Iterative Hydrodynamics

Solving relativistic hydrodynamics with a Taylor series technique

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Motivation

The idea from AdS/CFT [*Janik 2010*]

- metric for asymptotic AdS space in Feffermann-Graham coordinates:

$$G_{AB} = \frac{1}{z^2} \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}$$

- 5D Einstein-eqs. as eqs for $g_{\mu\nu}$
  [*cf. de Haro, Skenderis, Solodukhin, 2000*]
- expansion of $g_{\mu\nu}$ w.r.t. z^2
- boundary conditions
- construction of $g_{\mu\nu}$ order by order (holographic reconstruction)

Application to diagonal metric

- diagonal metric with Bjorken symmetry:

$$ds^2 = \frac{1}{z^2}\left(-e^{a(z,\tau)}d\tau^2 + e^{b(z,\tau)}\tau^2 d\eta^2 + e^{c(z,\tau)}dx_\perp^2 + dz^2\right)$$

- boundary metric = Minkowski: $a|_{z=0} = b|_{z=0} = c|_{z=0} = 0$
- boundary energy-momentum tensor $\Longrightarrow a_{(4)} = \epsilon(\tau)$

Result

$a(z,\tau) = z^4 a^{(4)} + z^6 a^{(6)} + z^8 a^{(8)} + \dots$ with:

$$
\begin{aligned}
a^{(4)} &= -\epsilon(\tau) \\
a^{(6)} &= -\frac{\dot{\epsilon}(\tau)}{4\tau} + \frac{\ddot{\epsilon}(\tau)}{12} \\
a^{(8)} &= \frac{\epsilon(\tau)^2}{6} + \frac{\tau\dot{\epsilon}(\tau)^2}{6} + \frac{\tau^2\ddot{\epsilon}(\tau)^2}{16} \\
&\quad + \frac{\dot{\epsilon}(\tau)}{128\tau^3} - \frac{\ddot{\epsilon}(\tau)}{128} - \frac{\dddot{\epsilon}(\tau)}{64\tau} - \frac{\ddddot{\epsilon}(\tau)}{384}
\end{aligned}
$$

$b(z,\tau), c(z,\tau)$ similar

what worked well here...

...could work elsewere, too

Goal

solving the equations for energy-momentum conservation

a) without approximations

b) without numerical errors (but there are other uncertainties)

Why?

- in general interesting
- application to many physical problems, e.g. rHICs
  - study systematic how deviations from initial conditions effect known solutions
  - getting analytical expressions for elliptic flow

Toy problem: solving df/dx = f iteratively:

- write f and f' in Taylor expansion:

$$\frac{df}{dx} = \sum_{k=0}^{\infty} \frac{1}{k!} f_{(k+1)} x^k = \sum_{k=0}^{\infty} \frac{1}{k!} f_{(k)} x^k = f$$

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- solution as Taylor expansion:

$$f(x) = a \sum_{k=0}^{\infty} \frac{1}{k!} x^k = a e^x \quad \text{, as expected}$$

Outlining the method

- $T^{\mu\nu}$: 10 different components

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- write down energy momentum conservation and put the time derivative at LHS
- expand $T^{\alpha 0}$ in Taylor series w.r.t. time

energy-momentum conservation:

$$0 = \nabla_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^\mu_{\nu\rho} T^{\rho\nu} + \Gamma^\nu_{\nu\sigma} T^{\mu\sigma}$$

solve for the time derivative

energy-momentum conservation

$$\frac{\partial}{\partial t} \sum_{m=0}^{\infty} \frac{1}{m!} T^{\mu 0}_{(m)} t^m = -\frac{\partial}{\partial x^a} \sum_{m=0}^{\infty} \frac{1}{m!} f^{\mu a}_{(m)} t^m - \Gamma^\mu_{\nu\rho} f^{\rho\nu} - \Gamma^\nu_{\nu\sigma} f^{\mu\sigma}$$

This must be valid in any order of t

In k^{th} order

$$T^{\mu 0}_{(k+1)} = -\frac{\partial}{\partial x^a} f^{\mu a}_{(k)} - \sum_{l=0}^{k} \binom{k}{l} \left(\Gamma^{\mu}_{(k-l)\nu\rho} f^{\rho\nu}_{(l)} + \Gamma^{\nu}_{(k-l)\nu\sigma} f^{\mu\sigma}_{(l)} \right)$$

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result

$$T^{\mu 0} = T^{\mu\,0}_{(0)} t^0 + T^{\mu\,0}_{(1)} t^1 + T^{\mu\,0}_{(2)} t^2 + \dots$$

- rearrange to get physical degrees of freedom, e.g. e, u^{μ}

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- functional form of initial conditions as simple as possible

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Solution

- f can be expanded up to the needed order
- ask a mathematican for which conditions convergence is good
- for 1st guess: look where the highest order term gets dominant
- good approximations are possible

recipe for ideal hydro

- energy-momentum tensor for ideal Hydro:
  $T^{\mu\nu} = (e + p)u^{\mu}u^{\nu} - pg^{\mu\nu}$

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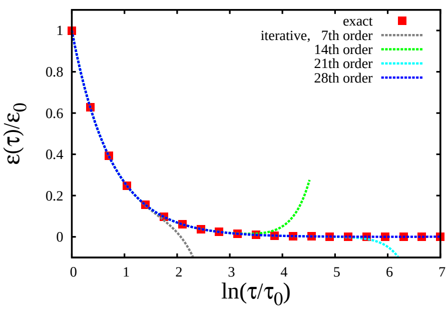
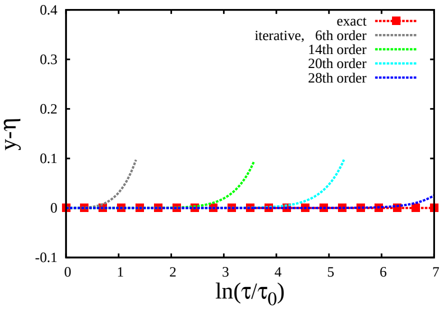
  $\Rightarrow T^{\mu\nu} = f^{\mu\nu}(T^{\alpha 0})$
- solve the problem up to sufficiently high order

Bjorken solution - no gradients in initial energy density

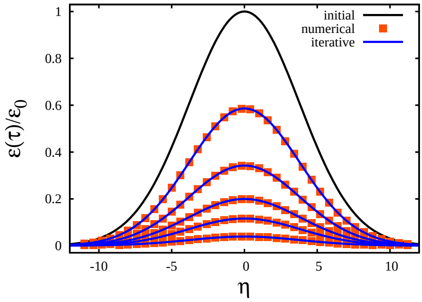


energy density with Bjorken initial conditions

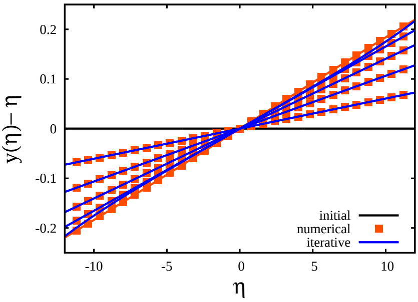


fluid rapidity with Bjorken initial conditions

logitudinal gradients in initial energy density, initial flow: Bjorken



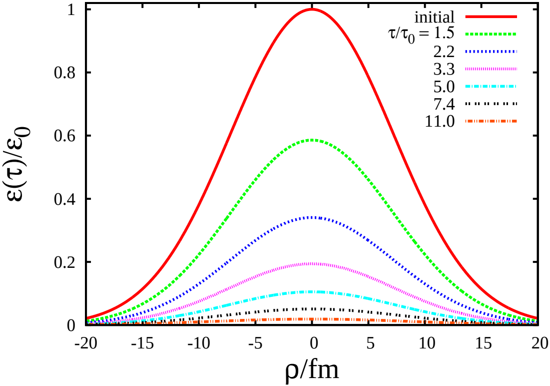
profiles of the energy density for Gaussian initial conditions up to $\ln(\tau/\tau_0) = 2.4$



profiles for the rapidity for Gaussian initial conditions up to $\ln(\tau/\tau_0) = 1.6$

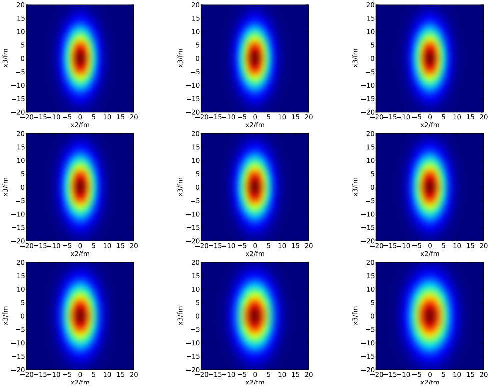
width from [*Eskola, Kajantie, Ruuskanen, EPJC (1997)*]

transversal gradients in initial energy density, initial flow: Bjorken

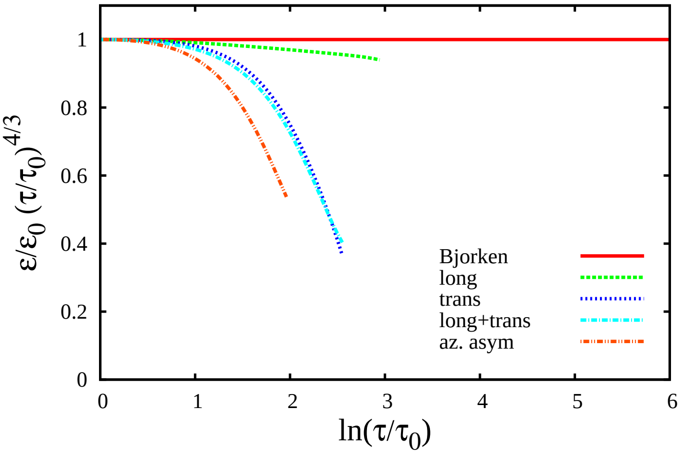


energy density profiles at different times

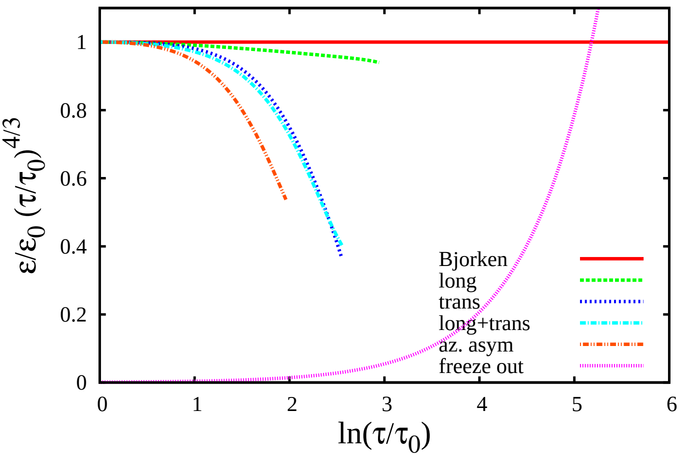
initial energy density with gradients in 3 dimensions, inital flow: Bjorken



energy density in the transversal plane at mid-rapidity: evolution of the initial asymetry



energy densities at the center at mid-rapidity: compairson between finite and infinite distribution of the energy density at $\tau = \tau_0$



energy densities at the center at mid-rapidity: Freeze out energy density ($T \approx 200 MeV$) for a system with $\epsilon_0 = 967 GeV/fm^3$ [*cf. Eskola, Kajantie (1996)*]

summary

- possible to calculate high derivatives

$\Rightarrow$ possible to get Taylor-expansion

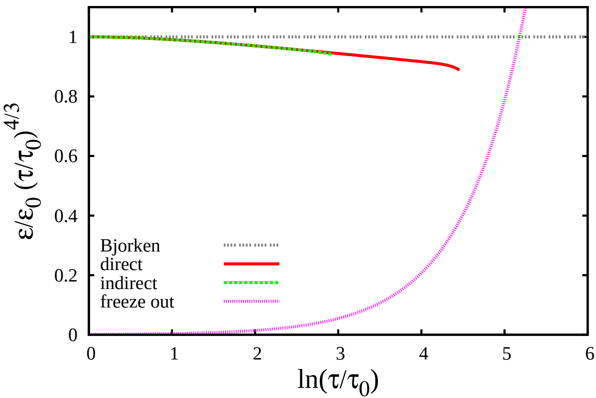
- reaches limits of a normal computer (Pentium i7, 4 GB RAM, Maple 9.5)
- some work left until freeze out reached
- not faster than numerical hydro codes, but analytical

outlook

- extend to viscous hydro (1st and 2nd order)
- extend to nontrivial EoS
- saving resources (time and memory) and go to higher orders
- triaxial expansion: study elliptic flow
- getting reference for checking numerical codes

Thank you for your attention!

speeding up, direct iteration



energy density at mid-rapidity: initial conditions with longitudinal gaussian shape. Freeze out energy density ($T \approx 200 MeV$) for a system with $\epsilon_0 = 967 GeV/fm^3$ [*cf. Eskola, Kajantie (1996)*]

Literature

- R. A. Janik, *"The dynamics of quark-gluon plasma and AdS/CFT,"* Lect. Notes Phys. **828** (2011) 147 [arXiv:1003.3291 [hep-th]].
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