, L}$$

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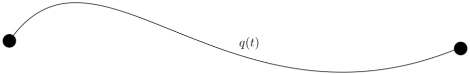

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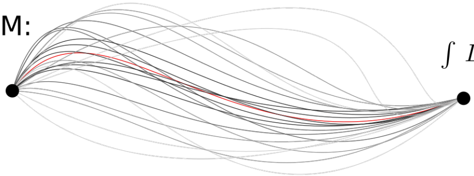

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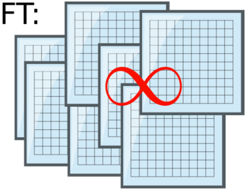
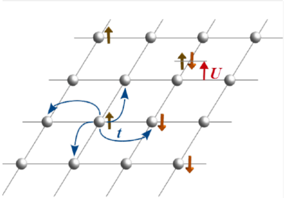
QFT:

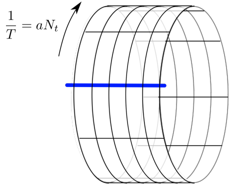


$$\int \mathcal{D}\phi(x) e^{\mathrm{i}\int \mathrm{d}^4x \mathcal{L}}$$

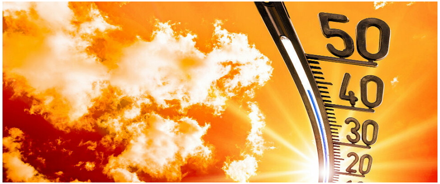


Lattice Simulation and Statistical mechanics

Zero-Temperature-LQCD:



Finite-Temperature-LQCD:

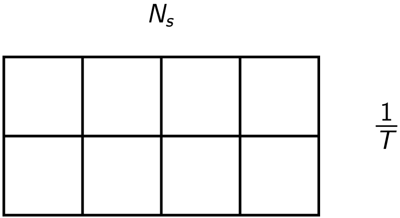


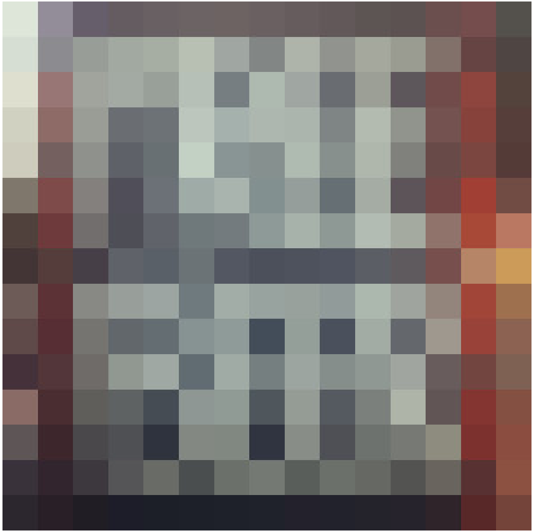
$$\int \mathcal{D}\phi(x) e^{\mathrm{i}\int \mathrm{d}^4 x \mathcal{L}} \xrightarrow{t \to \mathrm{i}\tau} \int \mathcal{D}\phi(x) e^{-S} \xrightarrow{\text{periodic boundary in a finite time}} \int \mathcal{D}\phi(x) e^{-\int_0^\beta \mathrm{d}t \int \mathrm{d}^3 x \mathcal{L}} = Z$$

Infinite space time volume of a QFT in Euclidean space time

Partition function of a grand canonical ensemble at finite temperature

The continuum limit





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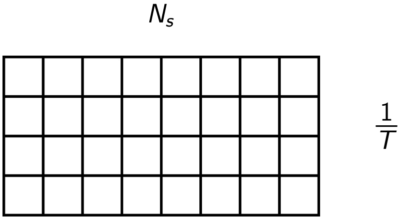

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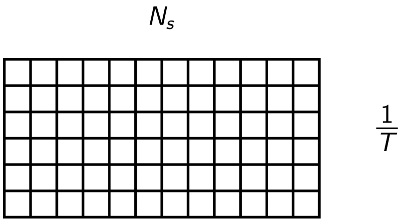

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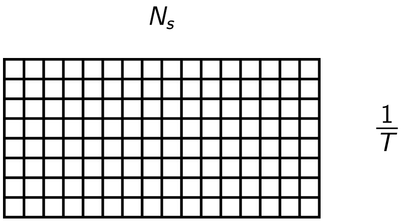

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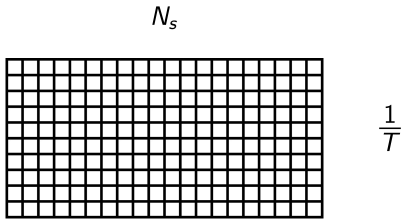

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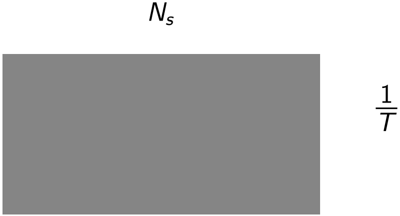
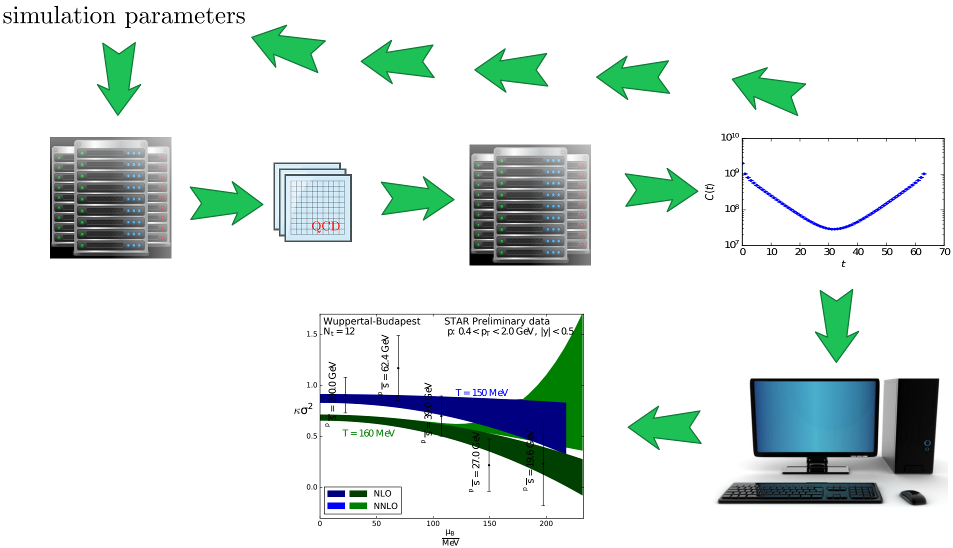


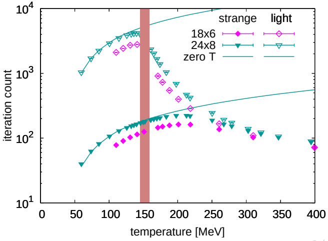
The work flow



Why aren't we finished yet?

- Simulations take a lot of computer time
- Not everything can be calculated directly. For example:
  - Only thermal equilibrium
  - Only simulations at $\mu_B = 0 \Rightarrow \langle n_B \rangle = 0$
    heavy ion collision experiments

1000 configurations on a $64^3 \times 16$ lattice cost about 1 million core hours



The sign problem

The QCD partition function:

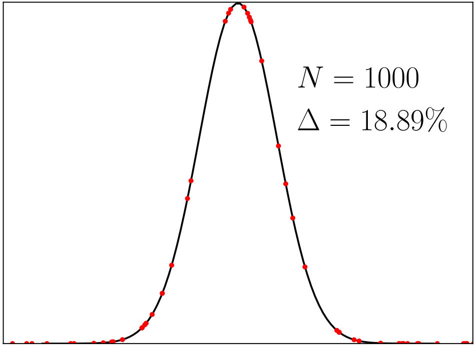
$$Z(V,T,\mu) = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar{\psi}\ e^{-S_F(U,\psi,\bar{\psi})-\beta S_G(U)}$$
$$= \int \mathcal{D}U\ \det M(U) e^{-\beta S_G(U)}$$

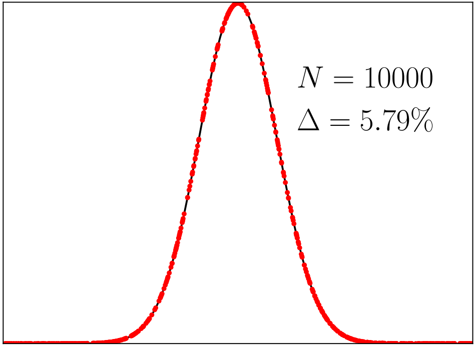
- For Monte Carlo simulations $\det M(U) e^{-\beta S_G(U)}$ is interpreted as Boltzmann weight
- If there is particle-antiparticle-symmetry $\det M(U)$ is real
- If $\mu^2 > 0$ $\det M(U)$ is complex

The sign problem

$$\int_{-\infty}^{\infty} \mathrm{d}x\ (100 - x^2)\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \approx \int_{-100}^{100} \mathrm{d}x\ (100 - x^2)\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} = \sum_{i=1}^{N}(100 - x_i^2)\frac{e^{-\frac{1}{2}x_i^2}}{\sqrt{2\pi}} \cdot \frac{200}{N}$$

The x_i are drawn from a uniform distribution in the interval $[-100, 100]$

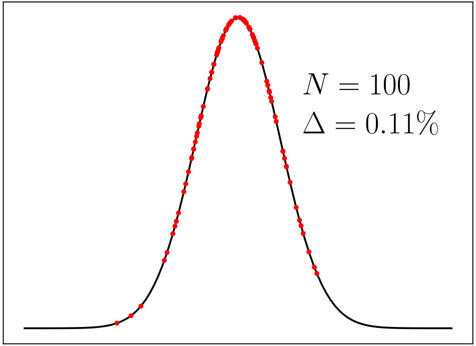


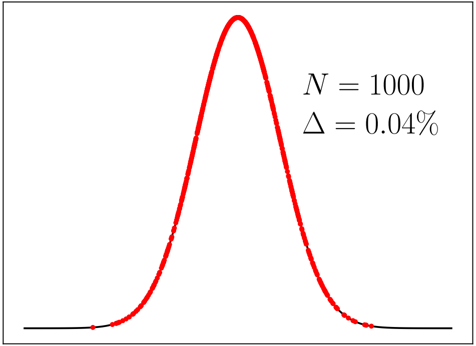


Importance sampling

$$\int_{-\infty}^{\infty} \mathrm{d}x\ (100 - x^2)\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} = \sum_{i=1}^{N}(100 - x_i^2) \cdot \frac{1}{N}$$

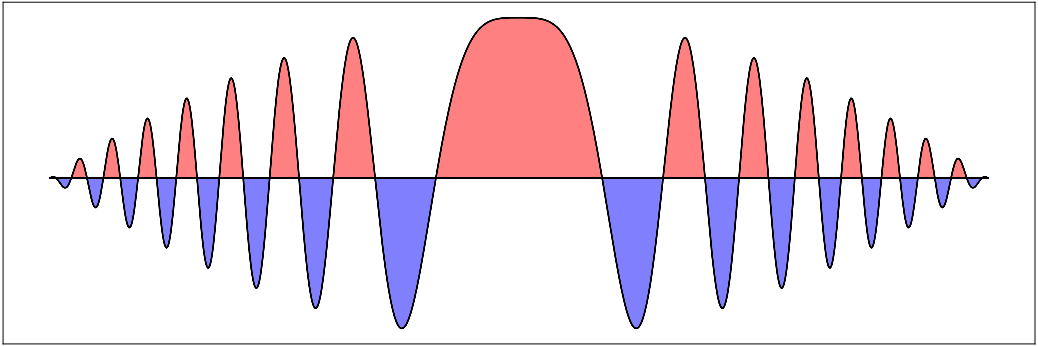
The x_i are drawn from a normal distribution





The sign problem

$$\int_{-\infty}^{\infty} \mathrm{d}x \, (100 - x^2) \frac{e^{-\frac{\mathrm{i}}{2} x^2}}{\sqrt{2\pi}}$$



Dealing with the sign problem

- (Sign) Reweighting techniques
- Canonical ensemble
- Lefshetz Thimble
- Density of state methods
- Dual variables
- Complex Langevin
- ...

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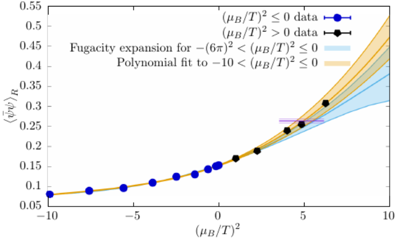

[Borsanyi:2021hbk]

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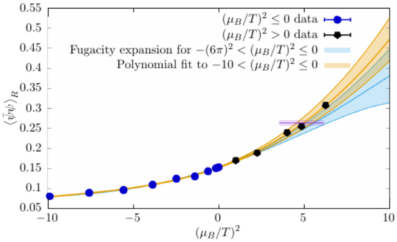
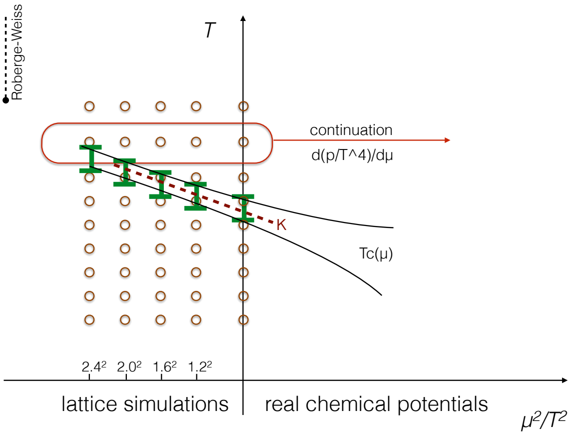


[Borsanyi:2021hbk]

- (Taylor) expansion
- Imaginary μ

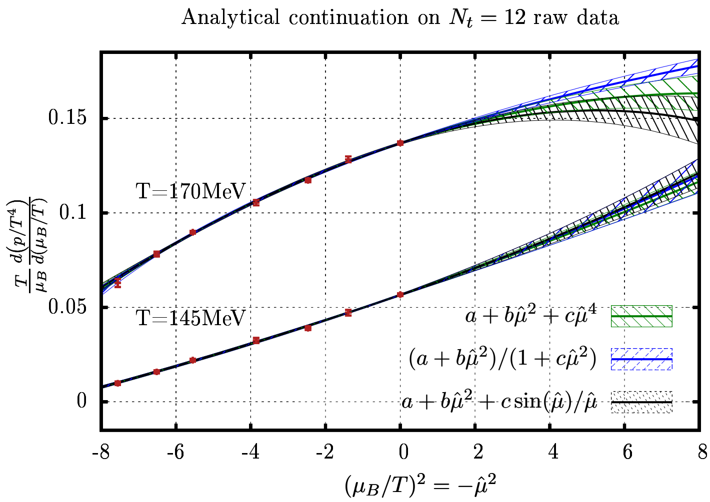
Analytic continuation from imaginary chemical potential



Common technique:

- [deForcrand:2002hgr]
- [Bonati:2015bha]
- [Cea:2015cya]
- [DElia:2016jqh]
- [Bonati:2018nut]
- [Borsanyi:2018grb]
- [Borsanyi:2020fev]
- [Bellwied:2021nrt]
- . . .

Different functions



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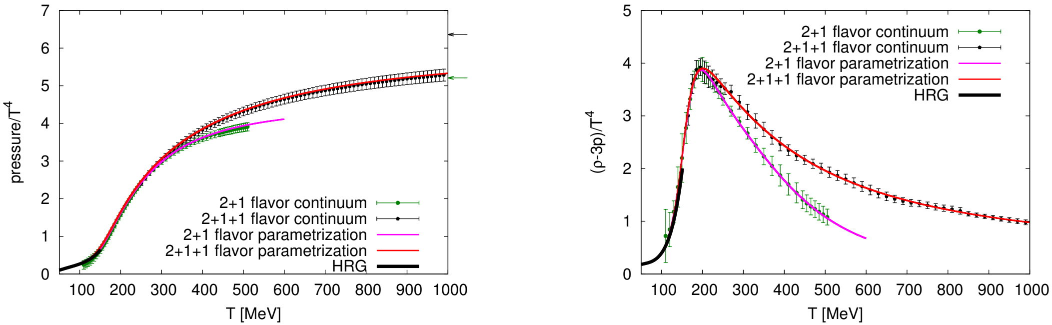
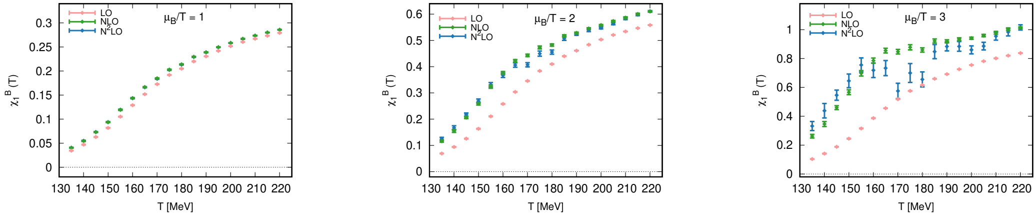
$\mu_B = 0$ and high T: Influence of the charm quark



[Borsanyi:2016ksw]

Trouble with the equation of state



[Borsanyi:2021sxv], [Borsanyi:2018grb], $N_t = 12$

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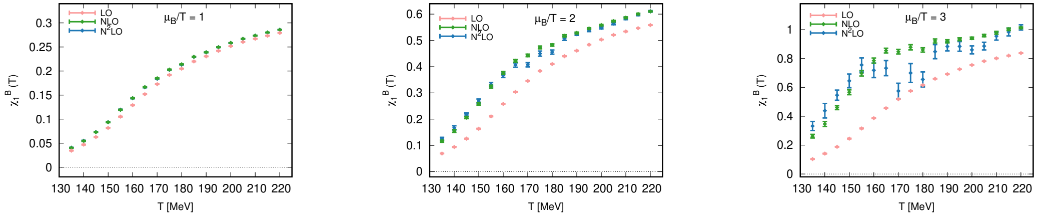

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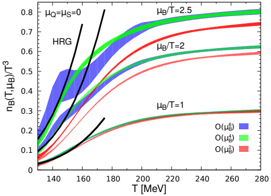

Taylor method

[Bazavov:2017dus]

[Bollweg:2022rps]

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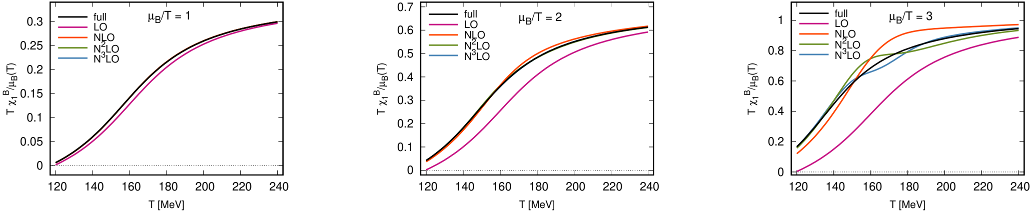

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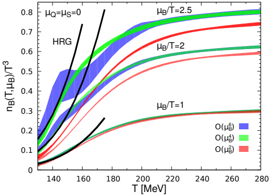

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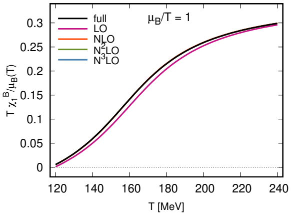
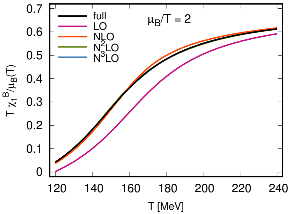
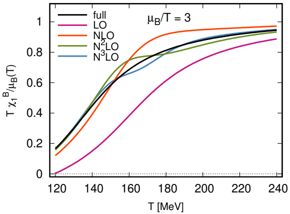
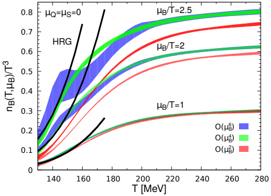






[Borsanyi:2021sxv]



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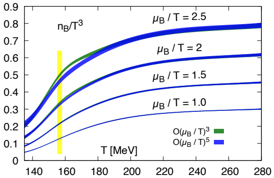
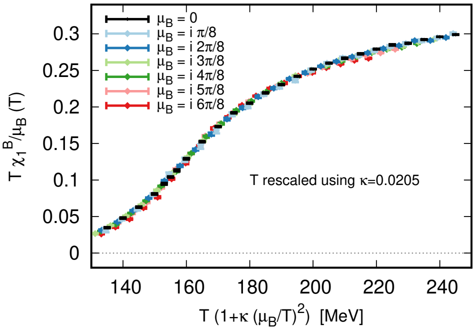
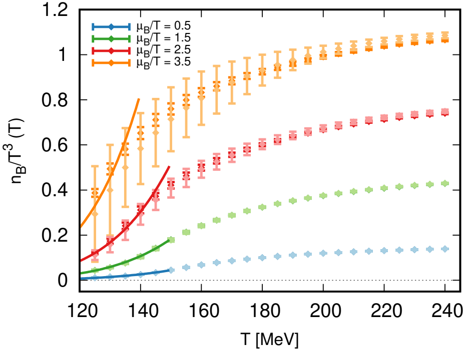


Results at $\mu_S = 0$

Find a different extrapolation scheme for extrapolating to higher μ_B.





- [Borsanyi:2021sxv]
- $N_t = 10, 12, 16$

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Strangeness Neutrality

Enforcing the conditions $\mu_Q = 0$ and $\chi_1^S = 0$:

$$\frac{\mathrm{d}\mu_S}{\mathrm{d}\mu_B} = -\frac{\chi_{11}^{BS}}{\chi_2^S}.$$

On this line, total derivatives with respect to the baryochemical potential read

$$\frac{\mathrm{d}}{\mathrm{d}\hat{\mu}_B} = \frac{\partial}{\partial\hat{\mu}_B} + \frac{\mathrm{d}\hat{\mu}_S}{\mathrm{d}\hat{\mu}_B}\frac{\partial}{\partial\hat{\mu}_S} = \frac{\partial}{\partial\hat{\mu}_B} - \frac{\chi_{11}^{BS}}{\chi_2^S}\frac{\partial}{\partial\hat{\mu}_S}.$$

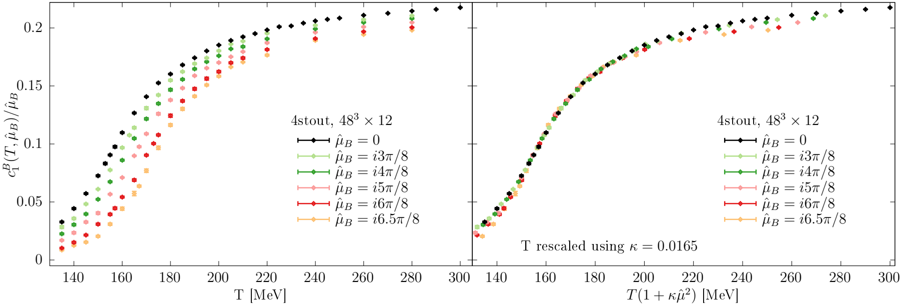
For the pressure we get:

$$c_n^B(T,\hat{\mu}_B) \equiv \left.\frac{\mathrm{d}^n\hat{p}(T,\hat{\mu}_B)}{\mathrm{d}\hat{\mu}_B^n}\right|_{\substack{\mu_Q=0\\ \chi_1^S=0}}.$$

The net baryon density is given by:

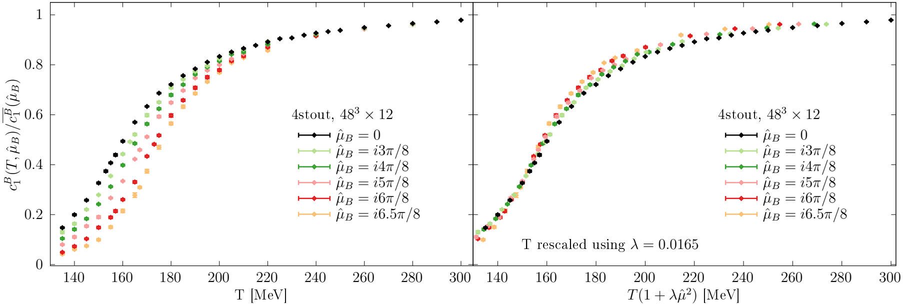
$$c_1^B(T,\hat{\mu}_B) = \chi_1^B - \frac{\chi_{11}^{BS}}{\chi_2^S}\chi_1^S = \chi_1^B$$

c_1^B



This rescaling will break down at large $T \longrightarrow$ rescaling with SBL

c_1^B



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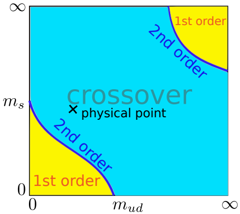
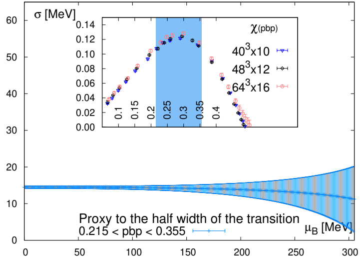
Why does the rescaling work?

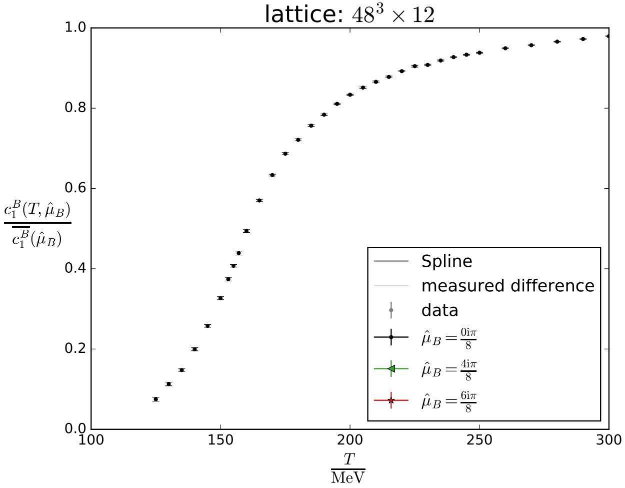
- It is an observation that it works
- It could be related to the critical scaling in the chiral limit
- If the universal contribution to EoS is large $\rightarrow$ single scaling variable
- If strength of transition is strongly Influenced by light quark masses $\rightarrow$ curves keep there shape
- Fits with the observation of constant width of the transition





[Borsanyi:2020fev]

Measuring the shift



c_1^B: net baryon density

$\overline{c_1^B}$: SBL of net baryon density

$$\Pi(T, \hat\mu_B, N_\tau) = \frac{T'(T, \hat\mu_B, N) - T}{T\hat\mu_B}$$

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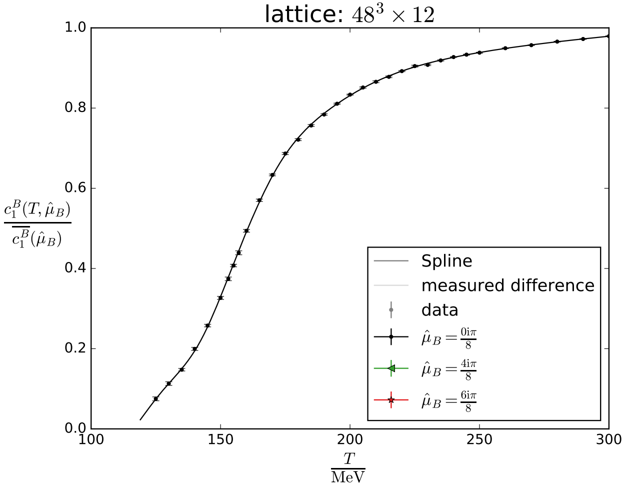

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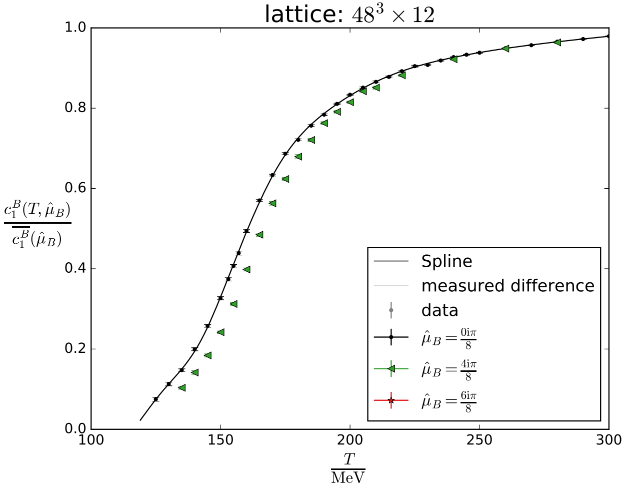

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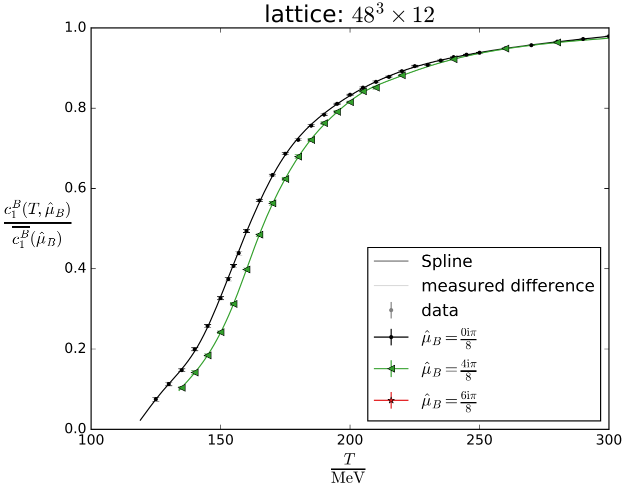

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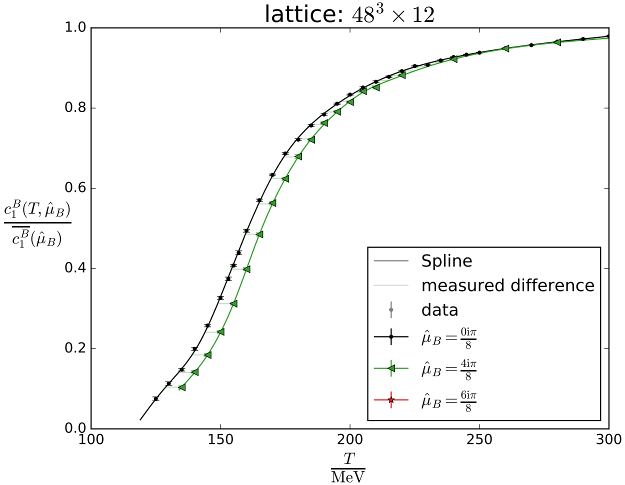

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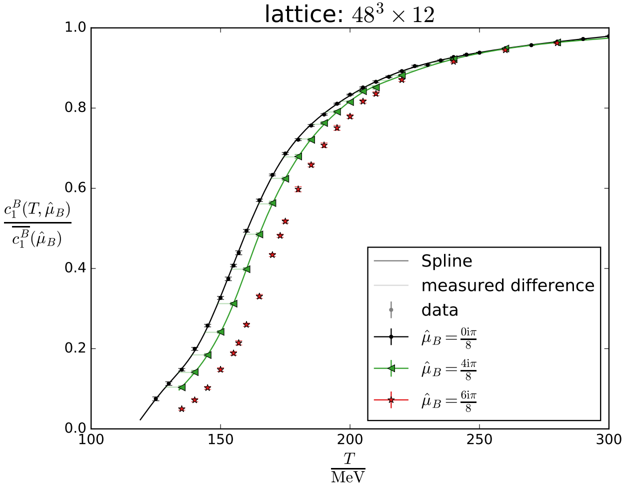

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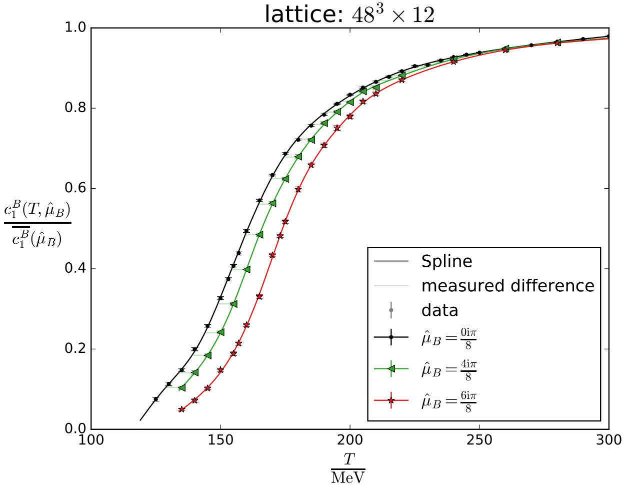

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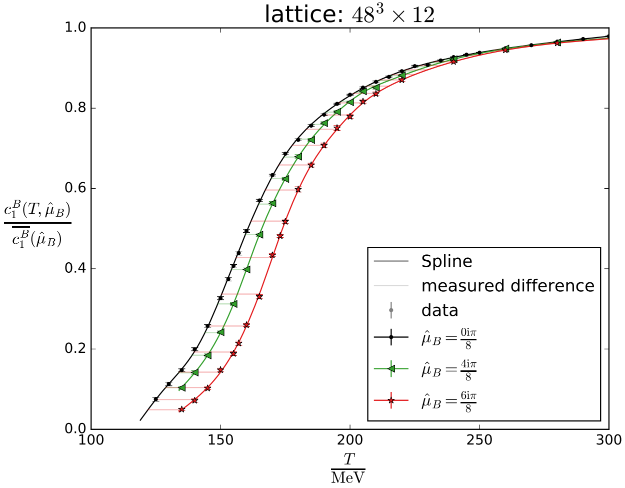

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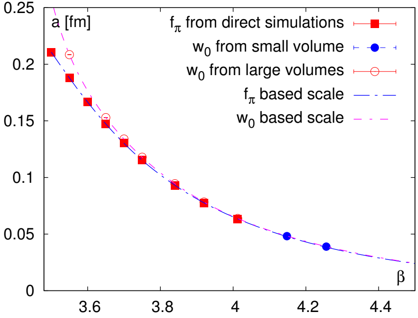
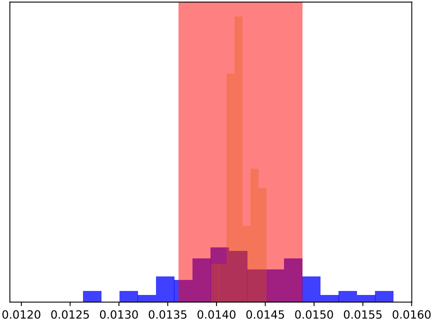
Lattice Setup



- Action: tree-level Symanzik improved gauge action, with four times stout smeared staggered fermions
- 2+1+1 flavour, on LCP with pion and kaon mass
- Simulation at $\langle n_S \rangle = 0$
- Continuum estimate from lattice sizes: $32^3 \times 8$, $40^3 \times 10$, $48^3 \times 12$ and $64^3 \times 16$
- $\frac{\mu_B}{T} = \mathrm{i}\frac{j\pi}{8}$ with $j = 0, 3, 4, 5, (5.5), 6$ and 6.5
- Two methods of scale setting: f_π and w_0, $Lm_\pi > 4$

Systematic Errors

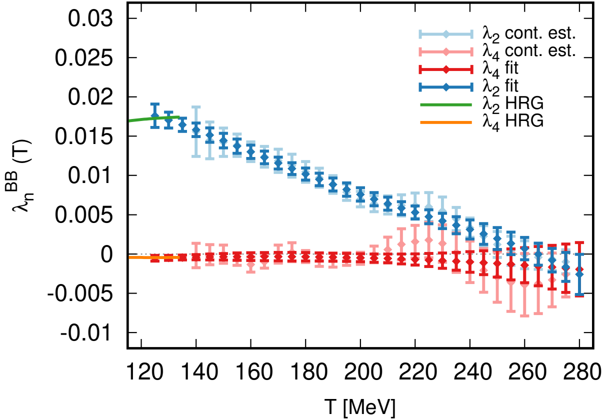
- 3 different sets of spline node points at μ_B=0
- 2 different sets of spline node points at finite imaginary μ_B
- w_0 or f_π based scale setting
- 2 different chemical potential ranges in the global fit: $\hat{\mu}_B \leq 5.5$ or $\hat{\mu}_B \leq 6.5$
- 2 functions for the chemical potential dependence of the global fit: linear or parabola
- including the coarsest lattice, $N_\tau = 8$, or not, in the continuum extrapolation.



histogram for λ_2 at $T = 155$ MeV

In total we perform 96 Fits. We weight every result with a $Q > 0.01$ uniformly

The expansion coefficients

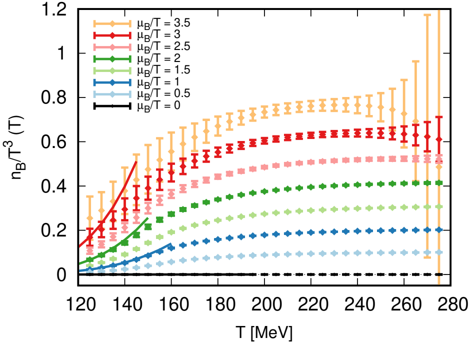


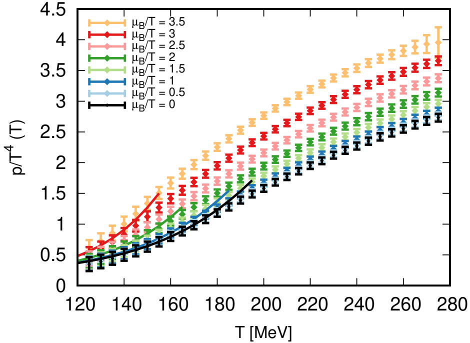
$$\Pi(T,\hat{\mu}_B,N_\tau) = \frac{T'(T,\hat{\mu}_B,N) - T}{T\hat{\mu}_B}$$

$$\Pi(T,\hat{\mu}_B,N_\tau) = \lambda_2^A + \lambda_4^A\hat{\mu}_B^2 + \lambda_6^A\hat{\mu}_B^4 + \frac{1}{N_\tau^2}\left(\alpha^A + \beta^A\hat{\mu}_B^2 + \gamma^A\hat{\mu}_B^4\right)$$

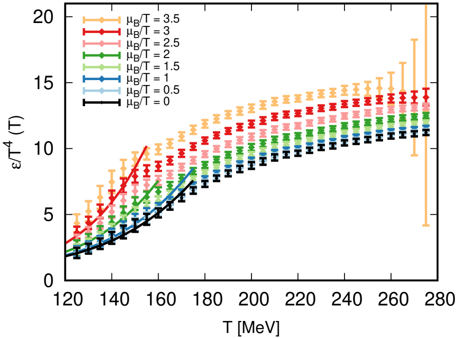
We make a fit to calculate derivatives and constrain it with the HRG.

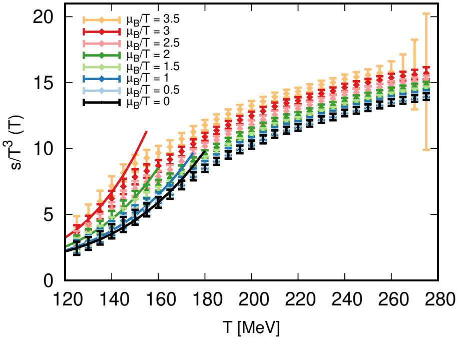
Results at $n_S = 0$ and $\mu_Q = 0$





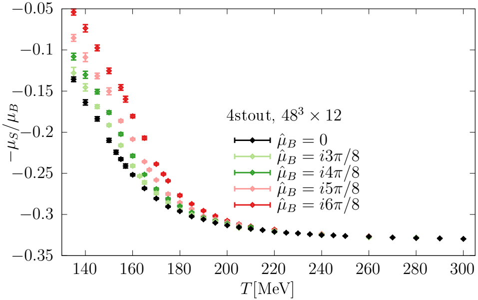
Results at $n_S = 0$ and $\mu_Q = 0$ II

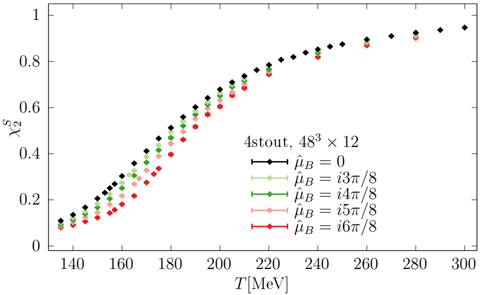




More strangeness

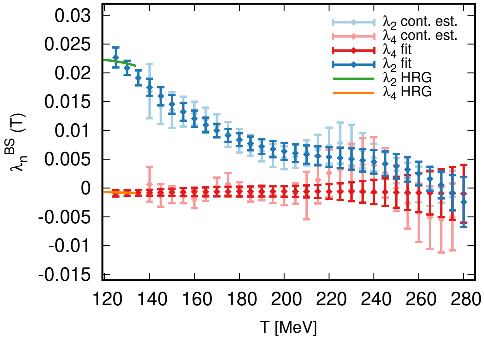
Two more observables:

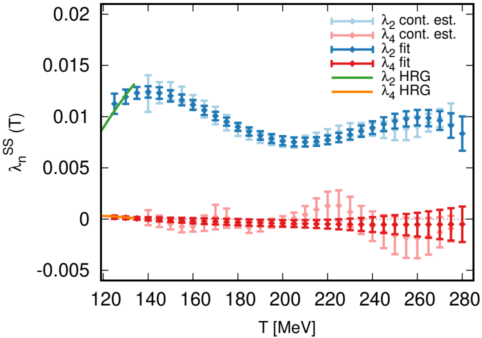




More strangeness

Two more expansion:





Beyond strangeness neutrality

$$\Delta\hat{\mu}_S \equiv \hat{\mu}_S - \hat{\mu}_S^{\star},$$

the dimensionless strangeness and baryon densities become:

$$\chi_1^S(\hat{\mu}_S) \approx \chi_2^S(\hat{\mu}_S^{\star})\Delta\hat{\mu}_S$$
$$\chi_1^B(\hat{\mu}_S) \approx \chi_1^B(\hat{\mu}_S^{\star}) + \chi_{11}^{BS}(\hat{\mu}_S^{\star})\Delta\hat{\mu}_S,$$

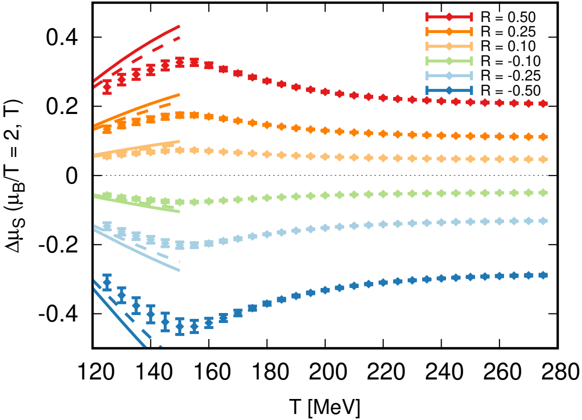
where we only kept the linear leading order terms in $\Delta\hat{\mu}_S$. We will express thermodynamic quantities in terms of the strangeness-to-baryon fraction:

$$R = \frac{\chi_1^S}{\chi_1^B} = \frac{\chi_2^S(\hat{\mu}_S^{\star})\Delta\hat{\mu}_S}{\chi_1^B(\hat{\mu}_S^{\star})\Delta\hat{\mu}_S + \chi_{11}^{BS}(\hat{\mu}_S^{\star})}.$$

Inverting this equation we get:

$$\Delta\hat{\mu}_S = \frac{R\hat{\chi}_1^B(\hat{\mu}_S^{\star})}{\chi_2^S(\hat{\mu}_S^{\star}) - R\chi_{11}^{BS}(\hat{\mu}_S^{\star})}.$$

Beyond strangeness neutrality



$$R = \frac{\chi_1^S}{\chi_1^B}$$

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Strange Baryon density

Expanding the baryon density:

$$\frac{\chi_1^B(T,\hat{\mu}_B,R)}{\chi_1^B(T,\hat{\mu}_B,R=0)} \approx 1 + R\frac{\chi_{11}^{BS}(T,\hat{\mu}_B,R=0)}{\chi_2^S(T,\hat{\mu}_B,R=0)}$$

where all quantities on the right hand side are along the strangeness neutral line.
At the strangeness neutral line the $\mathcal{O}(R)$ correction of the pressure vanishes. The leading order correction gives:

$$\hat{p}(T,\hat{\mu}_B,R) \approx \hat{p}(T,\hat{\mu}_B,R) + \frac{1}{2}\frac{\mathrm{d}^2\hat{p}}{\mathrm{d}R^2}(T,\hat{\mu}_B)\,R^2,$$

where

$$\frac{\mathrm{d}^2\hat{p}}{\mathrm{d}R^2}(T,\hat{\mu}_B) = \frac{\left(\chi_1^B(T,\hat{\mu}_B)\right)^2}{\chi_2^S(T,\hat{\mu}_B)}.$$

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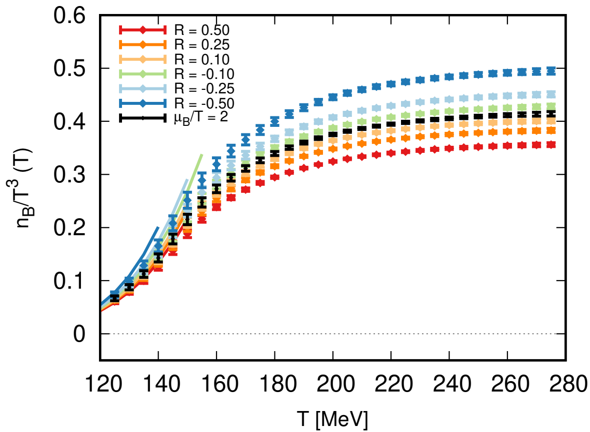

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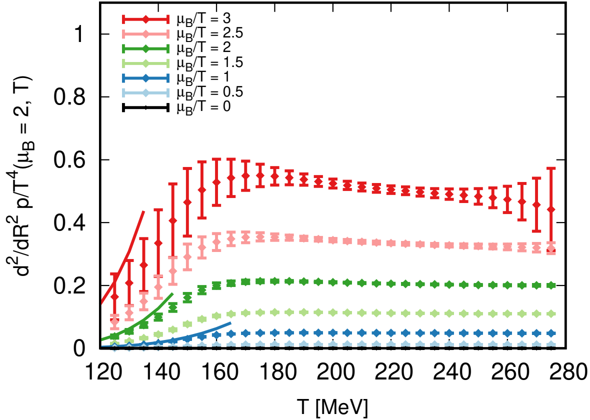
Strange Pressure



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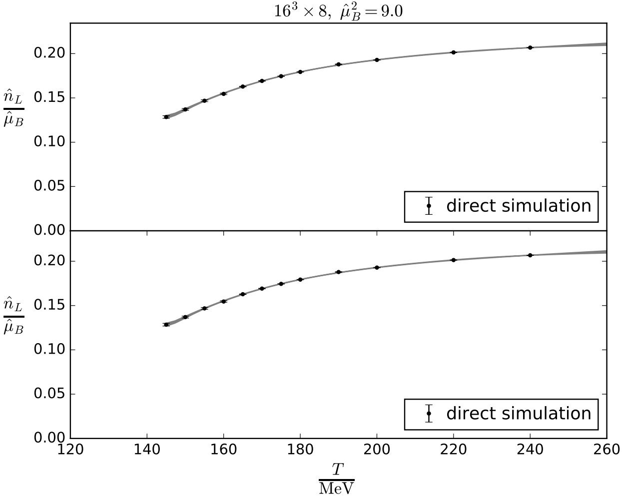
Does it work? - Check in a small volume [Borsanyi:2022soo]



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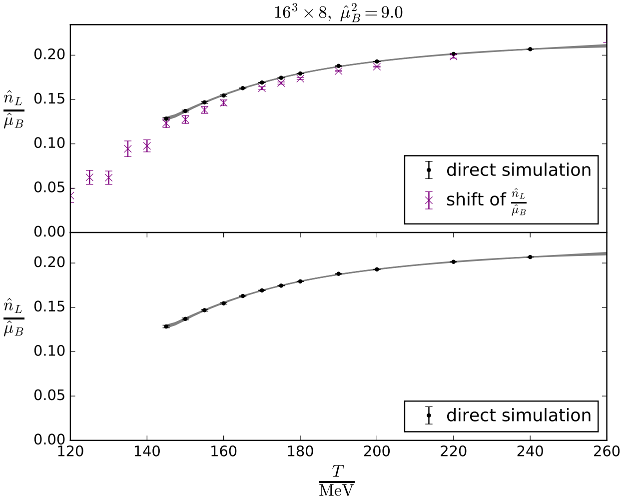

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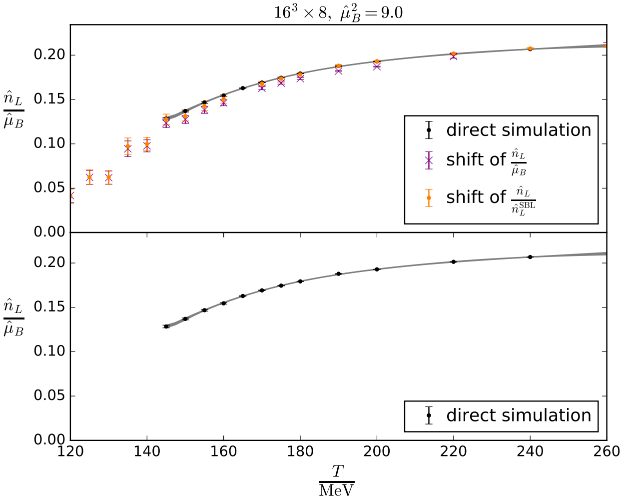

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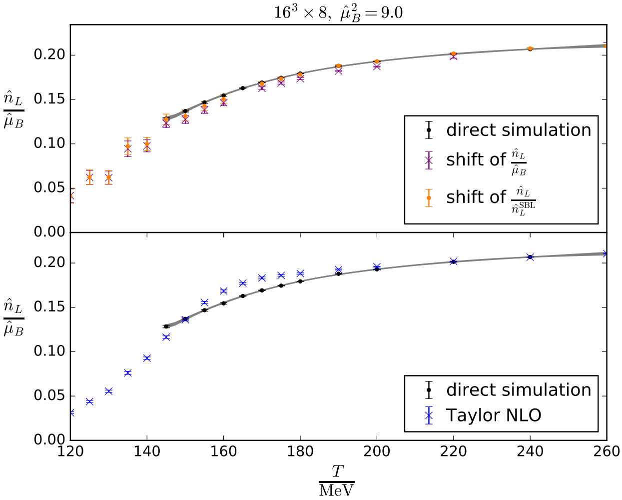

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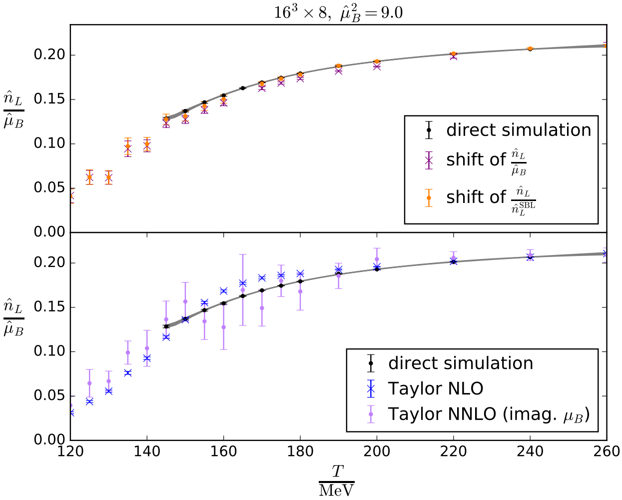

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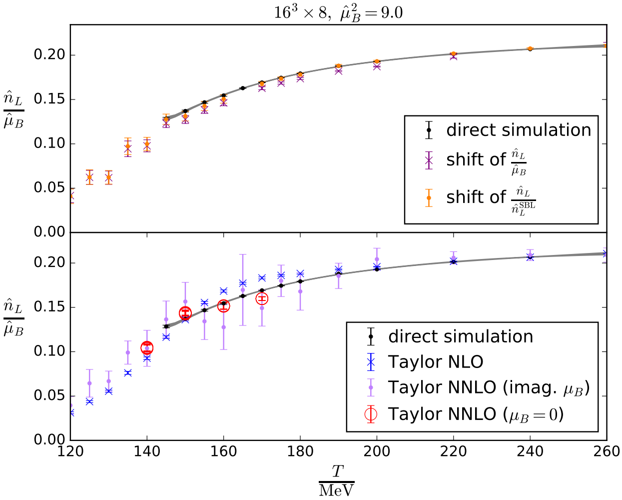

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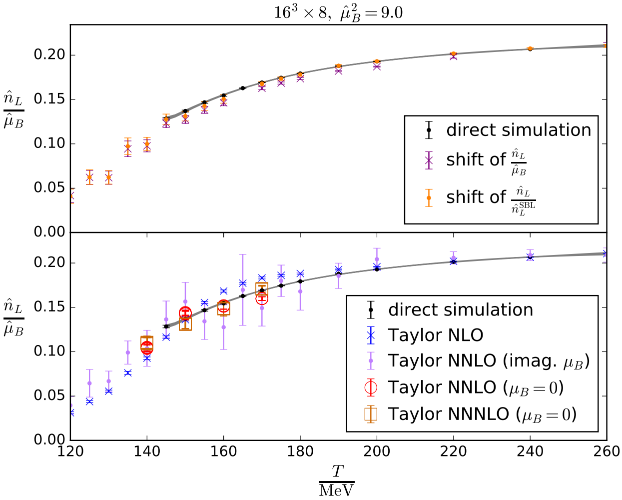

Outlook

- Update of the Equation of State at $\mu = 0$
- Addition of a magnetic field to the Equation of state at $\mu \neq 0$
- Further investigation of strangeness effects