Inhomogeneous chiral symmetry breaking phases

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Michael Buballa
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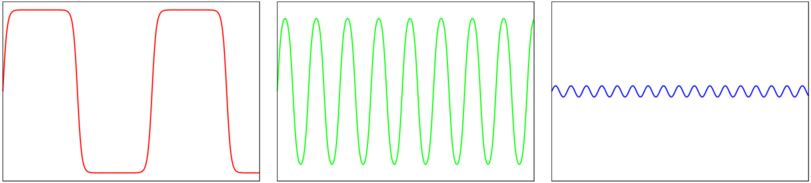

H-QM Helmholtz Research School Quark Matter Studies

HGS-HIRe for FAIR Helmholtz Graduate School for Hadron and Ion Research

Motivation: the QCD phase diagram (so far ?)

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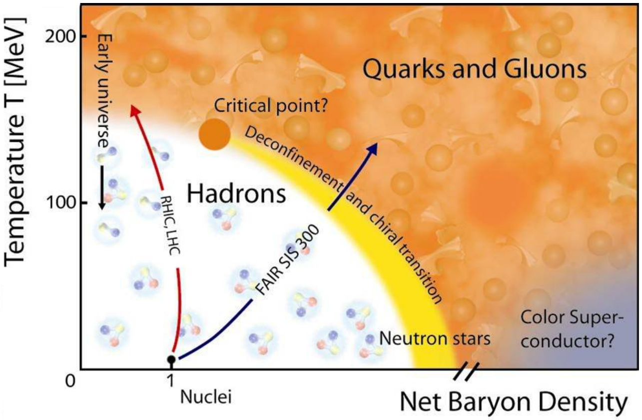

Why inhomogeneous phases ?

- Popular already for quite some time...
  - Overhauser pairing in nuclear matter
  - Pion condensation
  - (Color) superconductivity

- Recently rediscovered and revised
  - Studies of lower-dimensional models (GN_2, NJL_2, ...)
  - Quarkyonic chiral spirals
  - ...

The model

- Start from the usual $N_f = 2$ NJL Lagrangian

$$\mathcal{L}_{NJL} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi + G_s\left(\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}i\gamma^5\tau^a\psi\right)^2\right)$$

- Mean-field approximation
- Retain spatial dependence of the condensates

$$\langle\bar{\psi}\psi\rangle = S(\vec{x})\,, \qquad \langle\bar{\psi}i\gamma^5\tau^a\psi\rangle = P_a(\vec{x})$$

- Mean-field Lagrangian:

$$\mathcal{L}_{MF} = \bar{\psi}(x)\mathcal{S}^{-1}(x)\psi(x) - G_s\left(S(\vec{x})^2 + P(\vec{x})^2\right)$$

$$\mathcal{S}^{-1} = i\gamma^{\mu}\partial_{\mu} - m + 2G_s\left(S(\vec{x}) + i\gamma^5\tau^a P_a(\vec{x})\right) \equiv \gamma^0(i\partial_0 - \mathcal{H}_{MF})$$

Thermodynamic potential

$$\Omega(T,\mu;S(\vec{x}),P(\vec{x})) = -\frac{T}{V}\mathrm{Log}\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\exp\left(\int_{x\in[0,\frac{1}{T}]\times V}(\mathcal{L}_{MF}+\mu\bar{\psi}\gamma^0\psi)\right)$$

$$= -\frac{TN_c}{V}\sum_n \mathrm{Tr}_{D,f,V}\,\mathrm{Log}\left(\frac{1}{T}(i\omega_n+\mathcal{H}_{MF}-\mu)\right)+\frac{G_s}{V}\int_V\left(S(\vec{x})^2+P(\vec{x})^2\right)$$

- If we can calculate the eigenvalues $\{E_n\}$ of $\mathcal{H}_{MF}$, it's

$$\Omega(T,\mu;M(\vec{x})) = -\frac{TN_fN_c}{V}\sum_{E_n}\mathrm{Log}\left(2\cosh\left(\frac{E_n-\mu}{2T}\right)\right)+\frac{1}{V}\int_V\frac{|M(\vec{x})-m|^2}{4G_s}$$

Having defined $\quad M(\vec{x}) = m-2G_S\left(S(\vec{x})+iP(\vec{x})\right)$

Thermodynamic potential

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$$\Omega(T,\mu;S(\vec{x}),P(\vec{x})) = -\frac{T}{V}\mathrm{Log}\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\exp\left(\int_{x\in[0,\frac{1}{T}]\times V}(\mathcal{L}_{MF}+\mu\bar{\psi}\gamma^0\psi)\right)$$

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- **If** we can calculate the eigenvalues $\{E_n\}$ of $\mathcal{H}_{MF}$, it's

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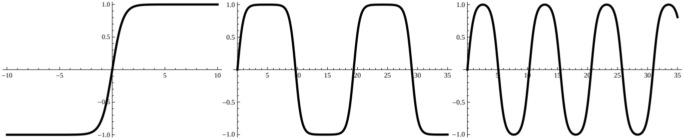
Having defined $\quad M(\vec{x}) = m-2G_S\left(S(\vec{x})+iP(\vec{x})\right)$

One-dimensional modulations: $M(\vec{x}) \to M(z)$

- Self-consistent real solutions known from studies of 1+1D Gross-Neveu model

  (M.Thies et al., Annals Phys. 314)

$$M(z) = \Delta \sqrt{\nu}\, \mathrm{sn}(\Delta z | \nu)$$



- Analytical expression for the eigenvalue spectrum of $\mathcal{H}_{MF}[M(z)]$
- Minimization of $\Omega[M(z)]$ w.r.t. two parameters (chiral limit): $\Omega(\Delta, \nu)$

  (D. Nickel, Phys.Rev.D 80)

Results: NJL (chiral limit)

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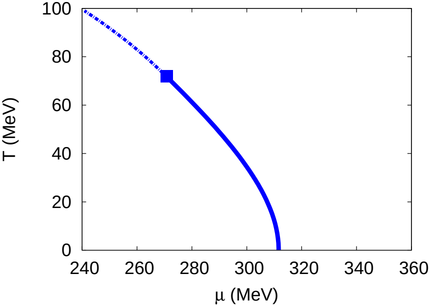

- Homogeneous only:

- First order phase transition
- ending at a critical point

Results: NJL (chiral limit)

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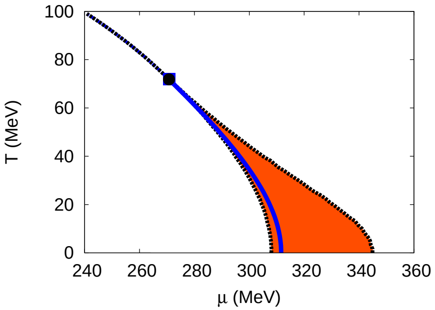


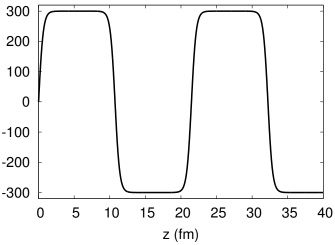
- Allow for inhomogeneous condensates:
- First order transition line covered by inhomogeneous phase
- All phase transitions are 2nd order
- Critical point → Lifschitz point

(D. Nickel, Phys.Rev.D80 074025, 2009 - arXiv:0906.5295)

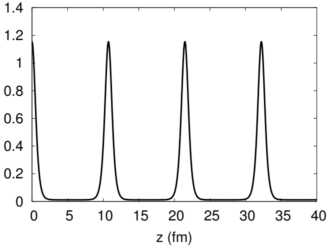
Mass and density profiles ($T = m = 0$)

$$M(z) = \Delta\sqrt{\nu}\,\mathrm{sn}(\Delta z|\nu) = \begin{cases} \Delta \tanh(\Delta z) & \text{for } \nu \to 1 \\ \Delta\sqrt{\nu}\sin(\Delta z) & \text{for } \nu \to 0 \end{cases}$$

M(*z*)



Density (norm.)



μ = 307.5 MeV

Mass and density profiles ($T = m = 0$)

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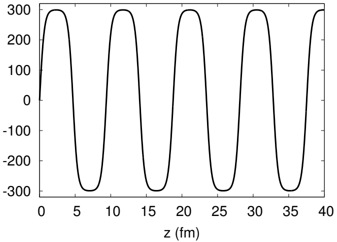
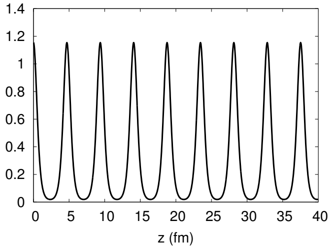
$M(z)$



Density (norm.)



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Mass and density profiles ($T = m = 0$)

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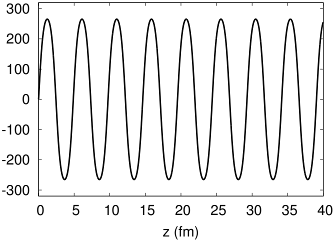
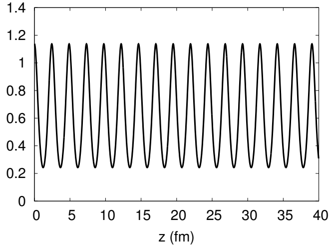
$M(z)$



Density (norm.)



μ = 309 MeV

Mass and density profiles ($T = m = 0$)

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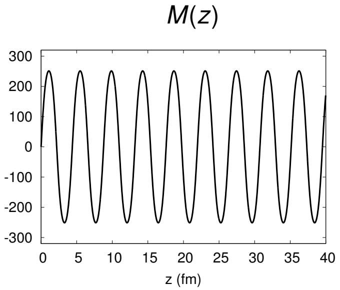

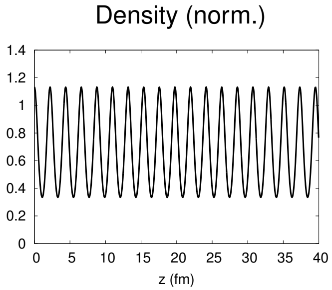

μ = 310 MeV

Mass and density profiles ($T = m = 0$)

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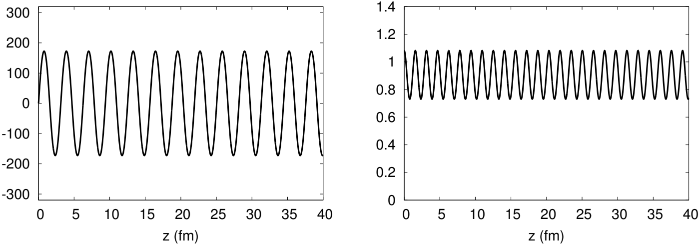
$M(z)$ Density (norm.)



μ = 320 MeV

Mass and density profiles ($T = m = 0$)

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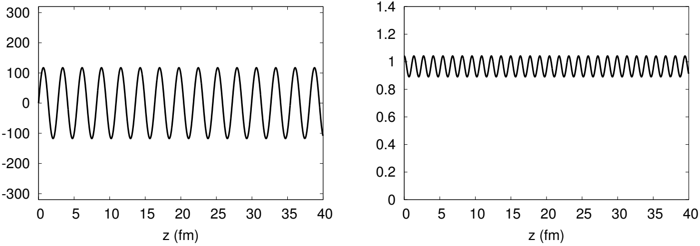

μ = 330 MeV

Mass and density profiles ($T = m = 0$)

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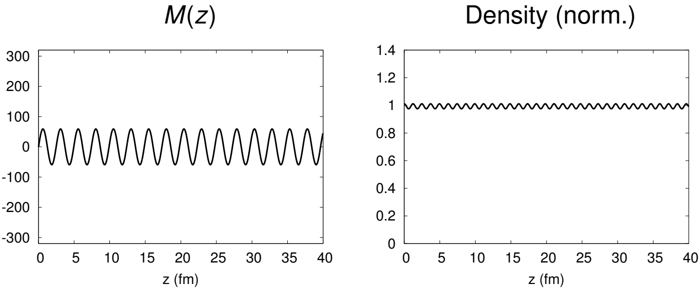
$$M(z) = \Delta\sqrt{\nu}\,\mathrm{sn}(\Delta z|\nu) = \begin{cases} \Delta\tanh(\Delta z) & \text{for } \nu \to 1 \\ \Delta\sqrt{\nu}\sin(\Delta z) & \text{for } \nu \to 0 \end{cases}$$



μ = 340 MeV

Mass and density profiles ($T = m = 0$)

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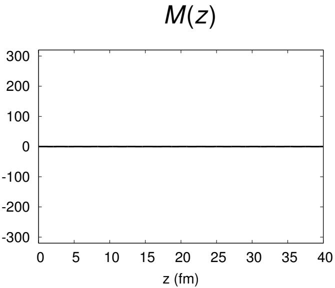

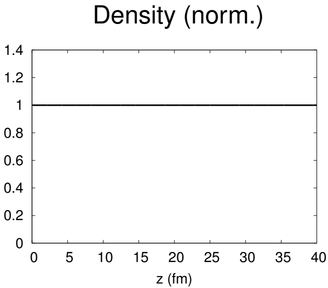

μ = 345 MeV

Vector interactions

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- Additional vector term: $\mathcal{L} = \mathcal{L}_{NJL} - G_V(\bar{\psi}\gamma^\mu\psi)^2$
- New mean field: $\bar{\psi}\gamma^\mu\psi \to \langle\bar{\psi}\gamma^\mu\psi\rangle \equiv n(\vec{x})\delta^{\mu 0}$ (density!)
- Introduce shifted chemical potential $\tilde{\mu}(\vec{x}) = \mu - 2G_V n(\vec{x})$
- Determine $\tilde{\mu}$ via $\frac{\delta\Omega}{\delta\tilde{\mu}} = 0$
- Sacrifice complete self-consistency: pick $\tilde{\mu} \equiv \langle\tilde{\mu}\rangle_z$ instead of $\tilde{\mu}(z)$
  - Most questionable in the inhomogeneous phase at low μ and T
  - More reliable close to the restored phase and the Lifshitz point

$$\Omega(T,\mu) \to \Omega(T,\tilde{\mu}) - \frac{(\mu-\tilde{\mu})^2}{4G_V}$$

Results: Vector interactions (Chiral limit)

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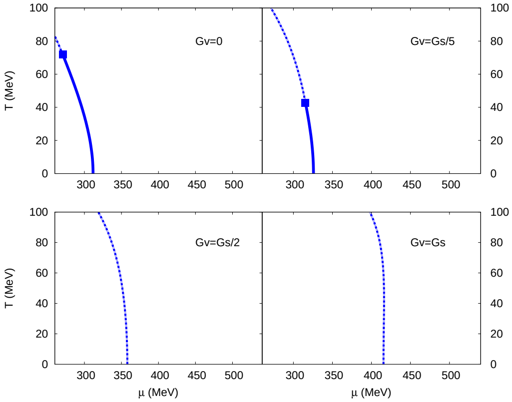

- Homogeneous:
- Shift towards higher μ
- Strong G_V-dependence of the critical point

Results: Vector interactions (Chiral limit)

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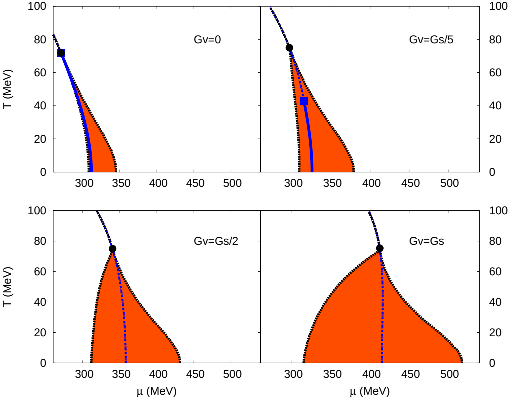
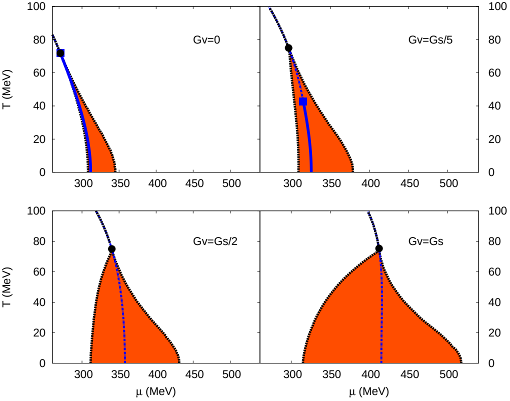
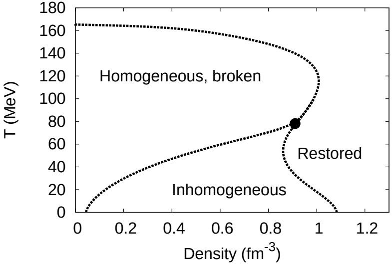


- Inhomogeneous:
- Stretch towards higher μ
- Lifshitz point at constant T
- Lifshitz and critical points split

Results: Vector interactions (Chiral limit)



T vs $\langle n \rangle$



G_V-independent!

Coupling with Polyakov Loop

- PNJL model:

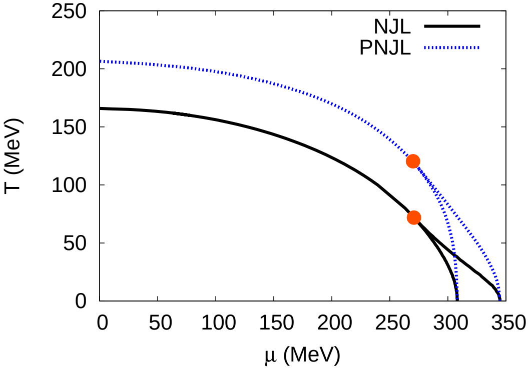
$$\mathcal{L}_{PNJL} = \bar{\psi}\left(i\gamma^{\mu}D_{\mu} - \hat{m}\right)\psi + G_s\left(\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}i\gamma^5\tau^a\psi\right)^2\right) - \mathcal{U}(L,\bar{L})$$

  - Covariant derivative: $D_{\mu} = \partial_{\mu} + iA_0\delta_{\mu 0}$
  - Polyakov loop: $L(\vec{x}) = \mathcal{P}exp[i\int_0^{1/T} d\tau A_4(\tau,\vec{x})]\,, \quad A_4(\tau,\vec{x}) = iA_0(t=-i\tau,\vec{x})$
  - Expectation values: $\ell = \frac{1}{N_c}\langle Tr_c L\rangle\,, \quad \bar{\ell} = \frac{1}{N_c}\langle Tr_c L^{\dagger}\rangle$
- Assumption: $\ell\,,\bar{\ell}$ space-time independent
- Main effect:

$$N_c T\log\left(1+e^{-\frac{E-\mu}{T}}\right) \to T\log\left(1+e^{-3(E-\mu)/T} + 3\,\ell\,e^{-(E-\mu)/T} + 3\,\bar{\ell}\,e^{-2(E-\mu)/T}\right)$$

- Thermally excited quarks are suppressed at small $\ell\,,\bar{\ell}$

Results: PNJL (Chiral limit)



- Suppression of thermal effects
- Phase diagram stretched in T

1D Modulations:
What have we learned?

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- Self-consistent 1D spatial modulations can be studied by relying on analytical results from the study of the Gross-Neveu model
- Inhomogeneous 1D phases are favored over homogeneous ones in a region of the phase diagram
- Extensions of the model (vector interactions, Polyakov loop) enhance the size of the inhomogeneous region
- The phase diagram is qualitatively altered !
- Only 2nd order phase transitions remain
- Vector interactions: $G_V > 0 \rightarrow$ CP disappears!

(SC, M. Buballa and D. Nickel, Phys.Rev.D**82)**

Large N_C

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- Arbitrary number of colors
- Practical implementation: modified PNJL model
- Assume $\ell = \bar{\ell}$, expand for small ℓ
  (McLerran, Redlich, Sasaki 2008)

$$\Omega_{med} = -2N_c N_f T \int dE \tilde{\rho}(E)[\Theta(E_p - \mu)\ell\left(e^{-\beta(E_p-\mu)} + e^{-\beta(E_p+\mu)}\right)$$
$$+ \Theta(\mu - E_p)\{\beta(\mu - E_p) + \ell\left(e^{-\beta(\mu-E_p)} + e^{-\beta(\mu+E_p)}\right)\}]$$

- Approximation works best in confined (small ℓ) phase

Large N_C - Preliminary results

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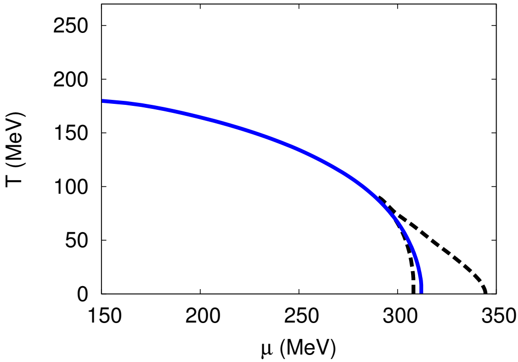

$N_C = 3$

Large N_C - Preliminary results

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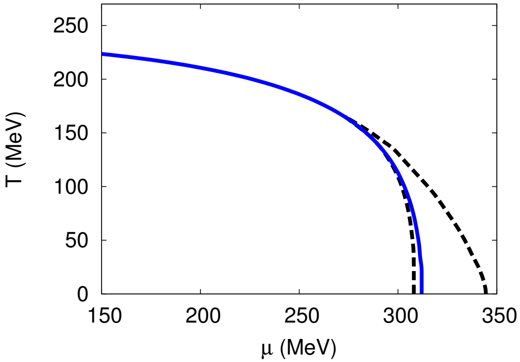

$N_C = 10$

Large N_C - Preliminary results

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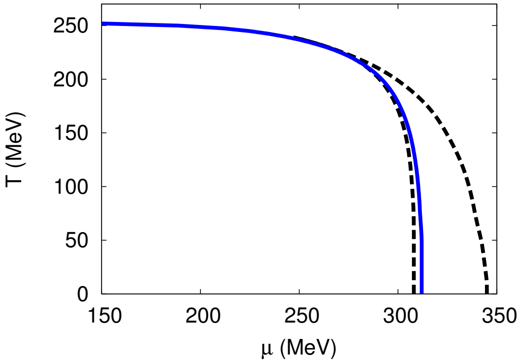

$N_C = 50$

Comparison with 2SC phase
(with D. Nowakowski)

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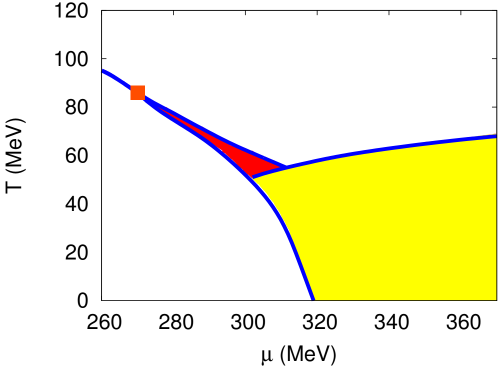

Higher dimensional modulations

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- No analytical results to help us this time
- Brute-force diagonalization of

$$\mathcal{H} = \gamma^0 \left[i\vec{\gamma} \cdot \vec{\partial} + m - 2G(S + i\gamma^5 P) \right]$$

- Expand $M(\vec{x})$ in a Fourier series:

$$M(\vec{x}) = \sum_q M_q \exp(i\mathbf{q} \cdot \mathbf{x})$$

- In momentum space:

$$\mathcal{H}_{p_{in},p_{out}} = \begin{pmatrix} -\vec{\sigma} \cdot \vec{p_{in}} \delta_{p_{in},p_{out}} & \sum_{\vec{q}} M_q \delta_{p_{out},p_{in}+q} \\ \sum_{\vec{q}} M_q \delta_{p_{out},p_{in}-q} & \vec{\sigma} \cdot \vec{p_{in}} \delta_{p_{in},p_{out}} \end{pmatrix}$$

- The inhomogeneous condensate couples different momenta

2D modulations

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- Assume spatial periodicity of the order parameter $M(x, y)$
- Start exploiting the lattice symmetries:
- $M(x, y)$ couples only selected momenta
- Project onto the Brillouin zone $\rightarrow \mathcal{H}(k)$ becomes block-diagonal

$$\mathcal{H}_{\vec{p_{in}},\vec{p_{out}}} = \sum_{\vec{k}\in BZ} \mathcal{H}_{\vec{q_{in}},\vec{q_{out}}}(\vec{k}) \quad , \; \vec{q_{in}}, \vec{q_{out}} \in RL$$

- Diagonalize $\mathcal{H}$ numerically
- Minimize $\Omega\left[\{M_q\}, \{q\}\right]$

$$\mathcal{H}_{p_m,p_n} = \begin{pmatrix} \square & & & \square & & & & & & \\ & \square & & & \square & & & & & \\ & & \square & & & \square & & & & \\ \square & & & \square & & & \square & & & \\ & \square & & & \square & & & \square & & \\ & & \square & & & \square & & & \square & \\ & & & \square & & & \square & & & \square \\ & & & & \square & & & \square & & \\ & & & & & \square & & & \square & \\ & & & & & & \square & & & \square \end{pmatrix}$$

Some first results:
the egg carton ansatz

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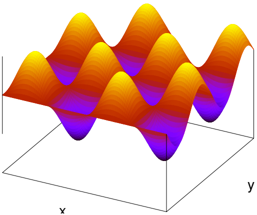
- Focus on a simple square crystal
- Ansatz for LOFF-type modulation

$$M(x, y) = \Delta sin(Qx)sin(Qy)$$



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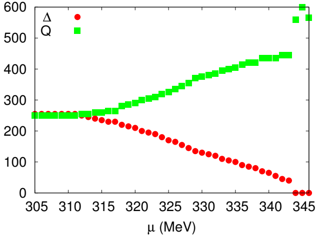
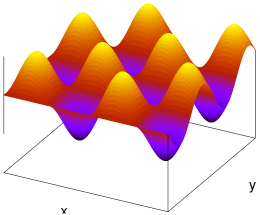
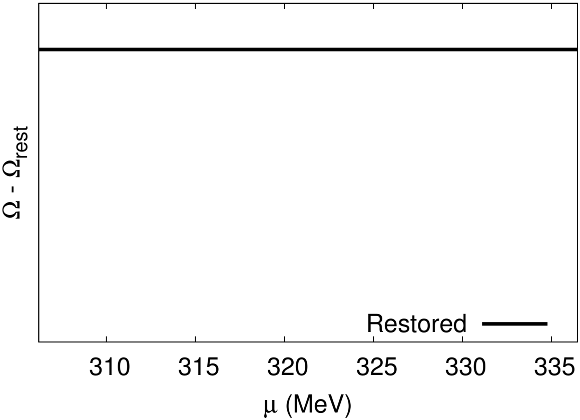




Some first results: thermodynamic potential, $T = 0$



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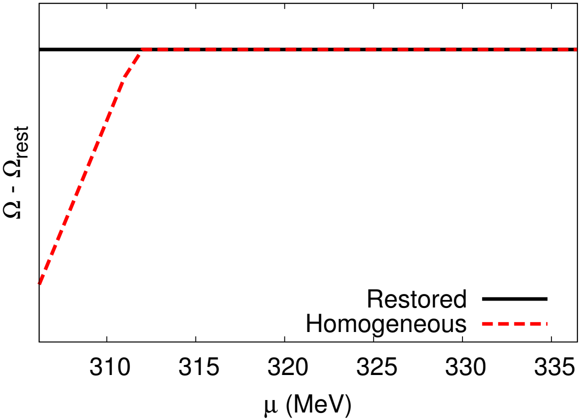

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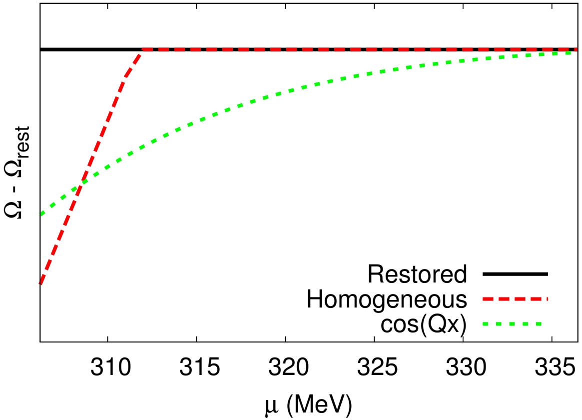
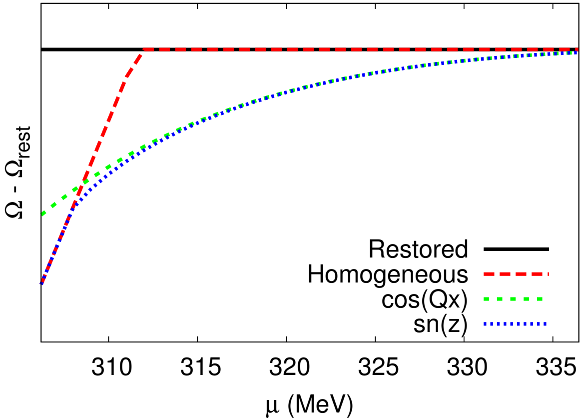
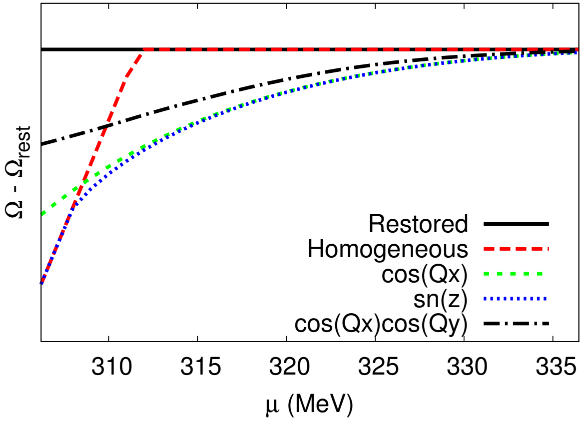


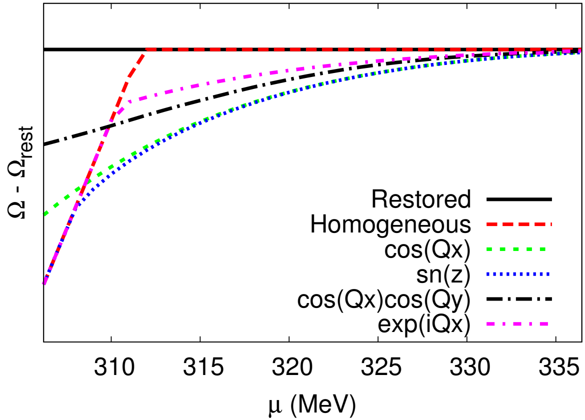
Some first results: thermodynamic potential, $T = 0$



Some first results: thermodynamic potential, $T = 0$



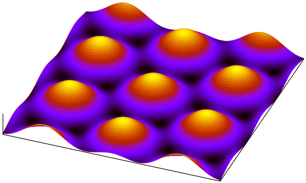
Some first results: thermodynamic potential, $T = 0$



Some first results:
Hexagon-symmetric modulation

- Now for something more elaborate: hexagonal symmetry

$$M(x,y) = \Delta \left[\cos(Qy) + 2\cos\left(\frac{\sqrt{3}}{2}Qx\right)\cos\left(\frac{Q}{2}y\right) \right]$$



Thermodynamic potential, $T = 0$

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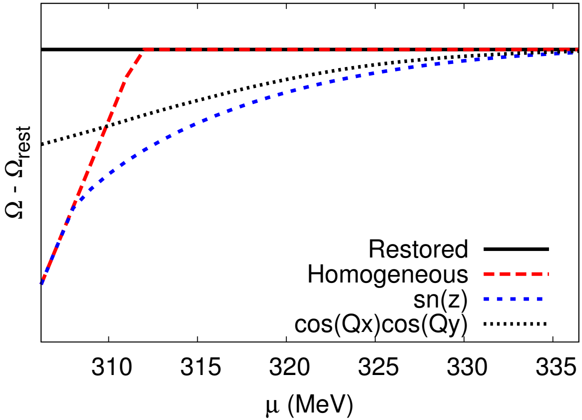
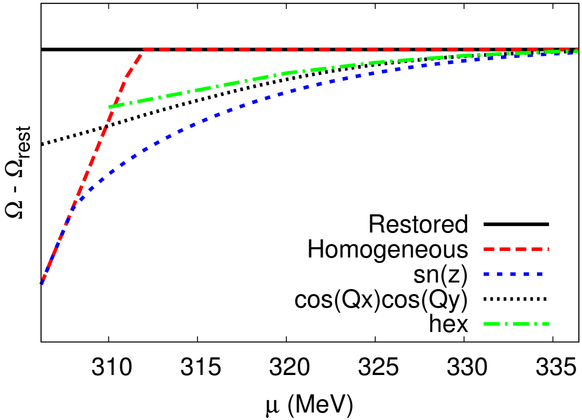


Thermodynamic potential, $T = 0$



Bottom line:

- 1D modulations are still fun to play with

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Bottom line:

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- 1D modulations are still fun to play with
- 2D modulations are a bit frustrating !!
- Still got a couple more modulation shapes to try ...

backup

What about pseudoscalar condensates?

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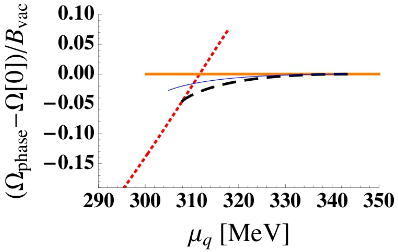
- Real modulations $\rightarrow P(x) = 0$

- Solitons: $M(z) \sim \Delta\sqrt{\nu}sn(z|\nu)$
- Chiral density wave: $M(z) = \Delta e^{iqz}$
- Homogeneous broken: $M(z) = \Delta$

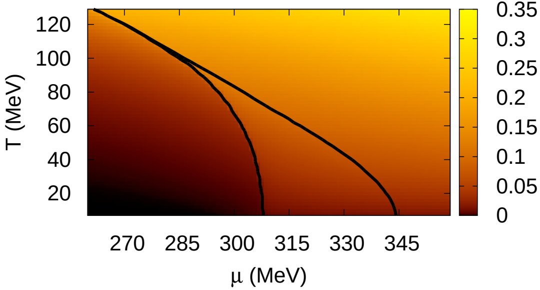


(D. Nickel, PRD 80)

- (Real) solitons are always favored over chiral density wave!

Polyakov loop expectation value

- How good is our approximation of constant $\ell, \bar{\ell}$?



- Inhomogeneous regime: $\ell, \bar{\ell} \leq 0.2$
- Effects of neglecting spatial variations of $\ell, \bar{\ell}$ presumably small

More phase diagrams: massive quarks

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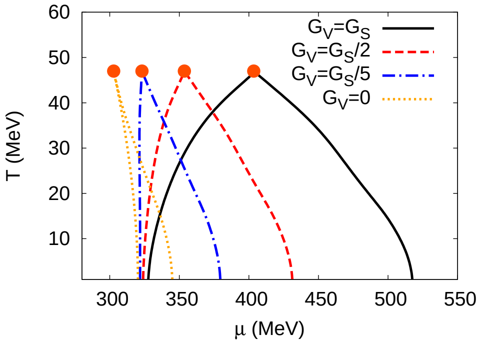
- Self-consistent solutions take the form

$$M(z) = \Delta\left(\sqrt{\nu}\,\mathrm{sn}(b|\nu)\mathrm{sn}(\Delta z|\nu)\mathrm{sn}(\Delta z + b|\nu) + \frac{\mathrm{cn}(b|\nu)\mathrm{dn}(b|\nu)}{\mathrm{sn}(b|\nu)}\right)$$

- Additional parameter: *b*
- Same qualitative features as $m = 0$
- Results for $m = 5$ MeV



Chiral spiral and vector interactions

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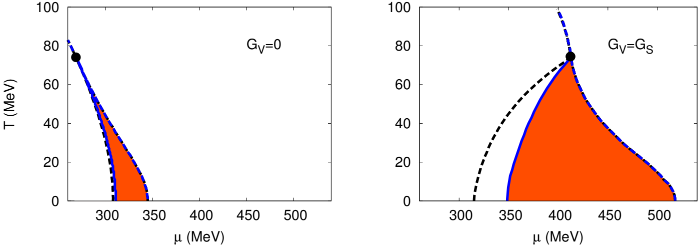
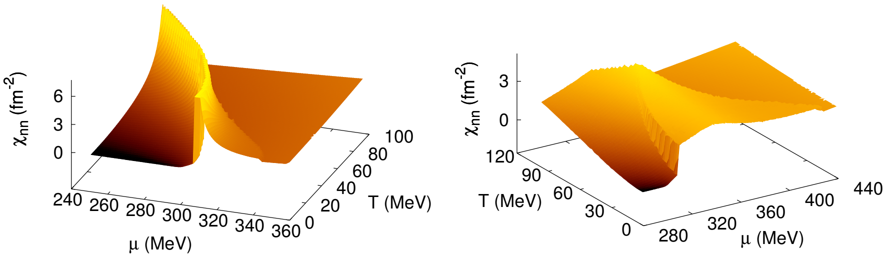
- How good is our $\tilde{\mu}(z) \to \langle \tilde{\mu}(z) \rangle$ approximation ?
- Cross-check: Chiral spiral $\to M(z) = \Delta e^{iqz} \;\to\; n(z) = \text{const.}$



- Same qualitative behaviour as the solitonic solutions
- Lifshitz point at the same position
- Different (1st order) homogeneous $\to$ inhomogeneous transition line

Quark number susceptibilities

$$\chi_{nn} = -\frac{\partial^2 \Omega}{\partial \mu^2} = \frac{\partial \bar{n}}{\partial \mu}$$



$G_V = 0$ $\quad$ $G_V = G_S/2$