Superfluid turbulence
& Nonthermal Fixed Points in Bose Gases

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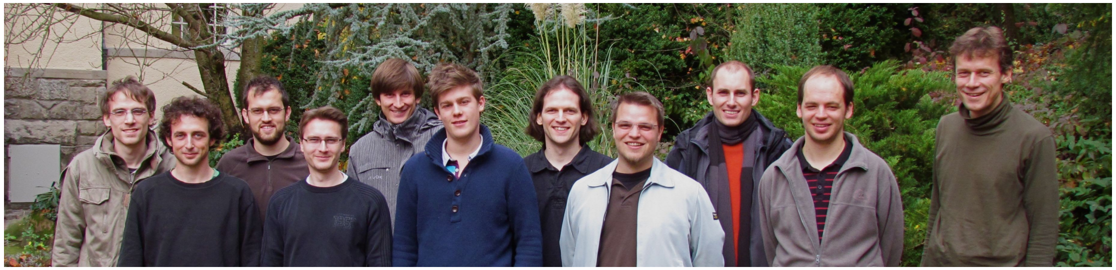
Center for Quantum Dynamics

EMMI

Thanks & credits to...



€€€...

...my work group in Heidelberg:

Boris Nowak
Maximilian Schmidt
Jan Schole
Dénes Sexty
Sebastian Bock
Sebastian Erne
Martin Gärttner
Steven Mathey
Nikolai Philipp
Martin Trappe
Jan Zill
Roman Hennig

...my former students:

Cédric Bodet (→ NEC), Alexander Branschädel (→ KIT Karlsruhe), Stefan Keßler (→ U Erlangen), Matthias Kronenwett (→ R. Berger), Christian Scheppach (→ Cambridge, UK), Philipp Struck (→ Konstanz), Kristan Temme (→ Vienna)

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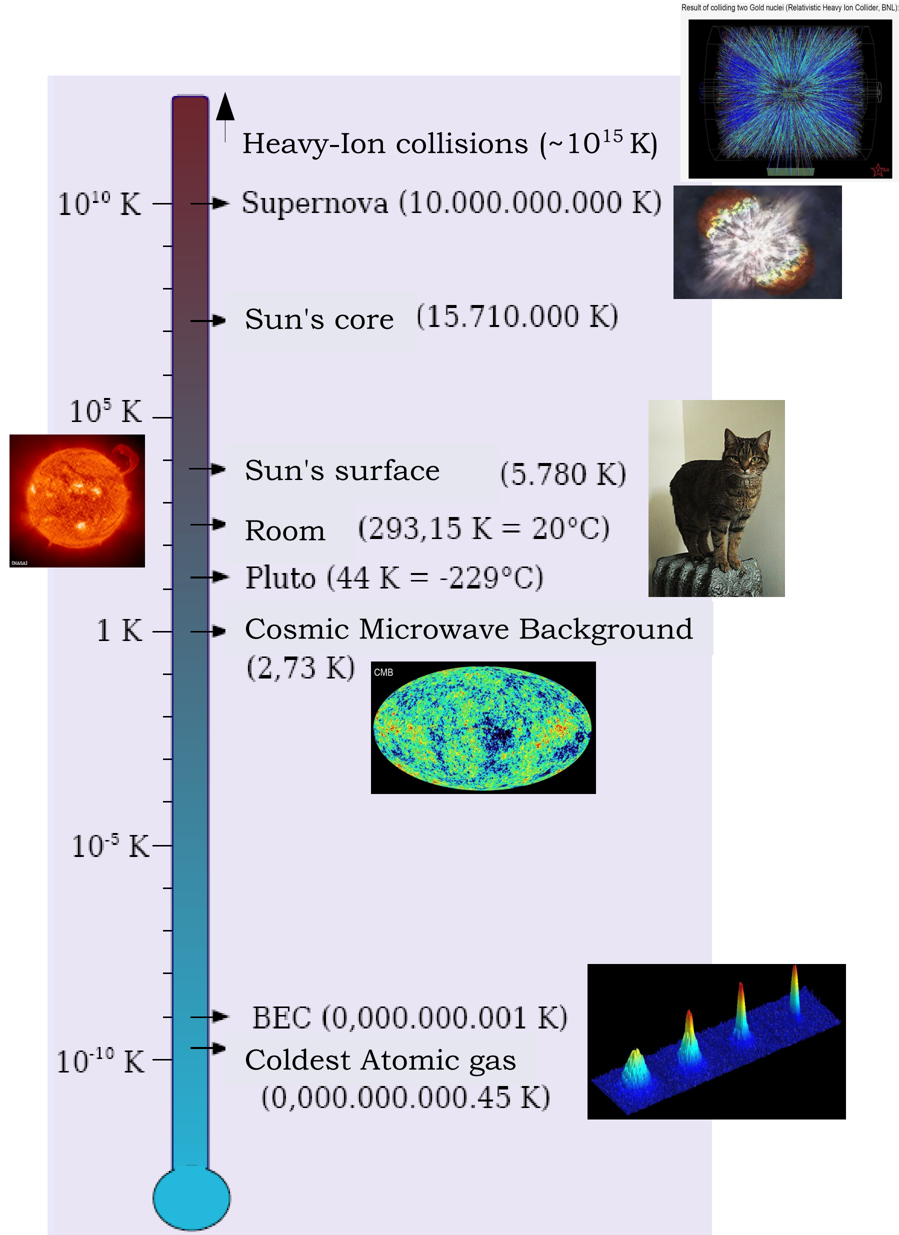
Non-equilibrium gases



Equilibration

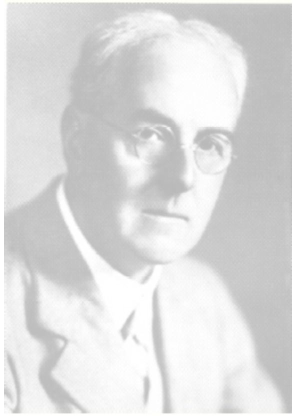
Transient state
e.g. Turbulence
Non-thermal fixed point

Classical Turbulence

Kinetic energy cascade

large scales (source)
$\rightarrow$ small scales (sink)

Classical Turbulence



Lewis F. Richardson
(1881-1953)

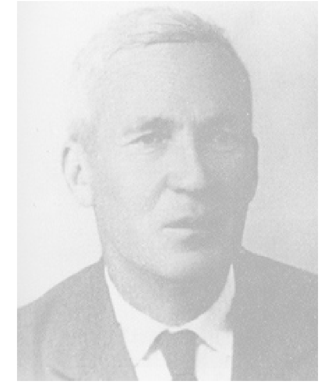
Kinetic energy cascade

large scales (source)
→ small scales (sink)

"Big whirls have little whirls that feed on their velocity,
and little whirls have lesser whirls and so on to viscosity."

(Richardson, 1920)

Classical Turbulence



Andrey N. Kolmogorov
(1903-1987)

Kinetic energy cascade

large scales (source)
$\rightarrow$ small scales (sink)

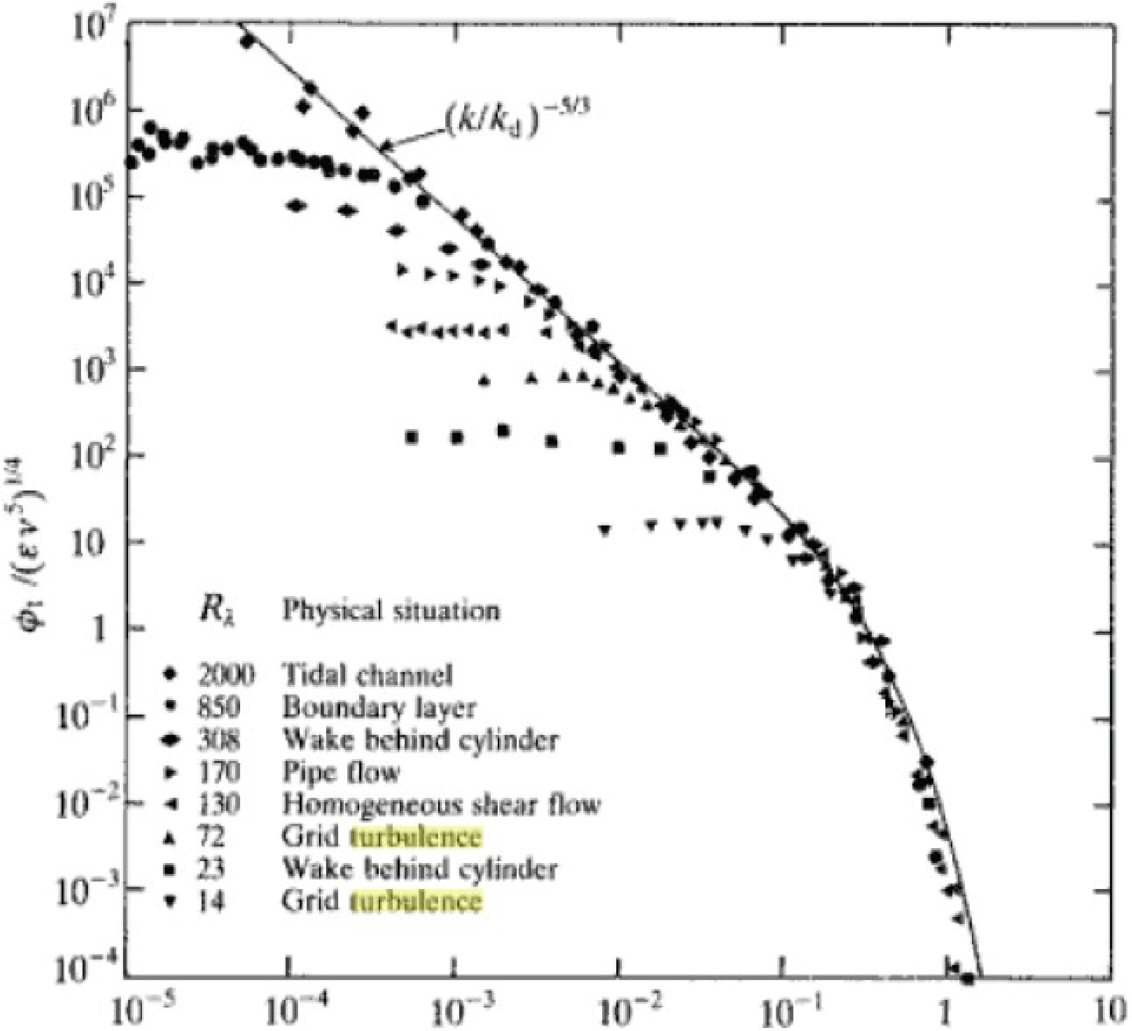
"Big whirls have little whirls that feed on their velocity,
and little whirls have lesser whirls and so on to viscosity."

(Richardson, 1920)

Kolmogorov (1941) $E(k) \sim k^{-5/3}$ (dynamical critical phenomenon)

Experiments

$\phi_1 = 2E_{\|}$



$k/k_d = k\eta$

Superfluid Turbulence

Vortices in a Na condensate

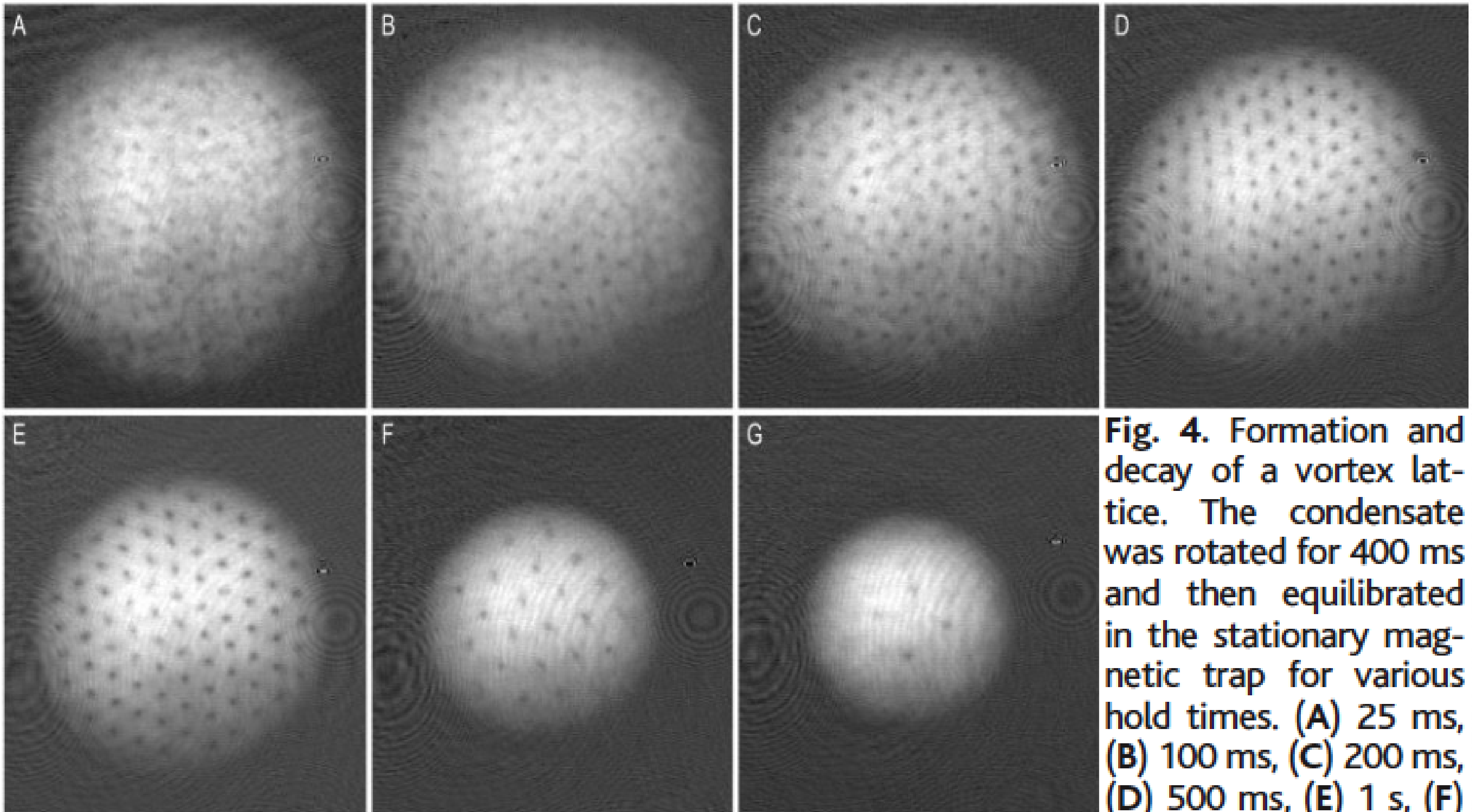


Fig. 4. Formation and decay of a vortex lattice. The condensate was rotated for 400 ms and then equilibrated in the stationary magnetic trap for various hold times. (A) 25 ms, (B) 100 ms, (C) 200 ms, (D) 500 ms, (E) 1 s, (F) 5 s, (G) 10 s,

J. R. Abo-Shaeer, C. Raman, J. M. Vogels, W. Ketterle
20 APRIL 2001 VOL 292 SCIENCE

Superfluid Hydro of a Dilute Gas

The Gross-Pitaevskii Eq. in the classical regime,

$$i\frac{\partial\Psi(\boldsymbol{\rho},t)}{\partial t}=\left(-\frac{\nabla^2}{2}+g|\Psi(\boldsymbol{\rho},t)|^2\right)\Psi(\boldsymbol{\rho},t)$$

is equiv. to

$$\frac{\partial}{\partial t}n+\nabla\cdot(n\mathbf{u})=0$$

$$\frac{\partial}{\partial t}\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}=-\nabla Q$$

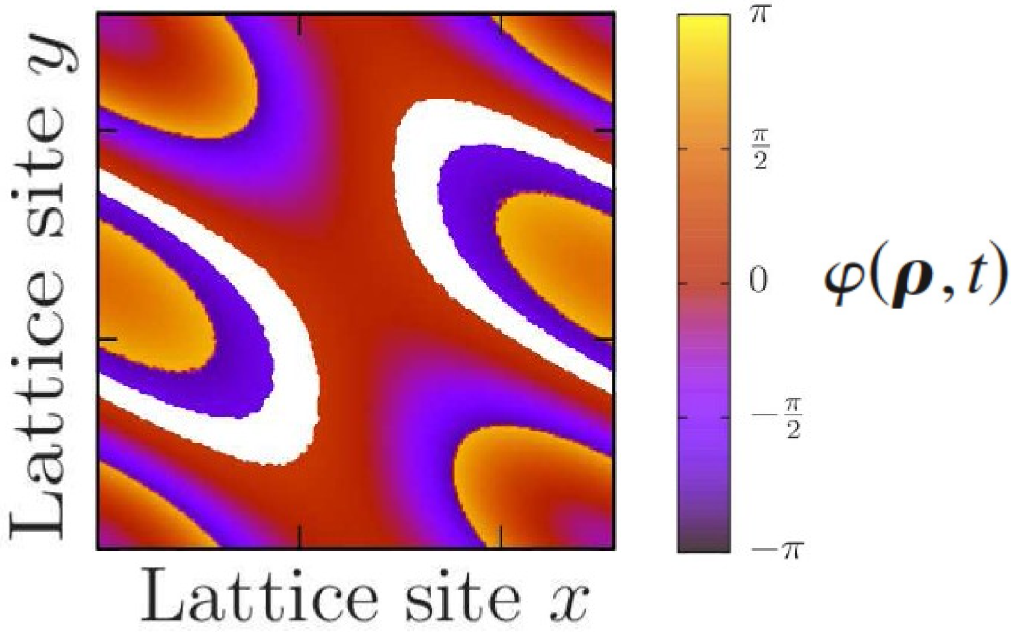
(Euler eq.)

with defs. $\Psi(\boldsymbol{\rho},t)=\sqrt{n(\boldsymbol{\rho},t)}\exp[i\varphi(\boldsymbol{\rho},t)]$

$Q=gn$ $\qquad$ $\mathbf{u}(\boldsymbol{\rho},t)=\nabla\varphi(\boldsymbol{\rho},t)$

Movie 1: Phase evolution

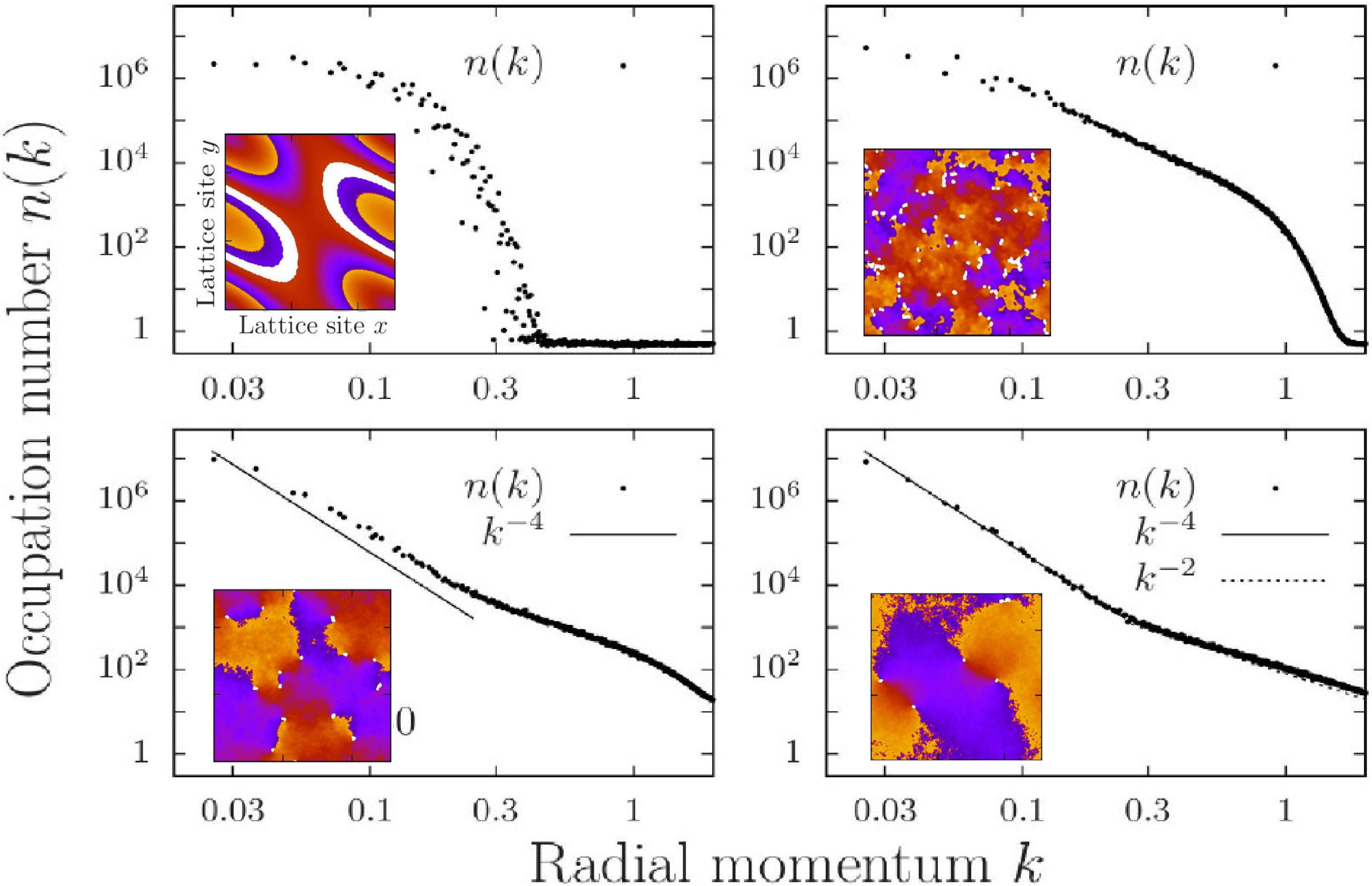
$$\Psi(\boldsymbol{\rho},t)=\sqrt{n(\boldsymbol{\rho},t)}\exp[i\varphi(\boldsymbol{\rho},t)]$$



Movie 2: Spectrum

$$n(\mathbf{k}) = \langle \Psi^*(\mathbf{k})\Psi(\mathbf{k})\rangle\big|_{\text{angle average}}$$

Spectrum in 2+1 D

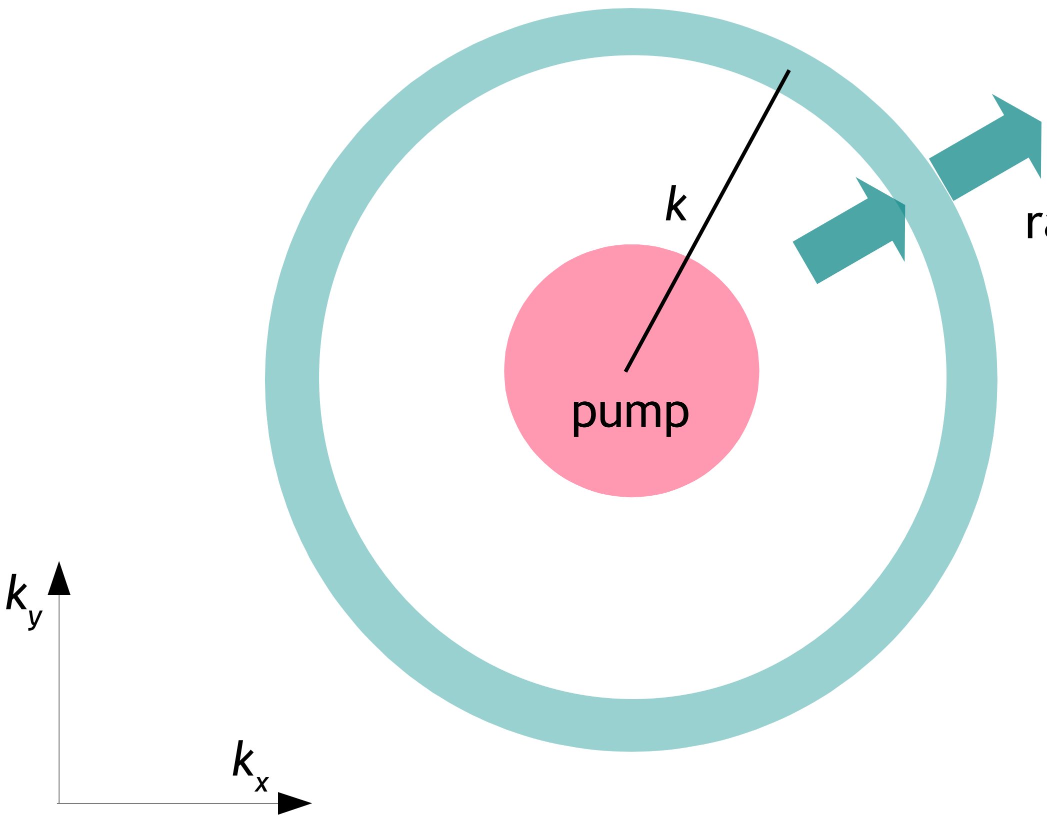


Wave Turbulence

Local radial flux only

Balance equation for radial flux

$$\partial_t \, n(k) = \; - \, \partial_k \, Q(k)$$



radial flux

rad. occupation no. n
rad. particle flux Q

Local radial flux only

Radial transport equation:

$$\partial_t\, n(k) = -\,\partial_k\, Q(k) \quad \sim \quad k^{d-1}\, J(k)$$

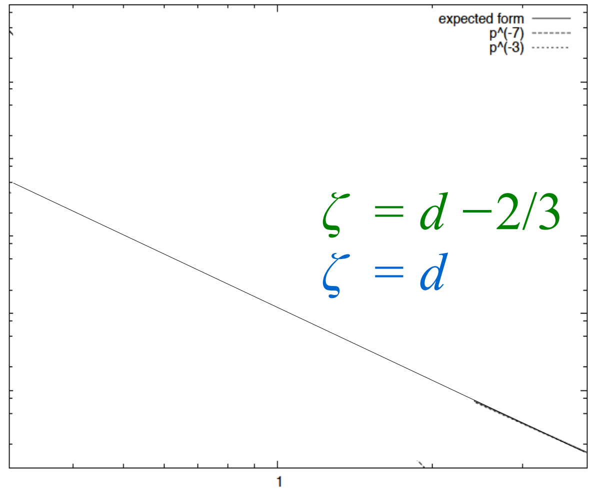
Quantum Boltzmann Scattering integral:

$$J(\mathbf{k},t) = g^2 \int \mathrm{d}^d p\, \mathrm{d}^d q\, \mathrm{d}^d r\, |T_{\mathbf{kpqr}}|^2 \delta(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r})$$
$$\times\ \delta(\omega_{\mathbf{k}}+\omega_{\mathbf{p}}-\omega_{\mathbf{q}}-\omega_{\mathbf{r}})$$
$$\times\ [(n_{\mathbf{k}}+1)(n_{\mathbf{p}}+1)n_{\mathbf{q}}n_{\mathbf{r}} - n_{\mathbf{k}}n_{\mathbf{p}}(n_{\mathbf{q}}+1)(n_{\mathbf{r}}+1)]$$

Scaling in d+1 Dim.

$$n \sim k^{-\zeta}$$

n



$\zeta = d - 2/3$

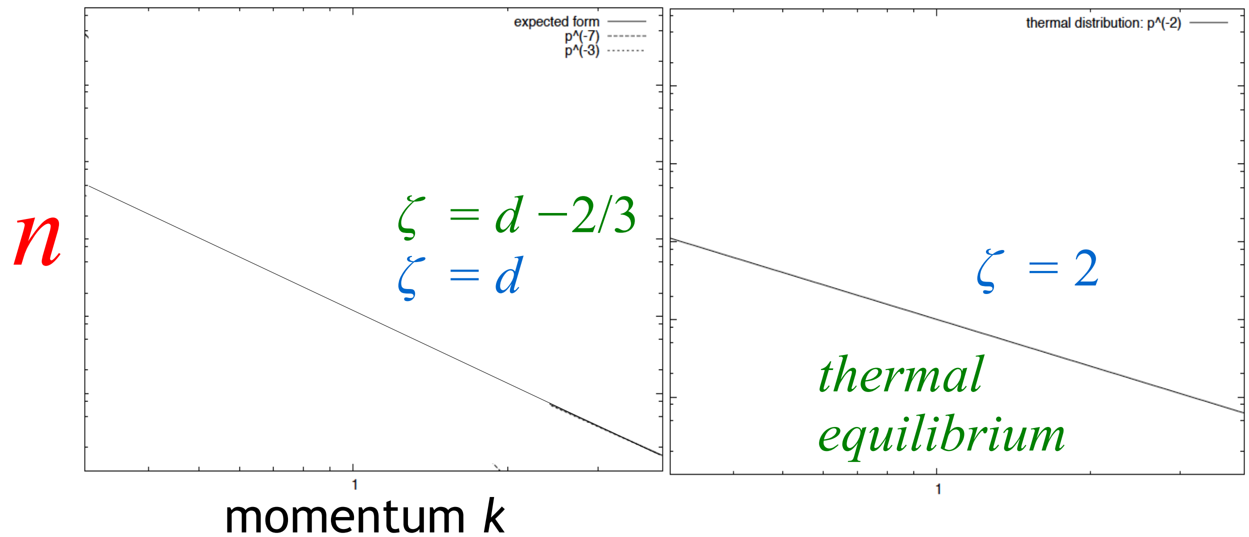
$\zeta = d$

momentum k

C. Scheppach, J. Berges, TG PRA **81** (10) 033611

Scaling in d+1 Dim.

$$n \sim k^{-\zeta}$$



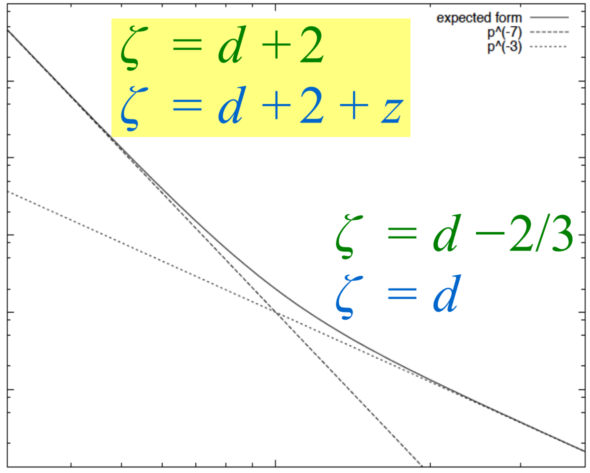


C. Scheppach, J. Berges, TG PRA **81** (10) 033611

Scaling in d+1 Dim.

$$n \sim k^{-\zeta}$$

n



momentum k



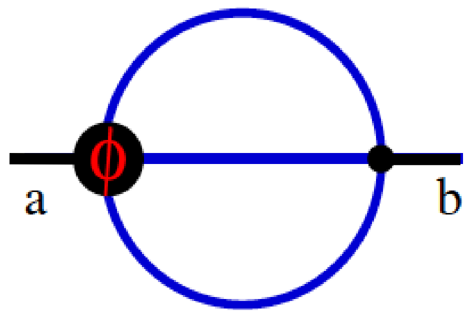
J. Berges, A. Rothkopf, J. Schmidt, PRL **101** (08) 041603
C. Scheppach, J. Berges, TG PRA **81** (10) 033611

Strong turbulence

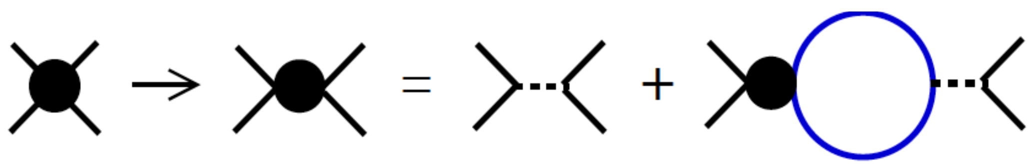
$p = (p_0, \mathbf{p})$:

$$J(p) := \Sigma^{\rho}_{ab}(p) F_{ba}(p) - \Sigma^{F}_{ab}(p) \rho_{ba}(p) \overset{!}{=} 0$$

$\Sigma_{ab}(x,y)$ =



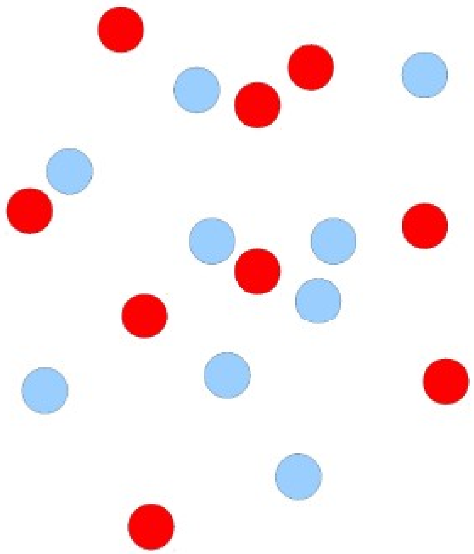
Vertex bubble resummation:
(~2PI to NLO in $1/N$)



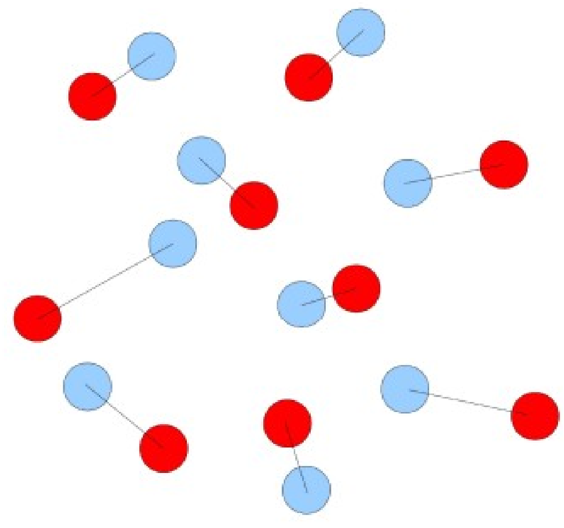
[Dynamics: J. Berges, (02); G. Aarts et al., (02); TG, Seco, Schmidt, Berges (05);
Kadan.Baym: "GW-Approximation", Hedin (65)]

Statistics of vortices

2D statistics of vortices



$$n_k \sim k^{-4}$$

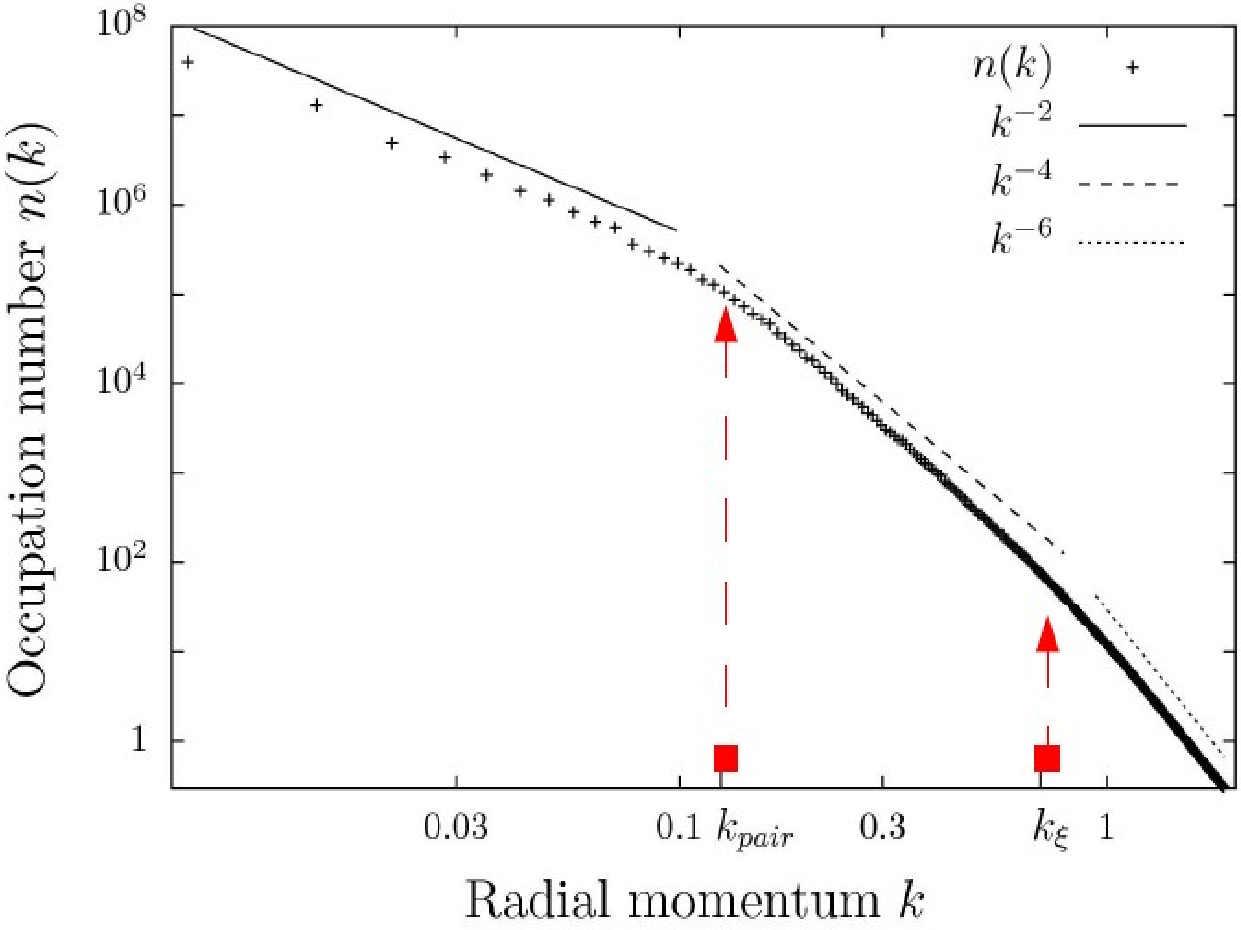


$$n_k \sim k^{-2}, \; k < k_{\text{pair}}$$

$$n_k \sim k^{-4}, \; k > k_{\text{pair}}$$

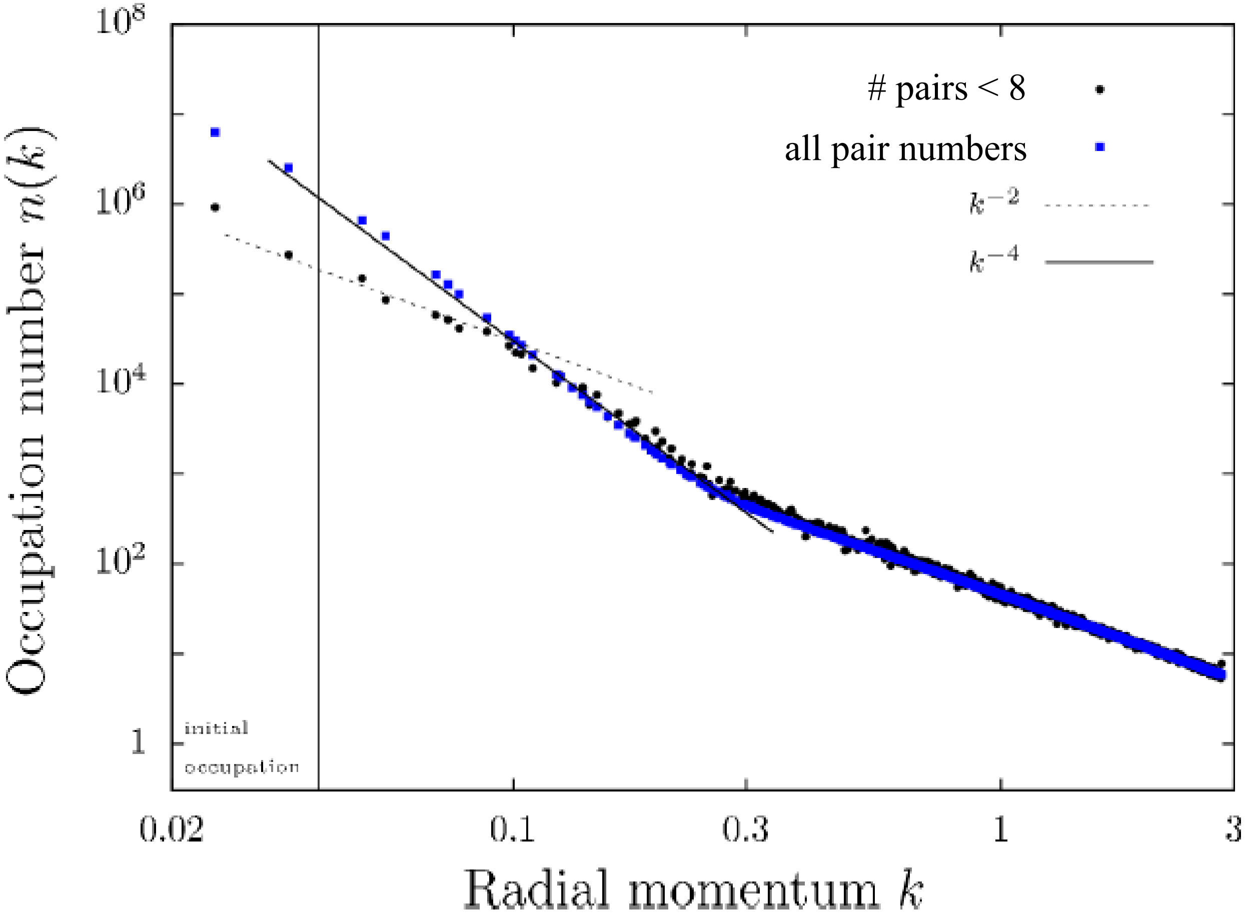
[B. Nowak, J. Schole, D. Sexty, T. Gasenzer, in prep.]

2D statistics of vortices



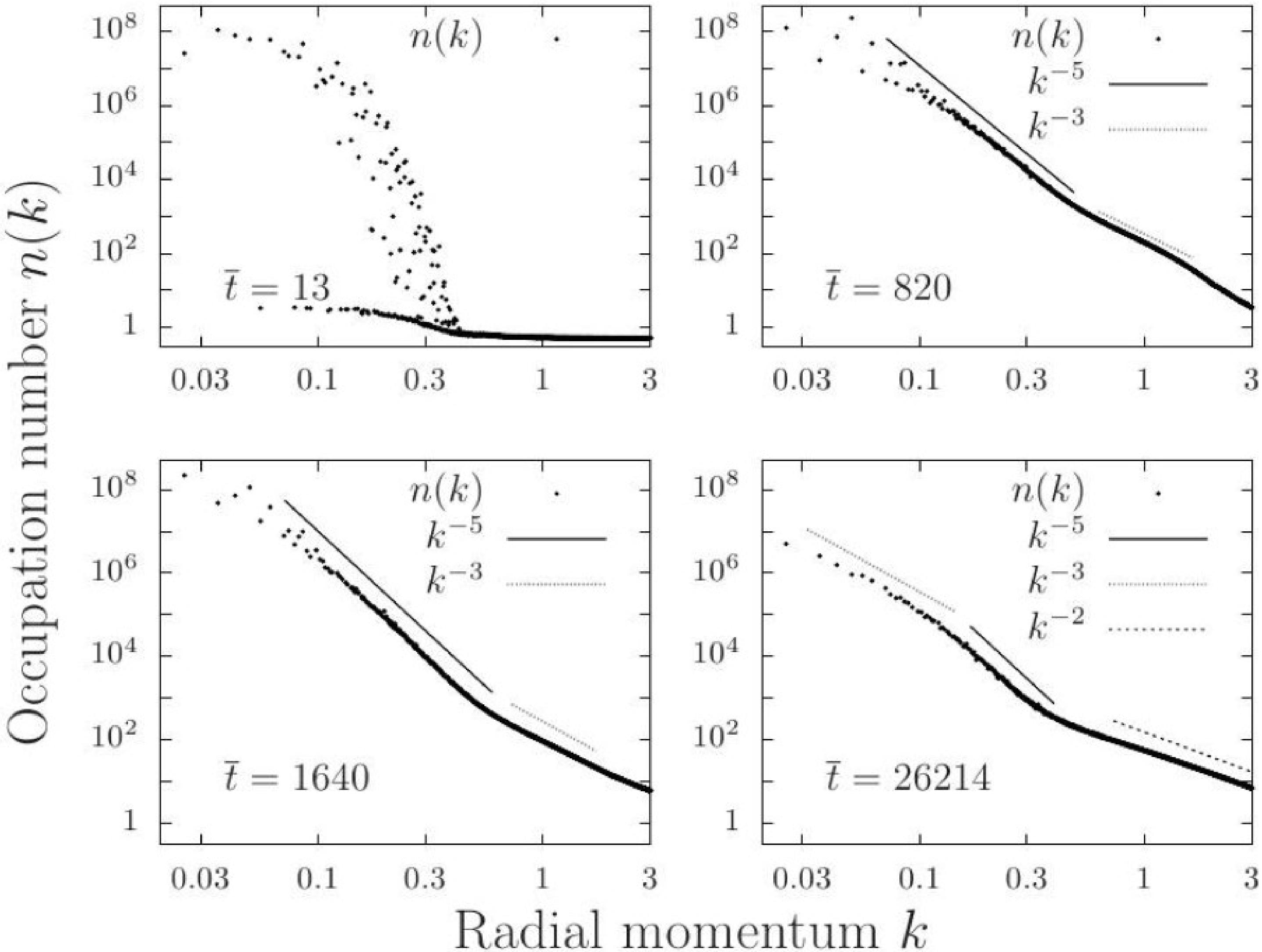
[B. Nowak, J. Schole, D. Sexty, T. Gasenzer, in prep.]

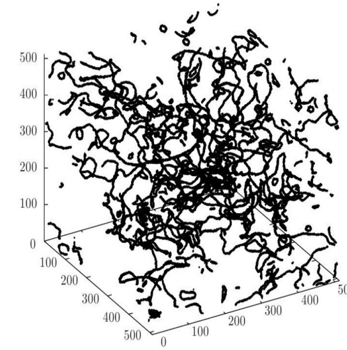
2D statistics of vortices



[B. Nowak, J. Schole, D. Sexty, T. Gasenzer, in prep.]

3D simulations



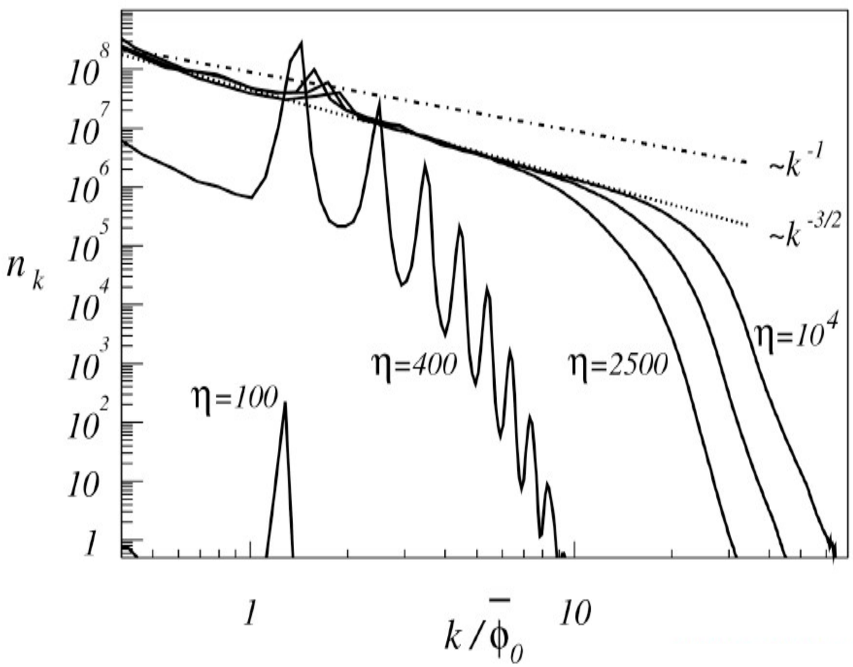


Relativistic scalar field

Turbulence in reheating after inflation

Simulations of the non-linear Klein-Gordon equation,

$$(\partial_t^2 - \partial_x^2)\varphi(x,t) + \lambda\varphi^3(x,t) = 0$$



Initial condition:

Highly occupied zero mode
Unoccupied modes with k>0

Turbulent spectrum emerges

Exponent: weak wave turbulence

Kofmann, Linde, Starobinsky (96)
Micha, Tkachev, PRL & PRD (04)

Strong Turbulence

Simulations of the non-linear Klein-Gordon equation, O(2) symmetry

$$(\partial_t^2 - \partial_x^2)\varphi(x,t) + \lambda\varphi^3(x,t) = 0$$

Initial condition: Highly occupied zero mode, Unoccupied modes with k>0

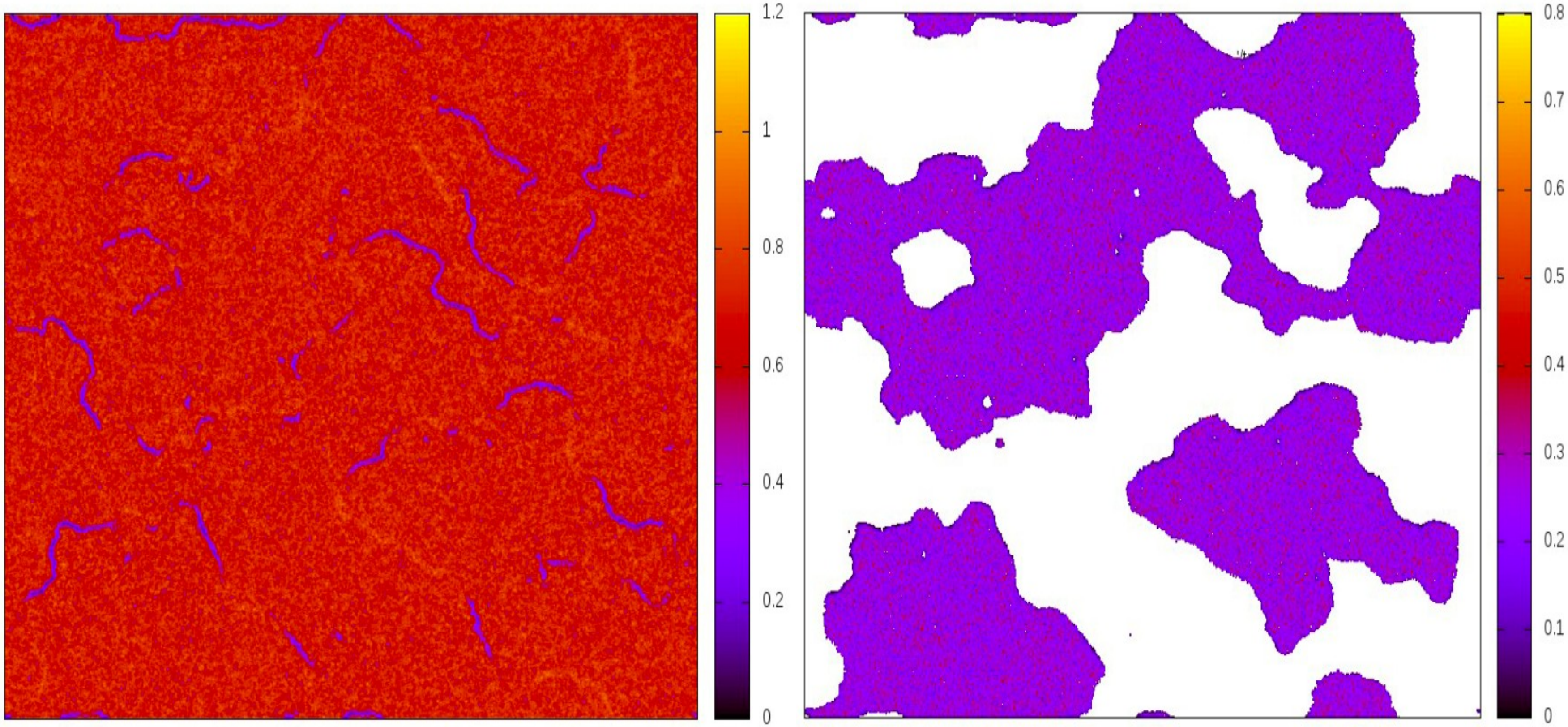
(video)

See also: http://www.thphys.uni-heidelberg.de/~sexty/videos

TG, B. Nowak, D. Sexty, arXiv:1108.0541 [hep-ph]

Strong Turbulence = Charge Separation

Modulus of complex field $|\varphi|$ vs. mean charge distribution

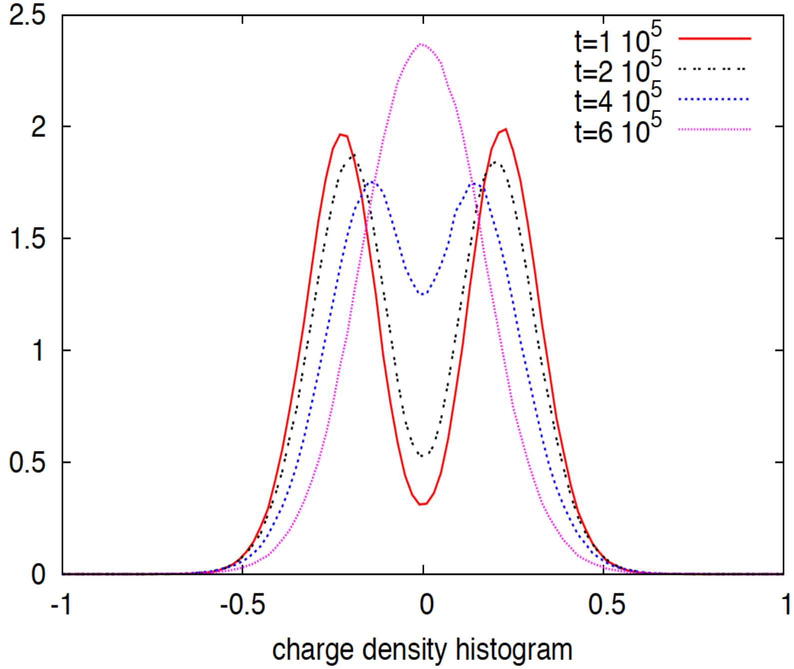


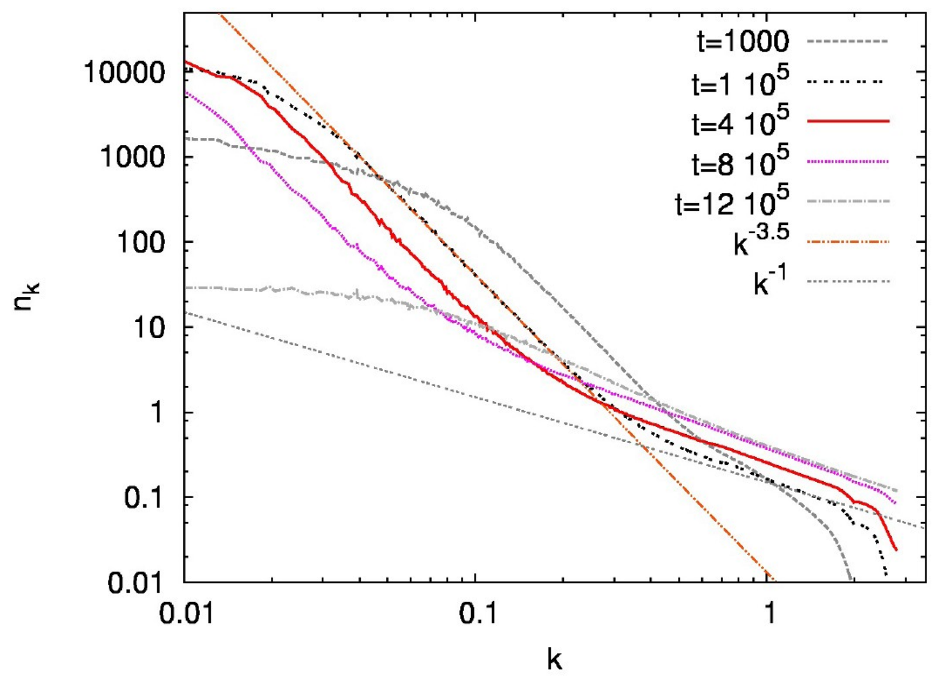
TG, B. Nowak, D. Sexty, arXiv:1108.0541 [hep-ph]
cf. also Tkachev, Kofman, Starobinsky, Linde (1998)

Strong Turbulence = Charge Separation

Charge density distribution vs. power spectrum

(d = 2, N = 2)

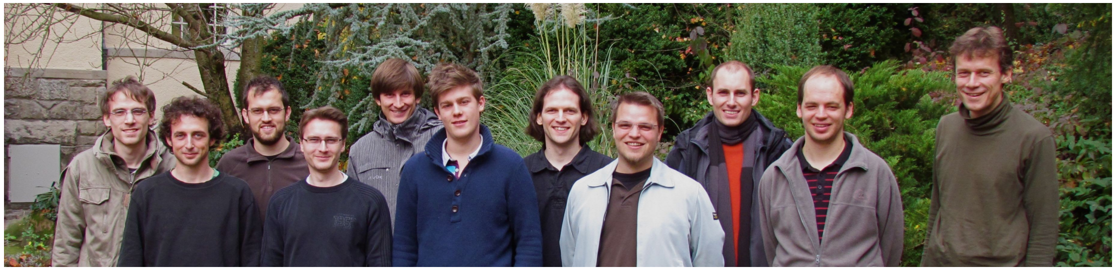




TG, B. Nowak, D. Sexty, arXiv:1108.0541 [hep-ph]


Have a non-turbulent flight home!

Thanks & credits to...



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€€€...

DFG Heisenberg-Programm Deutsche Forschungsgemeinschaft

Ruprecht-Karls-Universität Heidelberg

LGFG BaWue

DAAD Deutscher Akademischer Austausch Dienst German Academic Exchange Service

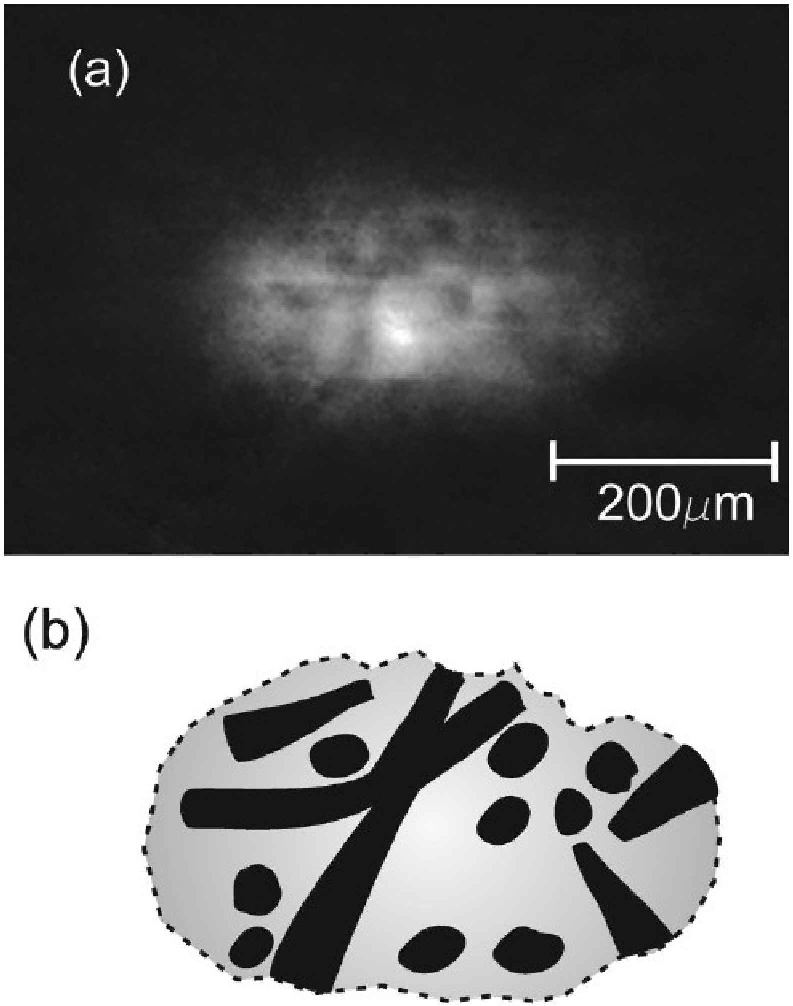
Center for Quantum Dynamics

EMMI

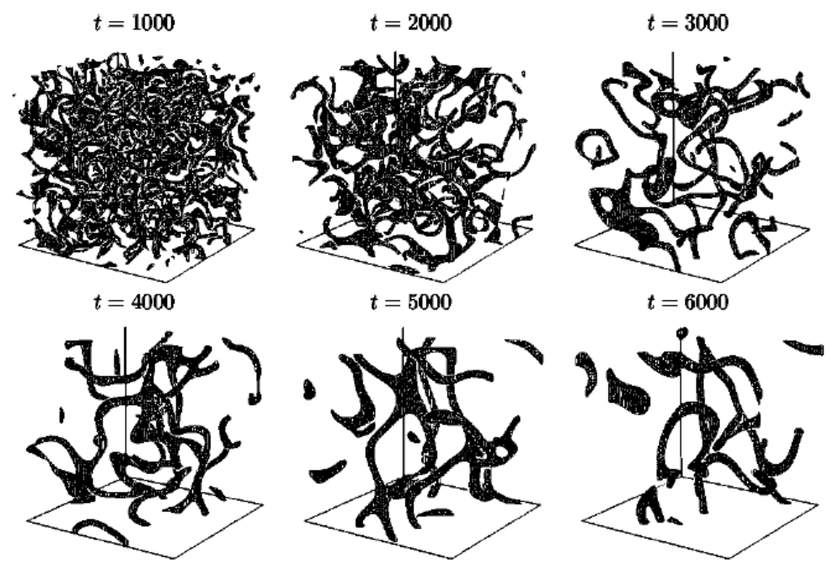
Helmholtz Gemeinschaft

Supplementary slides

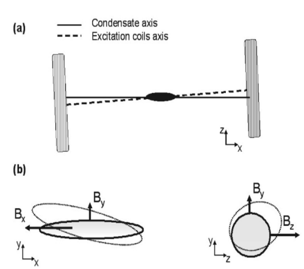
Vortex tangles in Bose Einstein Condensates



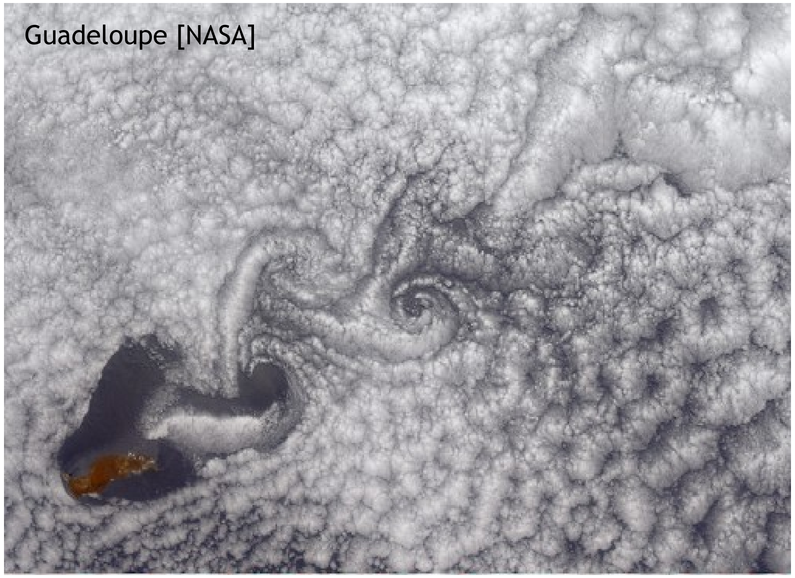
[E.A.L. Henn et al. PRL **103** (09)]



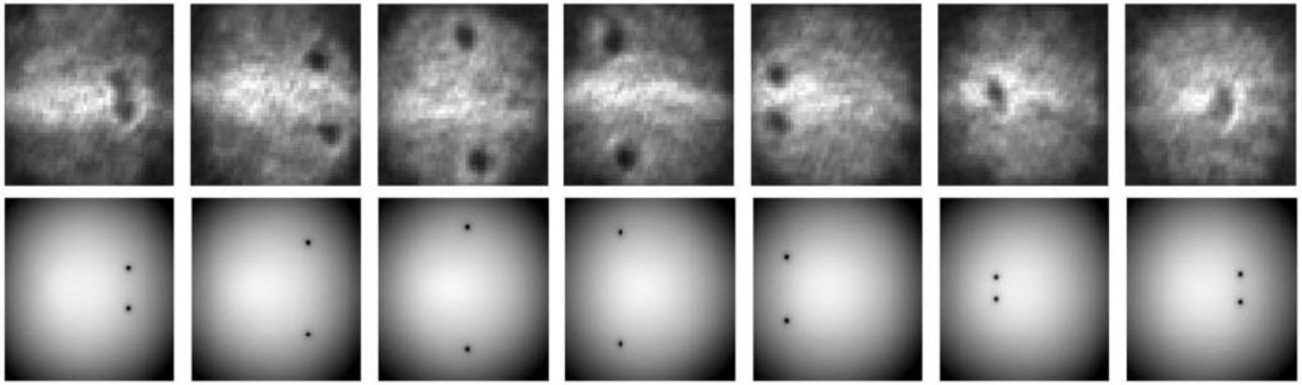
[N. Berloff & B. Svistunov, PRA (02)]

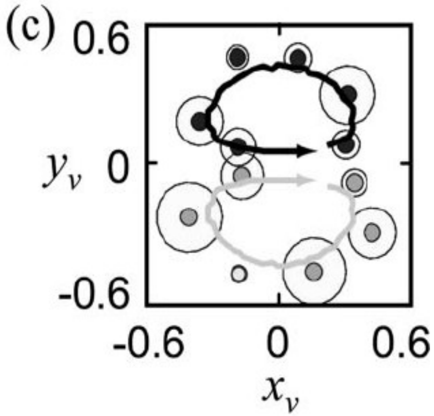


Vortex pairs



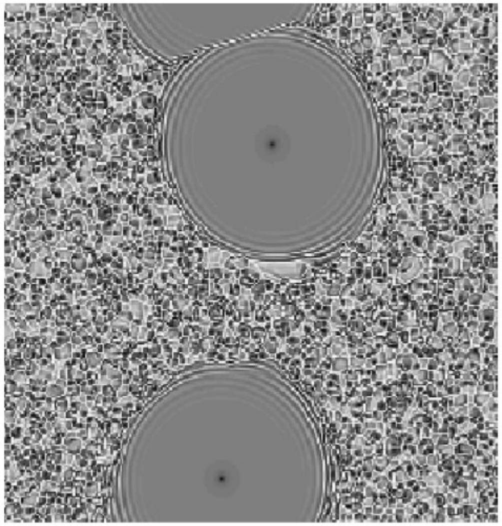
Tucson [AZ]





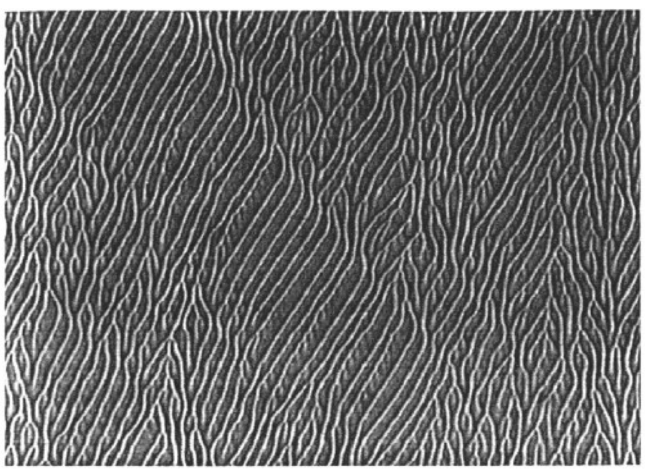
[T.W. Neely et al. PRL **104** (10)]

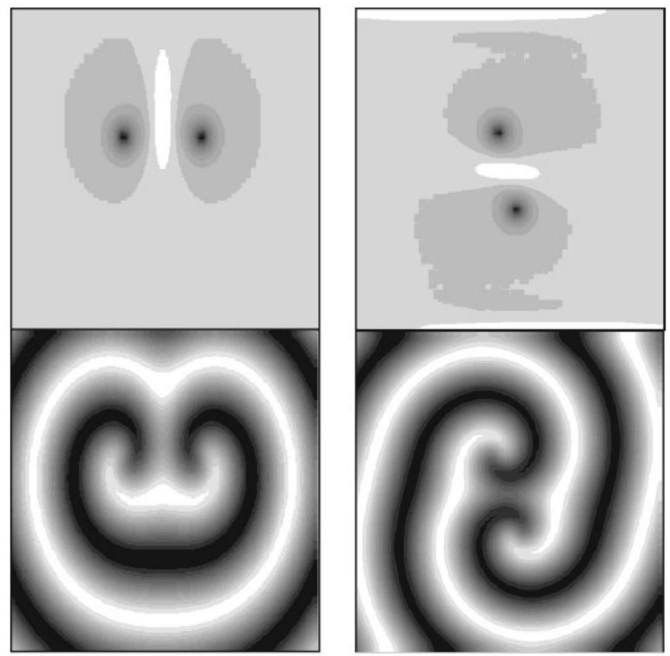
Nonlinear dynamics: Pattern formation



I. S. Aranson and L. Kramer: The complex Ginzburg-Landau equation

REVIEWS OF MODERN PHYSICS, VOLUME 74, JANUARY 2002





Nonlinear dynamics: Pattern formation

Visualization of spiral and scroll waves in simulated and experimental cardiac tissue

E M Cherry and F H Fenton
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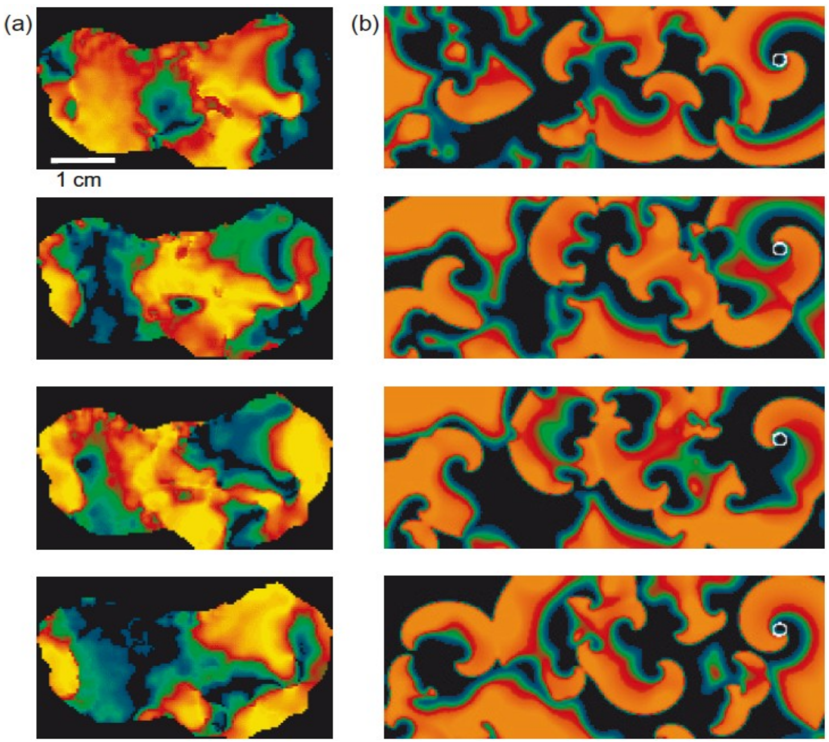
and
Max Planck Institute for Dynamics and Self-organization, Göttingen, Germany
E-mail: elizabeth.m.cherry@cornell.edu and flavio.h.fenton@cornell.edu

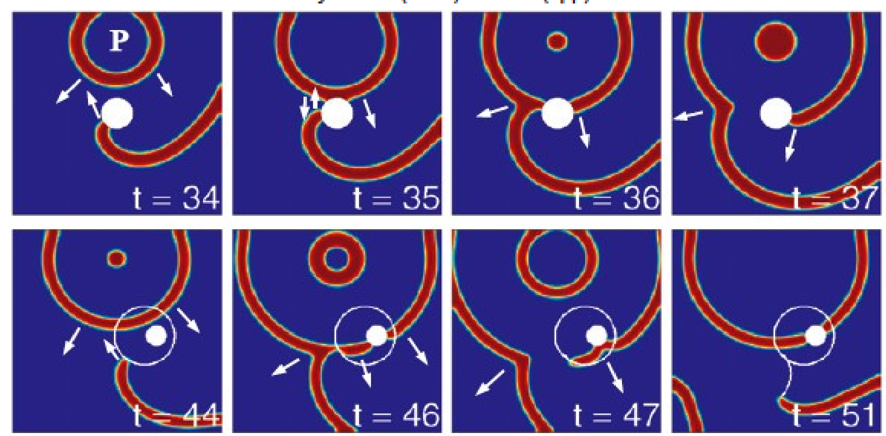
New Journal of Physics **10** (2008) 125016 (43pp)



Far field pacing supersedes anti-tachycardia pacing in a generic model of excitable media

Philip Bittihn[1,2,4], Gisela Luther[2], Eberhard Bodenschatz[2], Valentin Krinsky[2,3], Ulrich Parlitz[1] and Stefan Luther[2]
[1] Drittes Physikalisches Institut, Göttingen University, Friedrich-Hund-Platz 1, 37077 Göttingen, Germany
[2] Max Planck Institute for Dynamics and Self-Organization, Bunsenstraße 10, 37073 Göttingen, Germany
[3] Institut Non Linéaire de Nice, 1361 Rte des Lucioles, 06560 Valbonne/Sophia-Antipolis, France
E-mail: bittihn@physik3.gwdg.de

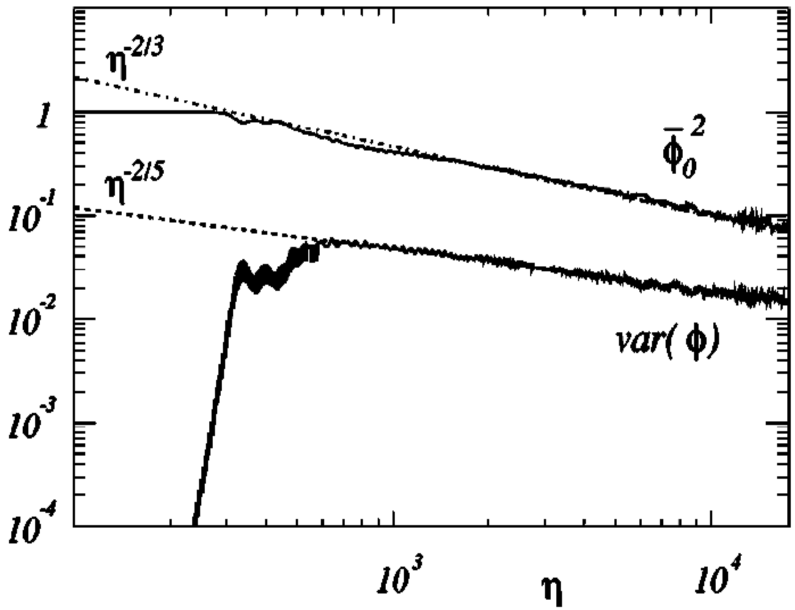
New Journal of Physics **10** (2008) 103012 (9pp)

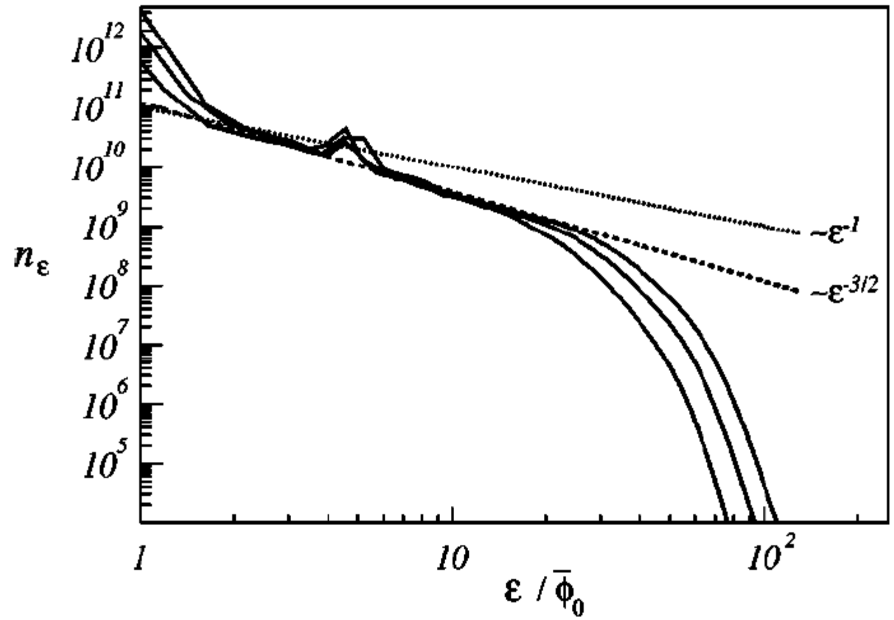


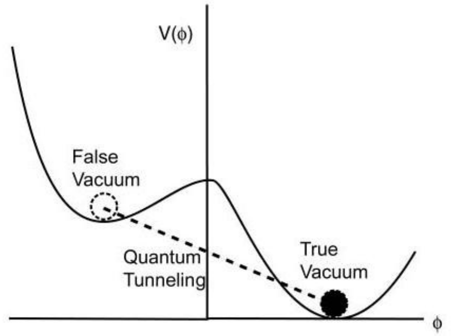
$$\frac{\partial u}{\partial t} = \varepsilon^{-1} u(1-u)\left(u - \frac{v+b}{a}\right) + \nabla^2 u$$

Barkley model

Wave turbulence







Turbulent thermalisation after universe inflation

[Micha & Tkachev, PRL **90** (03) 121301, PRD **70** (04) 043538]

Acoustic turbulence

Decomposition of Energy

$$E_{tot} = \int \left(\frac{1}{2} |\nabla \sqrt{n} e^{-i\varphi}|^2 + \frac{1}{2} g n^2 \right) d\boldsymbol{\rho}$$

$$= E_{kin} + E_q + E_{int}$$

$$\mathbf{u}(\boldsymbol{\rho}, t) = \nabla \varphi(\boldsymbol{\rho}, t)$$

$$E_{kin} = \frac{1}{2} \int |\sqrt{n}\mathbf{u}|^2 d\boldsymbol{\rho} \quad = E_{kin}^i + E_{kin}^c$$

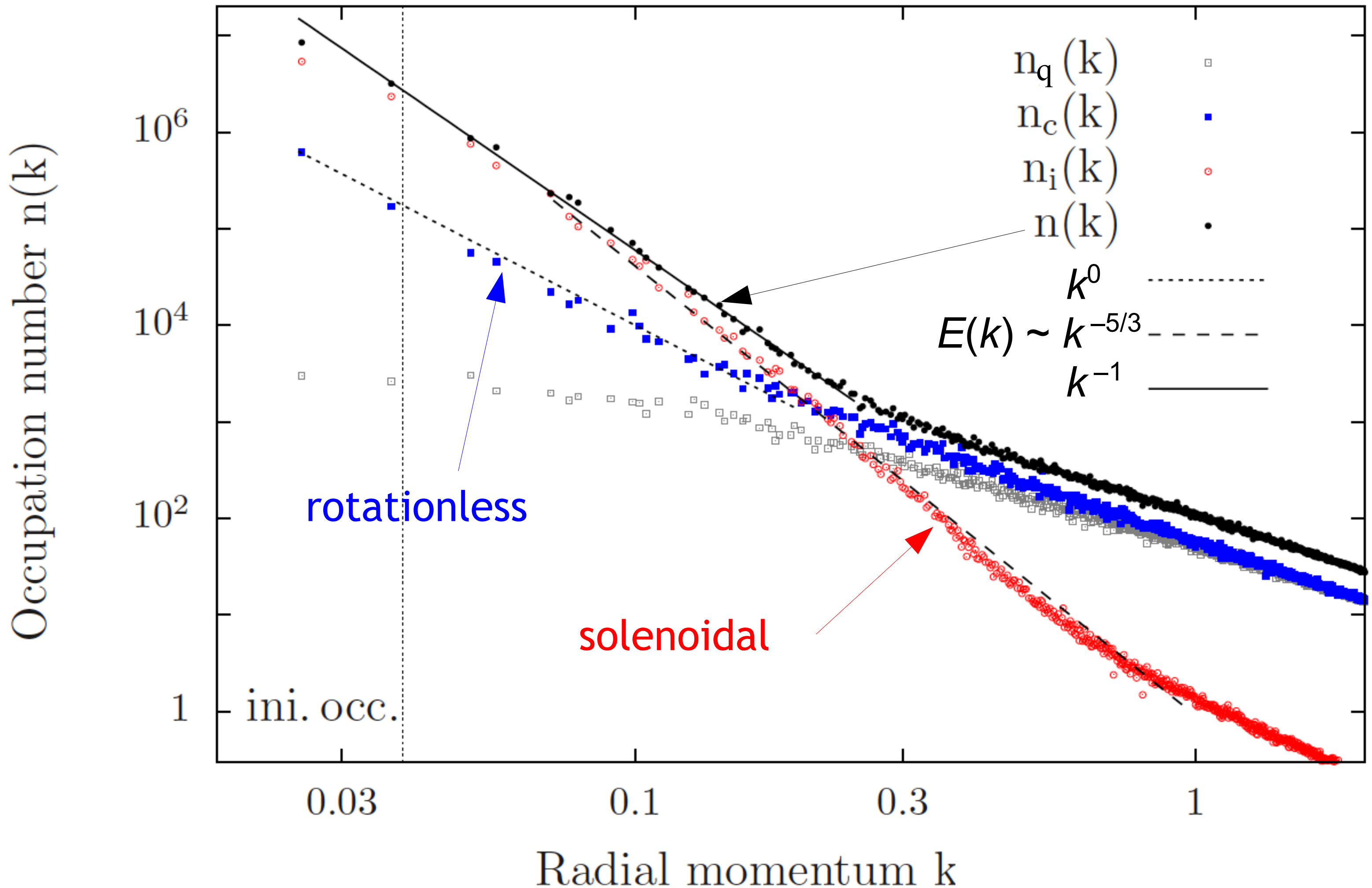
$$\nabla \times (\sqrt{n}\mathbf{u})^c = 0$$

$$\nabla \cdot (\sqrt{n}\mathbf{u})^i = 0$$

$$E_q = \frac{1}{2} \int (\nabla \sqrt{n})^2 d\boldsymbol{\rho}$$

Simulations in 2+1 D

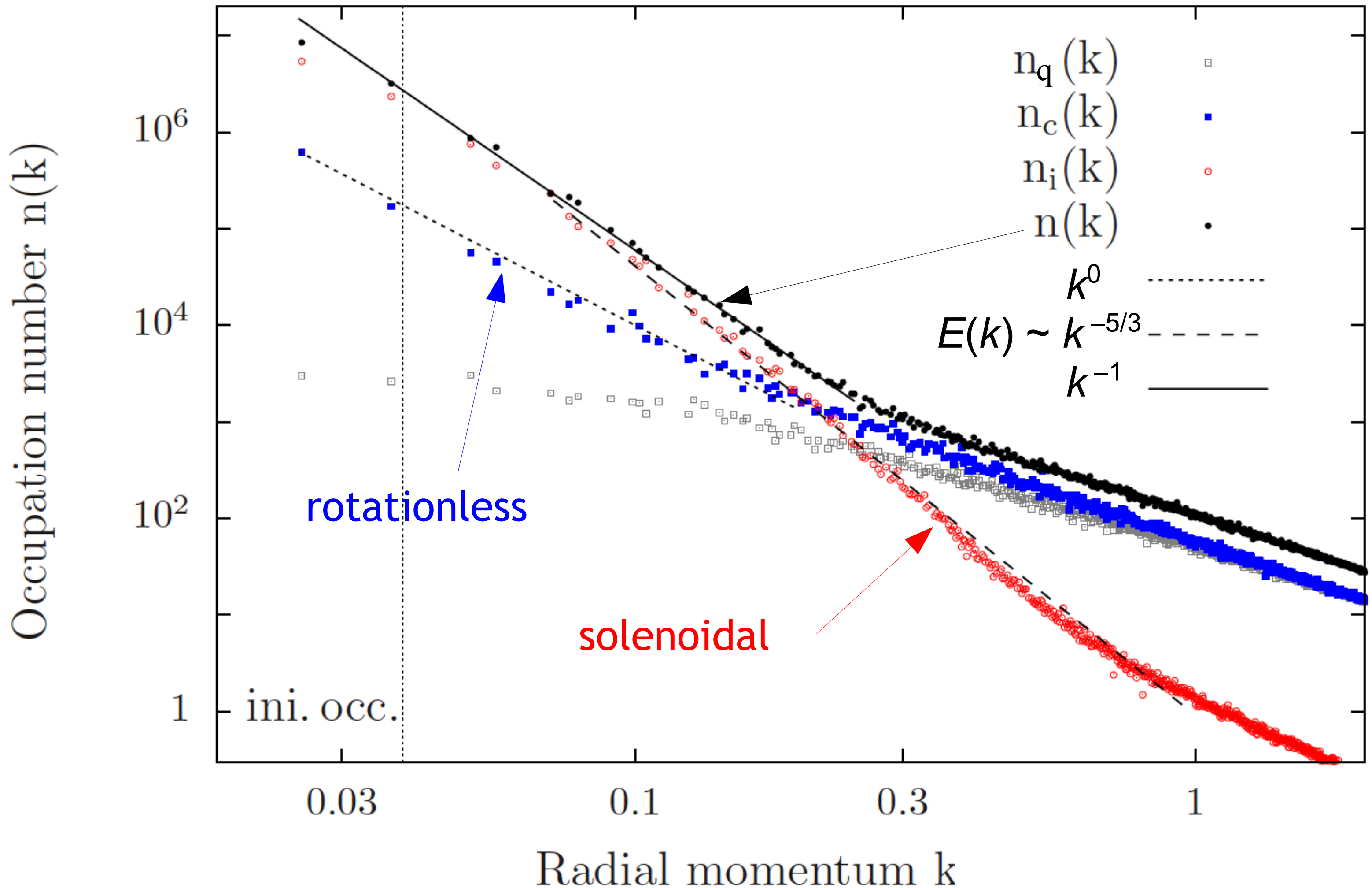
$$E(k) = \omega(k) k^{d-1} n(k)$$



B. Nowak, D. Sexty, TG (arXiv:1012.4437), PRB tbp.

Simulations in 2+1 D

$$E(\mathrm{k}) = \omega(\mathrm{k})\mathrm{k}^{d-1}\mathrm{n}(\mathrm{k})$$



B. Nowak, D. Sexty, TG (arXiv:1012.4437), PRB tbp.

Simulations in 2+1 D

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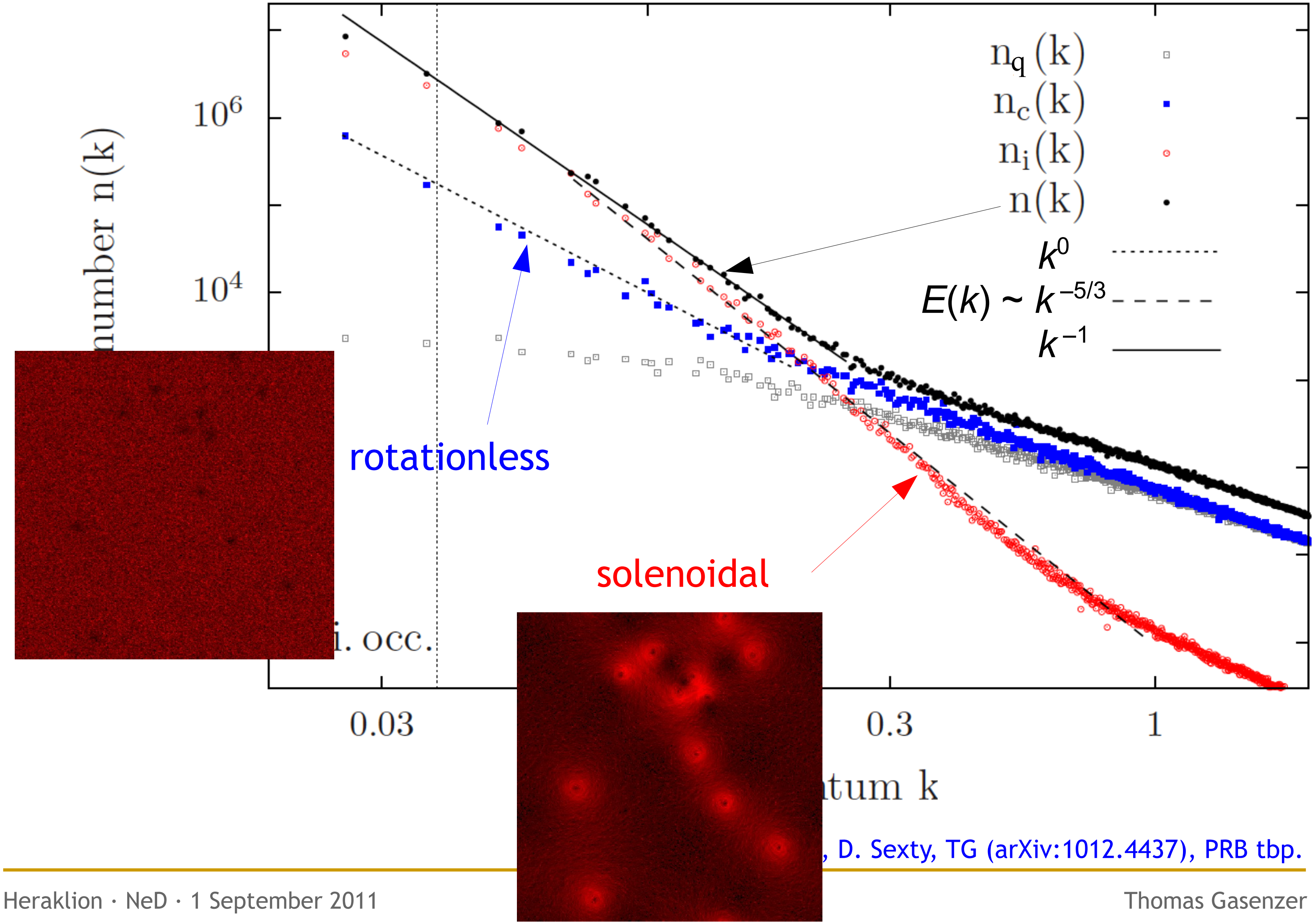
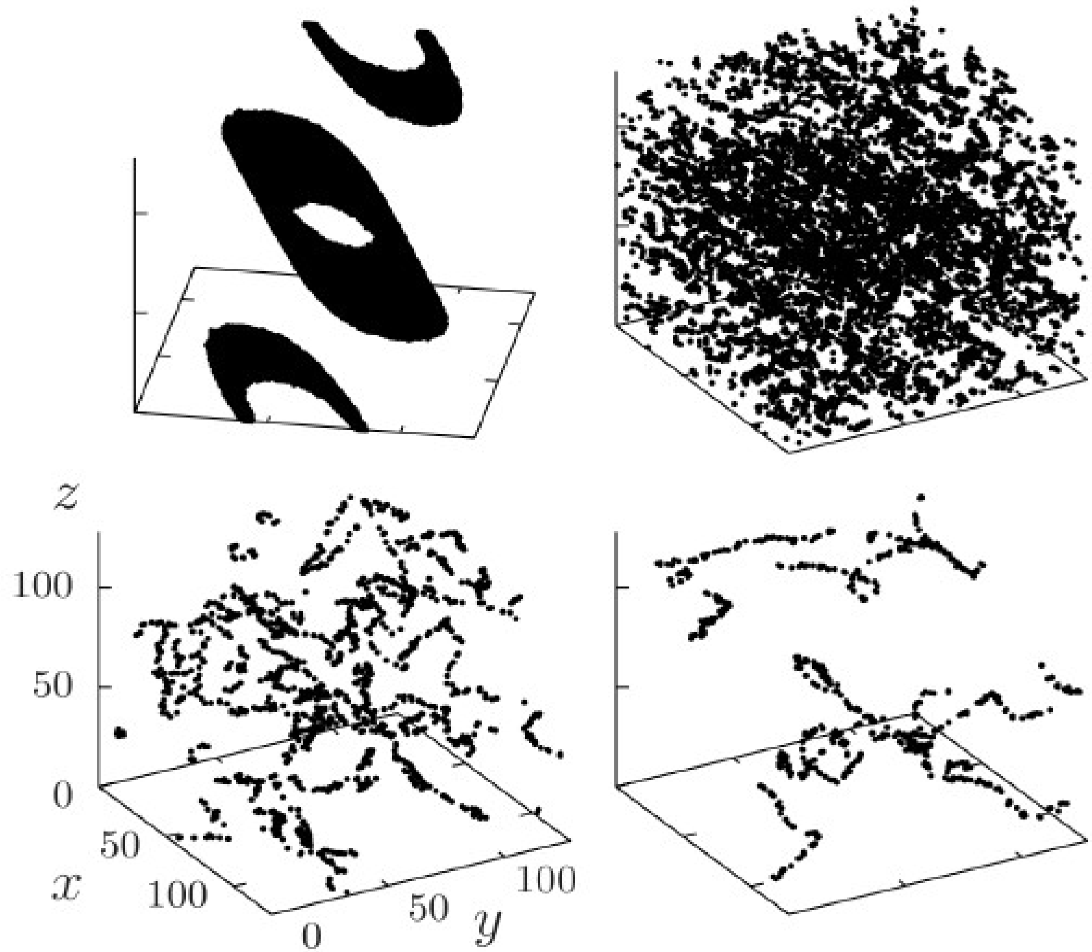


, D. Sexty, TG (arXiv:1012.4437), PRB tbp.

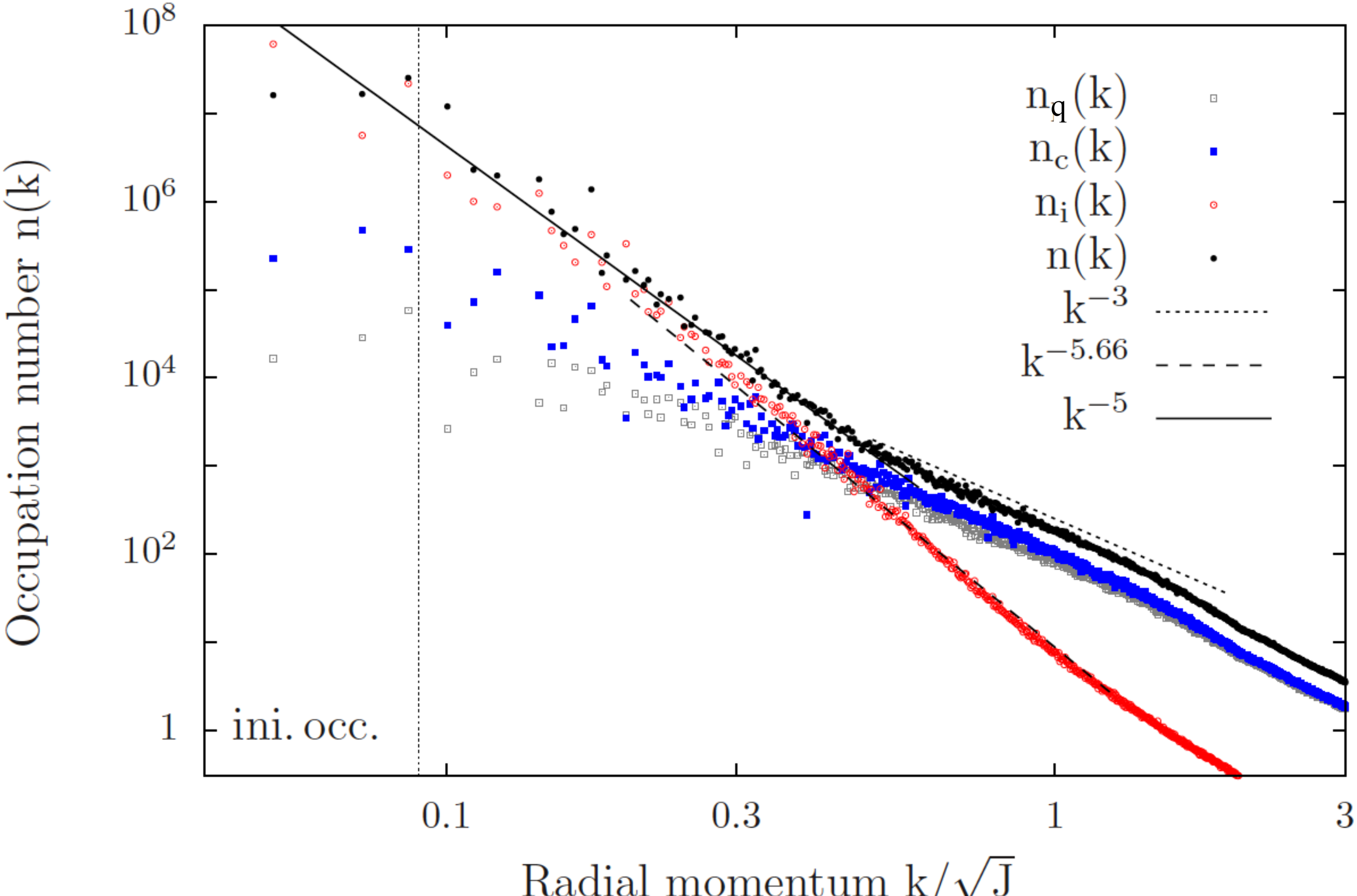
Simulations in 3+1 D



B. Nowak, D. Sexty, TG (arXiv:1012.4437)

Simulations in 3+1 D

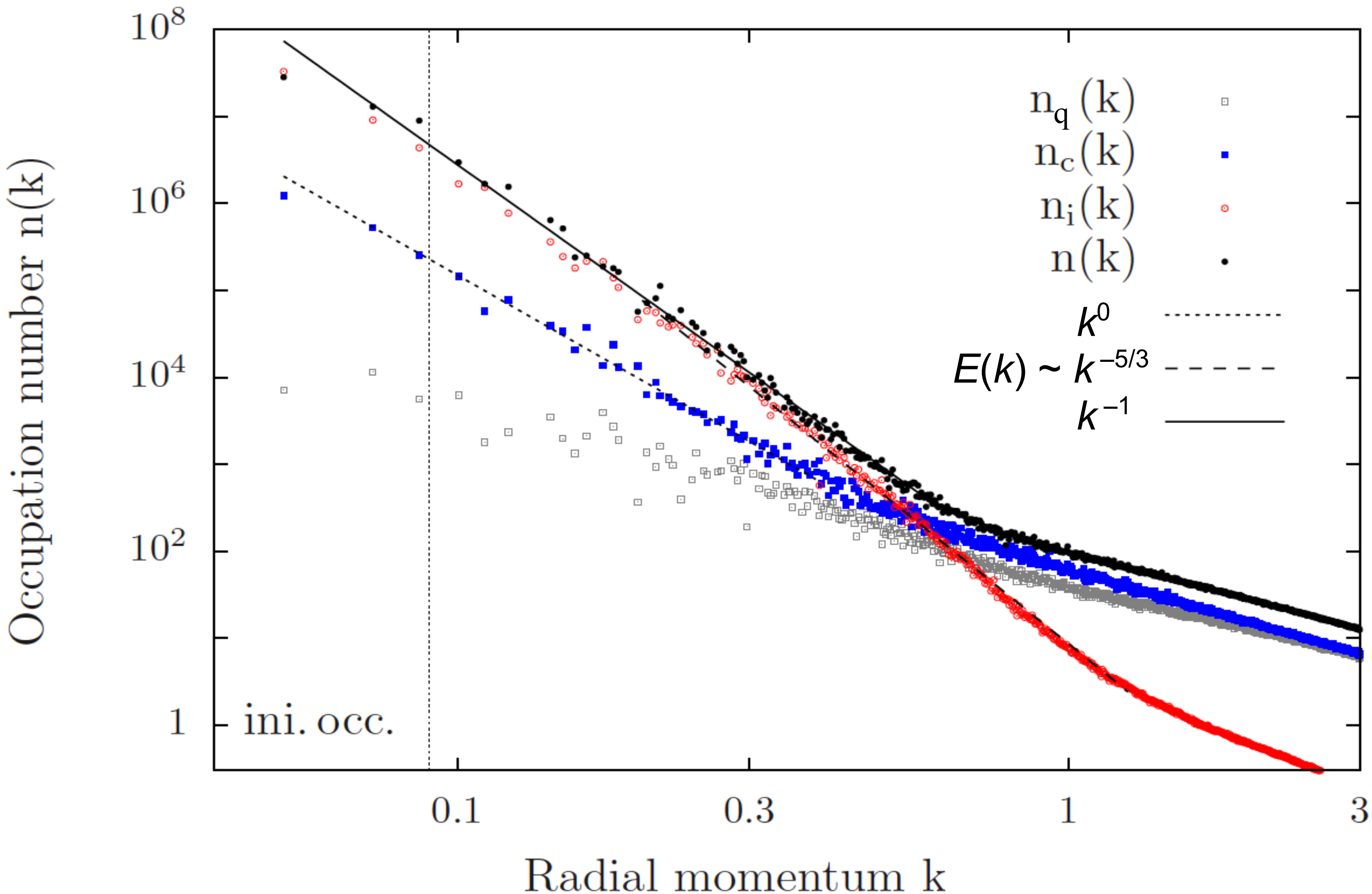
$$E(k) = \omega(k)k^{d-1}n(k)$$



B. Nowak, D. Sexty, TG (unpublished)

Simulations in 3+1 D

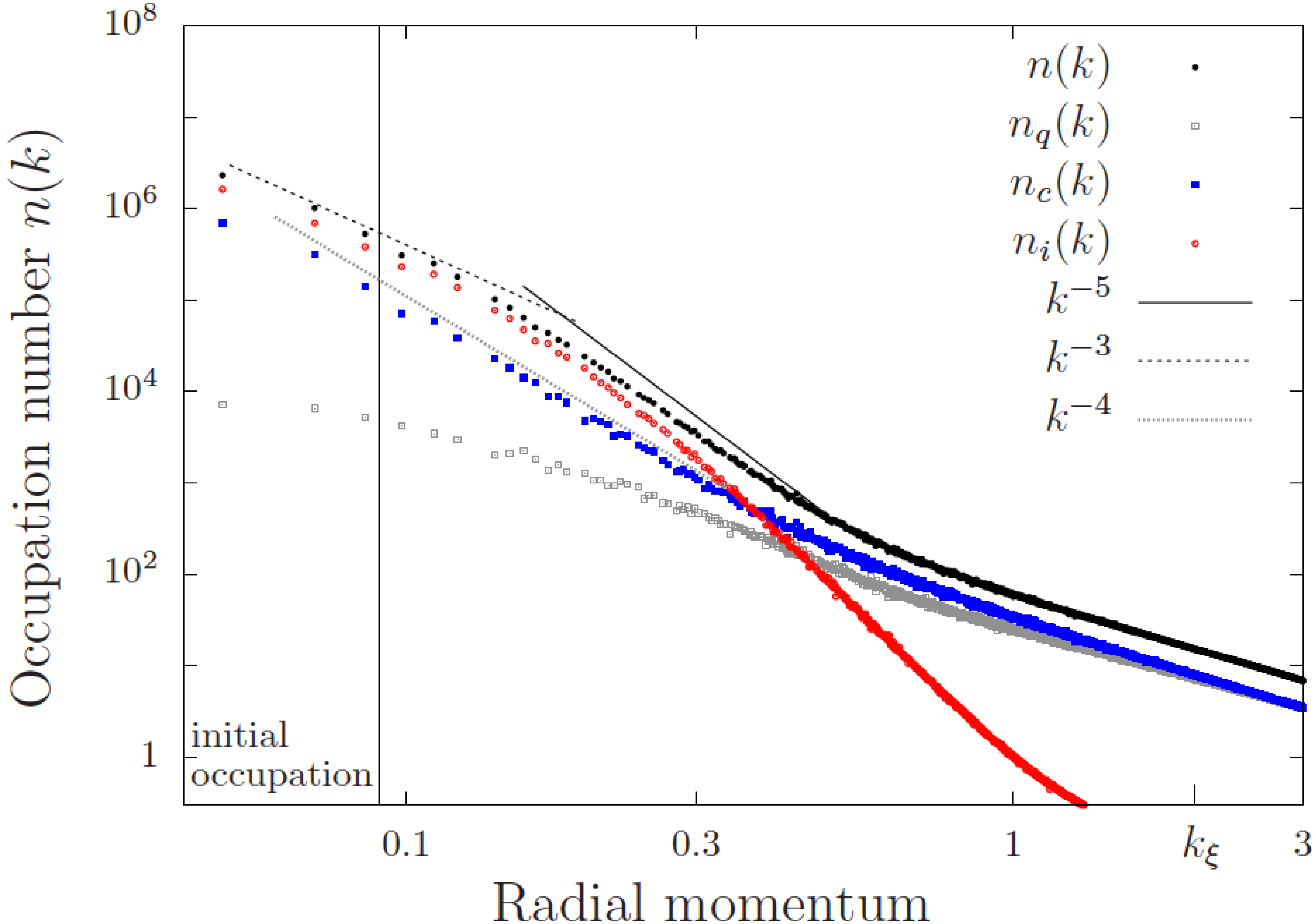
$$E(k) = \omega(k)k^{d-1}n(k)$$



B. Nowak, D. Sexty, TG (arXiv:1012.4437)

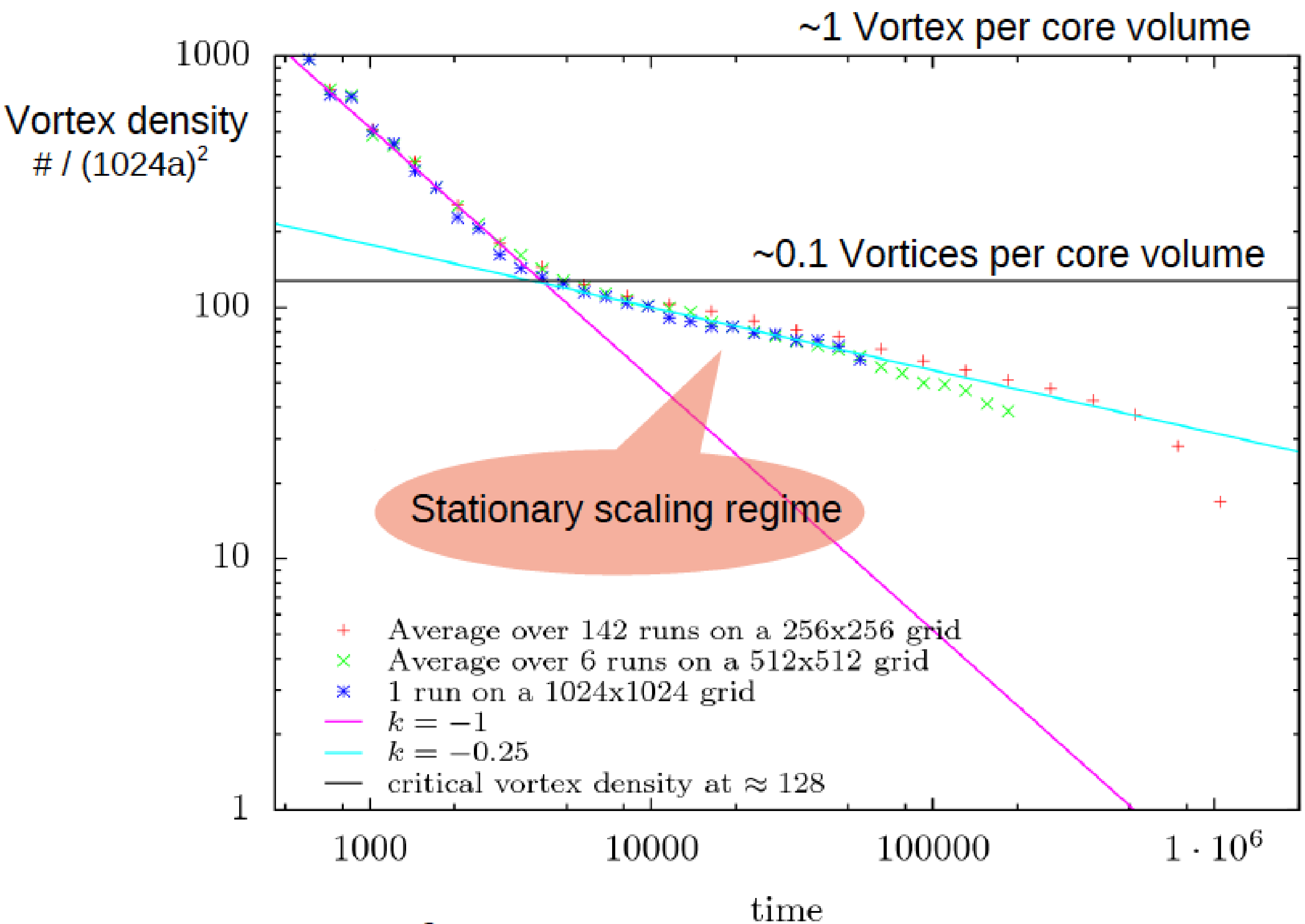
Late stage in 3+1 D

$$E(k) = \omega(k) k^{d-1} n(k)$$



J. Schole, B. Nowak, D. Sexty, TG (unpublished)

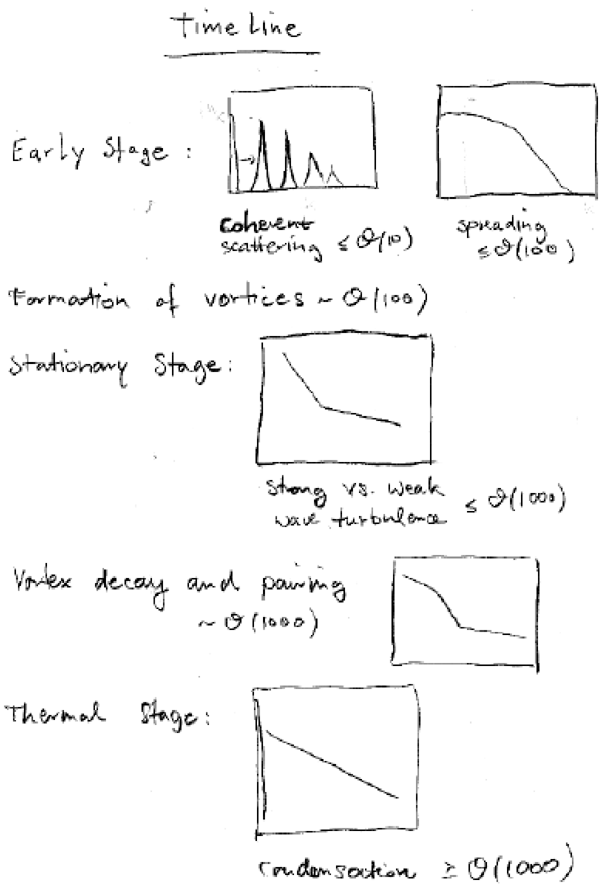
Time evolution of vortex density

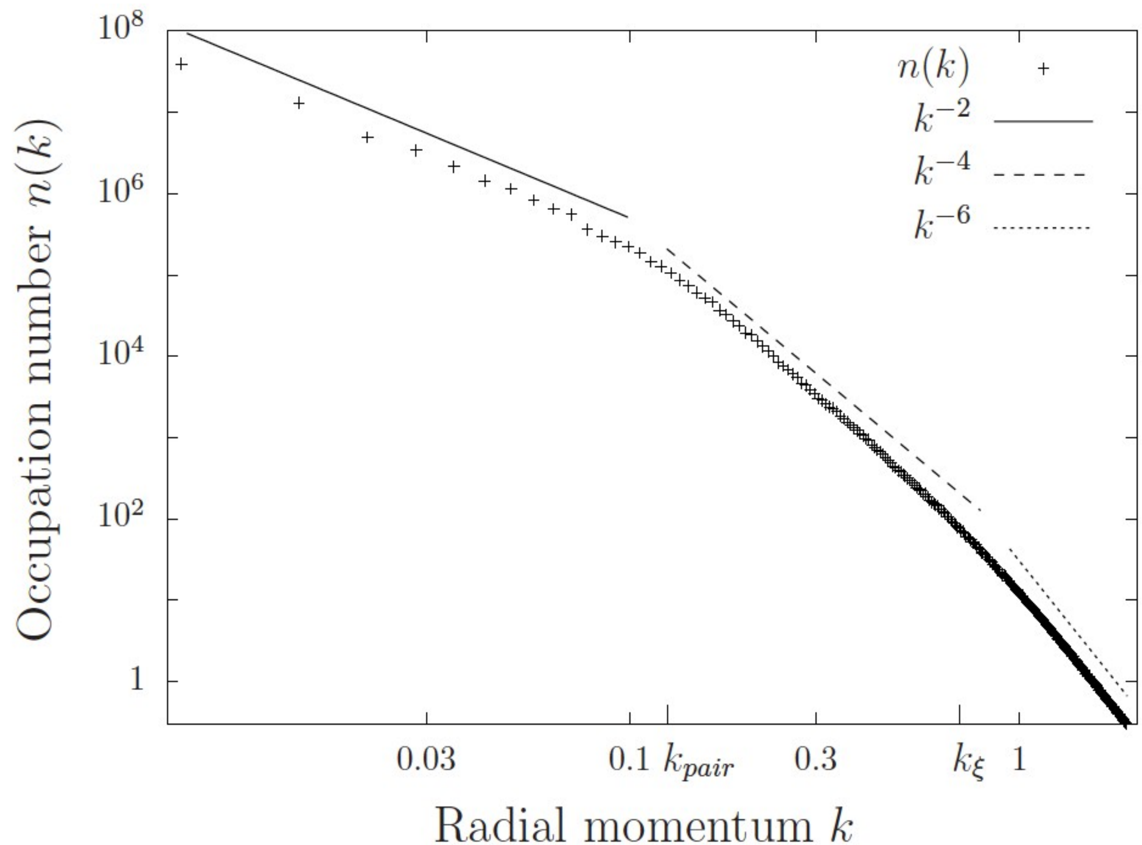


Core volume ~ $\pi(3\xi)^2$

J. Schole, B. Nowak, D. Sexty, TG (unpublished)

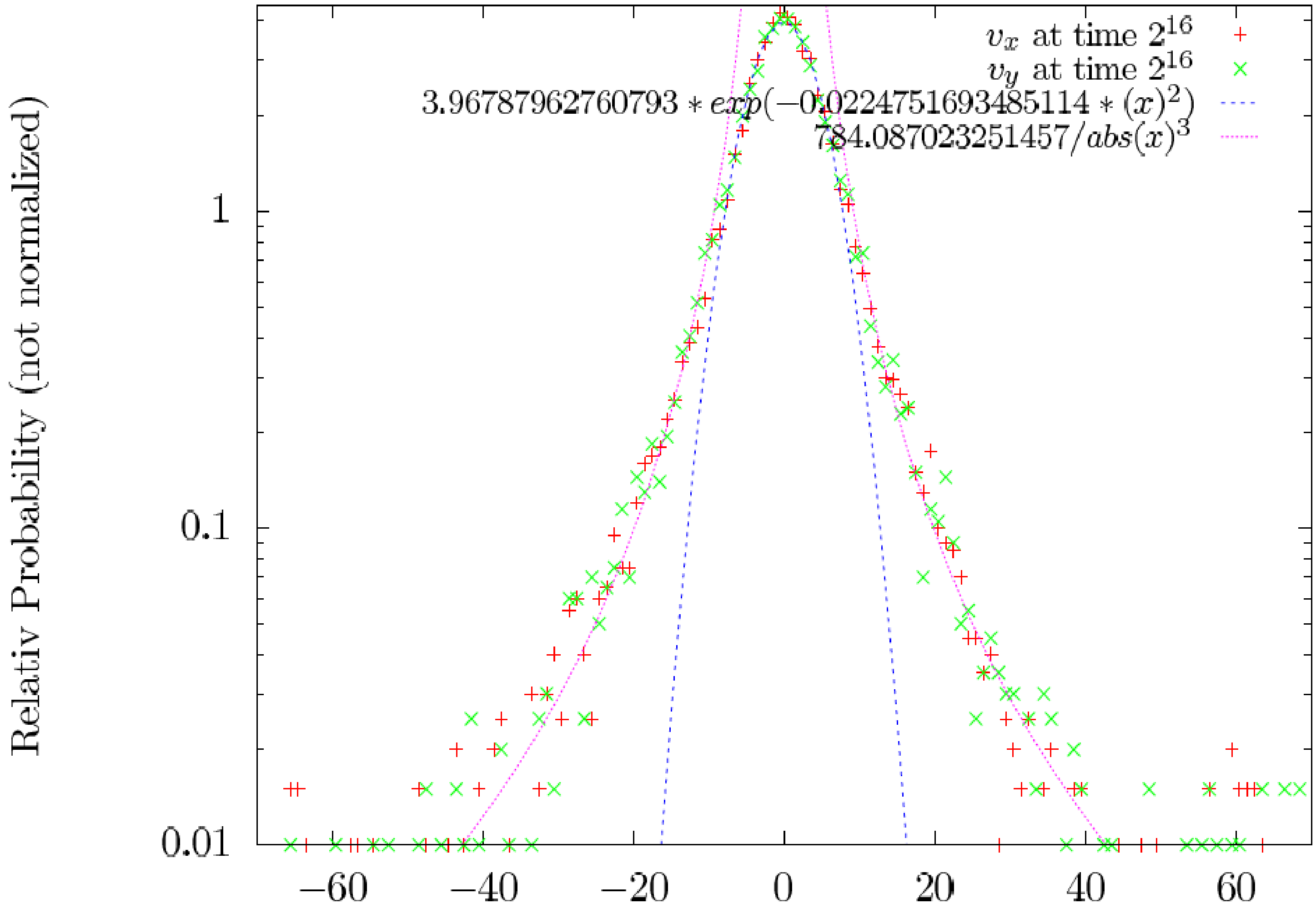
Time evolution of vortex density





J. Schole, B. Nowak, D. Sexty, TG (unpublished)

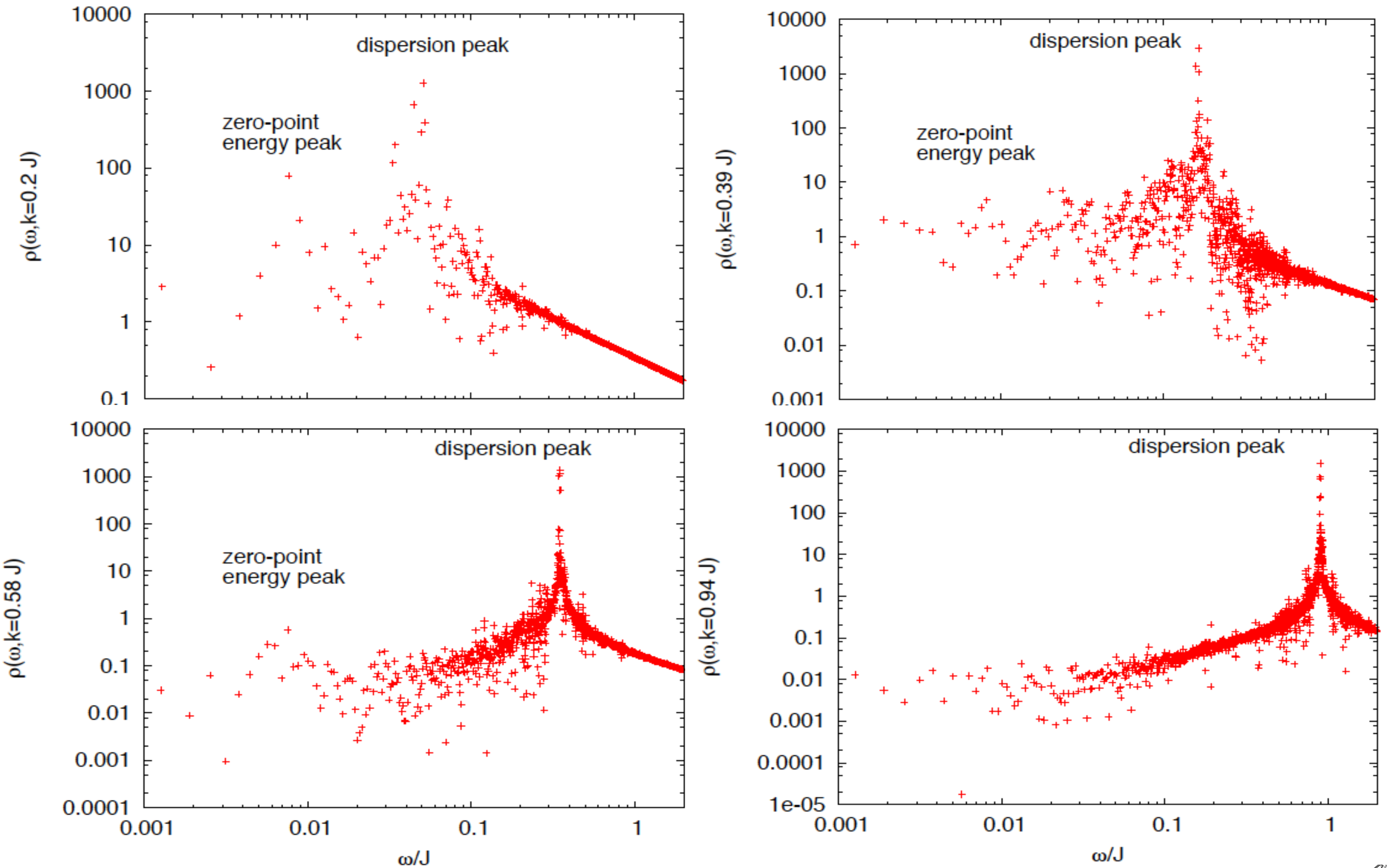
Vortex velocity distribution



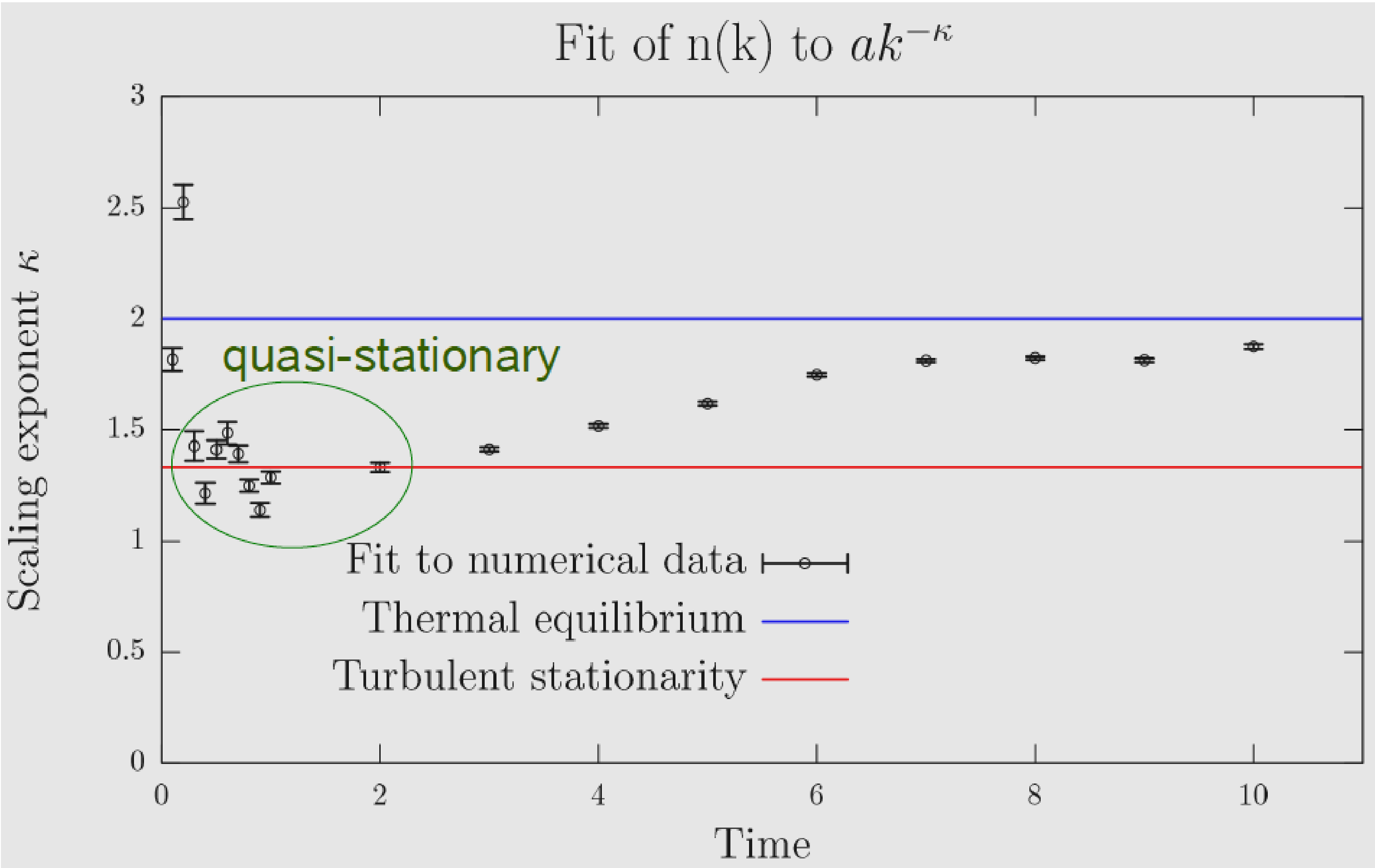
J. Schole, B. Nowak, D. Sexty, TG (unpublished)

Quantum Turbulence

Spectral functions

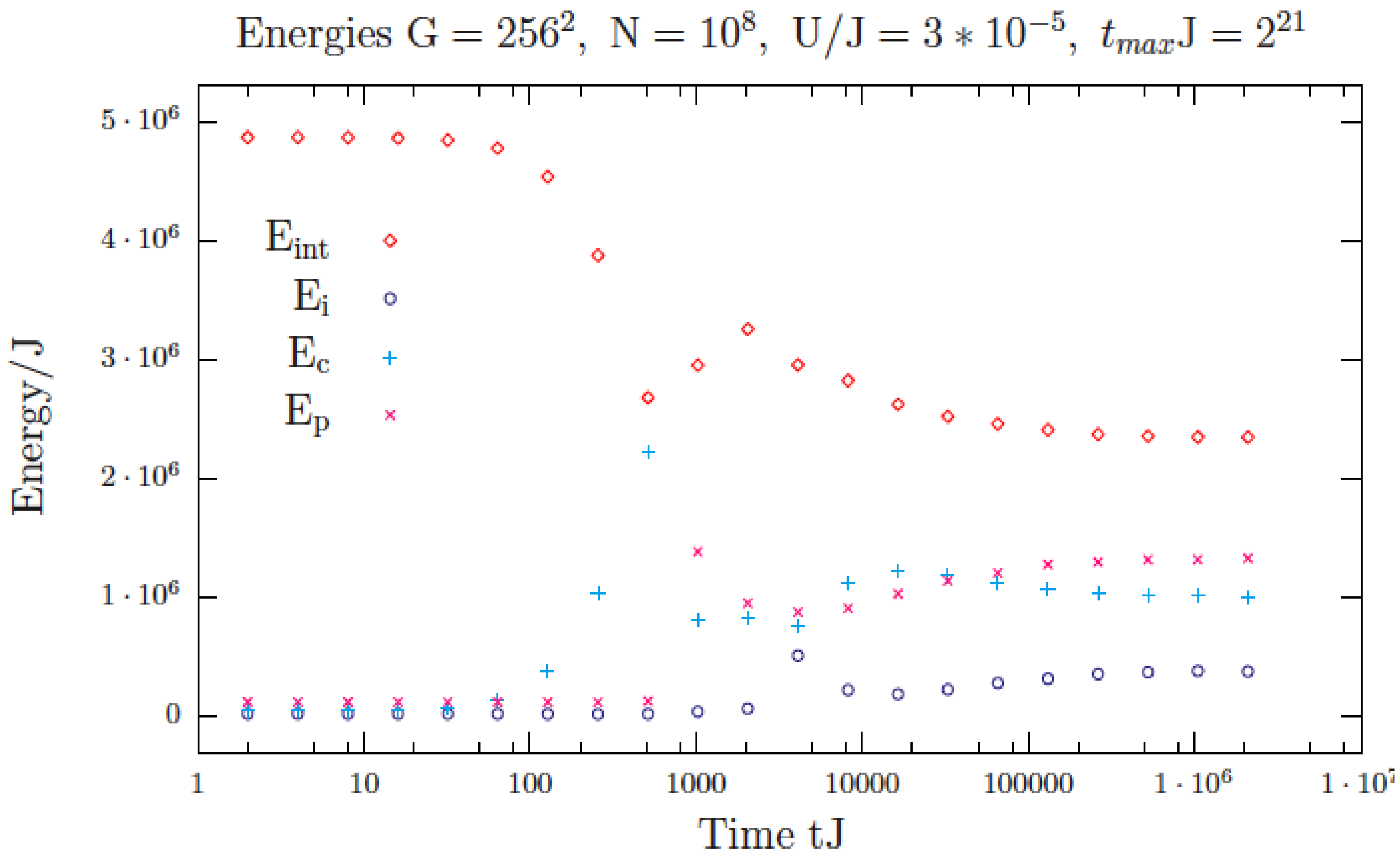


Simulations in 2+1 D (semi-classical)



B. Nowak, D. Sexty, TG (unpublished)

Energies



Energies $G = 256^2$, $N = 10^8$, $U/J = 3 * 10^{-5}$, $t_{max}J = 2^{21}$
Energy/J
$5 \cdot 10^6$
$4 \cdot 10^6$
$3 \cdot 10^6$
$2 \cdot 10^6$
$1 \cdot 10^6$
0
E_{int}
E_i
E_c
E_p
1
10
100
1000
10000
100000
$1 \cdot 10^6$
$1 \cdot 10^7$
Time tJ

Gross-Pitaevskii dynamics

Many-body Hamiltonian:

$$H = \int dx\, \hat{\Phi}^{\dagger}_{x} \left[-\frac{\hbar^2}{2m}\Delta + V_{\mathrm{ext}}(x) \right] \hat{\Phi}_{x} + \frac{1}{2}\int dx\, dy\, \hat{\Phi}^{\dagger}_{x}\hat{\Phi}^{\dagger}_{y}\, V_{x-y}\, \hat{\Phi}_{y}\hat{\Phi}_{x}$$

Gross-Pitaevskii, i.e. Classical Field Equation for "matter waves", from vNE:

$$i\partial_t \langle \mathcal{O} \rangle_t = \langle [\mathcal{O}, H] \rangle_t$$

from $\int V \langle \hat{\Phi}^{\dagger}\hat{\Phi}\hat{\Phi} \rangle$

$$\Rightarrow \quad i\hbar\partial_t \phi_x = \left[-\frac{\hbar^2}{2m}\Delta + V_{\mathrm{ext}}(x) + g|\phi_x|^2 \right] \phi_x,$$

$$g = \frac{4\pi\hbar^2 a}{m}$$

typical scattering length: $a \simeq 5\ \mathrm{nm}$

typical bulk density: $n \simeq 10^{14}\ \mathrm{cm}^{-3}$

$\Rightarrow$ diluteness parameter: $na^3 \simeq 10^{-5}$

(GPE valid for $\sqrt{na^3} \ll 1$

$\Leftrightarrow$ small condensate depletion)

Local radial flux only

With kinetic (Boltzmann) eq.

$$\partial_t n(k) = -\partial_k Q(k) \sim k^{d-1} J(k)$$

Scattering integral:

$$J(\mathbf{k},t) = g^2 \int \mathrm{d}^d p\, \mathrm{d}^d q\, \mathrm{d}^d r\, |T_{\mathbf{kpqr}}|^2 \delta(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r}) \quad 3d \;\; -d$$

$$\times\ \delta(\omega_\mathbf{k}+\omega_\mathbf{p}-\omega_\mathbf{q}-\omega_\mathbf{r}) \quad -2$$

$$\times\ [(n_\mathbf{k}+1)(n_\mathbf{p}+1)n_\mathbf{q}n_\mathbf{r} - n_\mathbf{k}n_\mathbf{p}(n_\mathbf{q}+1)(n_\mathbf{r}+1)] \quad -3\zeta$$

Radial flux density is k-independent, $Q(k) \equiv Q$, if:

$$d - 1 + 1 + 3d - d - 2 - 3\zeta = 0$$

Local radial flux only

$$n_k \sim k^{-\zeta}$$

Radial transport equation:

$$\partial_t n(k) = -\partial_k Q(k) \sim k^{d-1} J(k)$$

Quantum Boltzmann Scattering integral:

$$J(\mathbf{k},t) = g^2 \int \mathrm{d}^d p\, \mathrm{d}^d q\, \mathrm{d}^d r\, |T_{\mathbf{kpqr}}|^2 \delta(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r}) \quad 3d \;\; -d$$
$$\times\; \delta(\omega_\mathbf{k}+\omega_\mathbf{p}-\omega_\mathbf{q}-\omega_\mathbf{r}) \quad -2$$
$$\times\; [(n_\mathbf{k}+1)(n_\mathbf{p}+1)n_\mathbf{q}n_\mathbf{r} - n_\mathbf{k}n_\mathbf{p}(n_\mathbf{q}+1)(n_\mathbf{r}+1)] \quad -3\zeta$$

Radial flux density is k-independent, $Q(k) \equiv Q$, if:

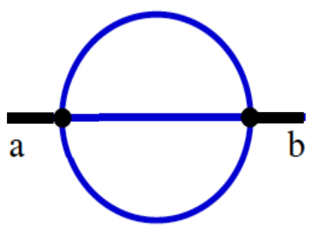
$$\cancel{d} - \cancel{1} + \cancel{1} + 3d - \cancel{d} - 2 - 3\zeta = 0 \;\Rightarrow\; \zeta = d - 2/3$$

Dynamical field theory

Kinetic (Quantum-Boltzmann) eq.: $\partial_t n_{\mathbf{k}} = J(\mathbf{k}, t)$

Dynamic (Schwinger-Dyson) eq.: (from 2PI effective action)

$$\partial_t n(p) \quad = \quad \Sigma^{\rho}_{ab}(p) F_{ba}(p) - \Sigma^{F}_{ab}(p) \rho_{ba}(p)$$

$\Sigma_{\mathrm{ab}}(x,y) =$ 

$p = (p_0, \mathbf{p})$:

$F_{ab}(x,y) = \frac{1}{2}\langle\{\Phi_a(x), \Phi_b(y)\}\rangle_c$ Statistical function: Occupation

$\rho_{ab}(x,y) = i\langle[\Phi_a(x), \Phi_b(y)]\rangle.$ Spectral function: Available modes

$n_{\mathrm{BE}}(\omega) = 1/(e^{\beta(\omega-\mu)} - 1)$

$$F^{(\mathrm{th})}_{ab}(\omega, \mathbf{p}) = -i\left(n_{\mathrm{BE}}(\omega) + \frac{1}{2}\right)\rho^{(\mathrm{th})}_{ab}(\omega, \mathbf{p})$$

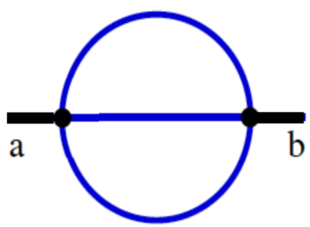
Dynamical field theory

Kinetic (Quantum-Boltzmann) eq.: $\partial_t n_{\mathbf{k}} = J(\mathbf{k}, t)$

Dynamic (Schwinger-Dyson) eq.: (from 2PI effective action)

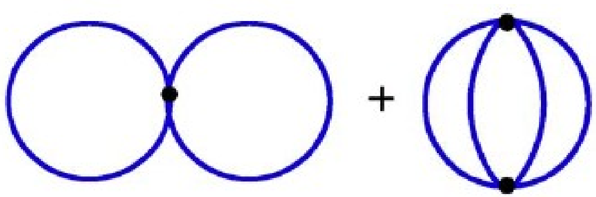
$$\partial_t n(p) \;=\; \Sigma^{\rho}_{ab}(p)F_{ba}(p) - \Sigma^{F}_{ab}(p)\rho_{ba}(p)$$

$\Sigma_{ab}(x,y)$ =



$p = (p_0, \mathbf{p})$:

$\Gamma_2^{3\mathrm{loop}}[\phi,G]$ =



Scaling solutions

We look for scaling solutions fulfilling stationarity condition $J(p) = 0$

Scaling ansatz:

$$\rho_{ab}(s^z p_0, s\mathbf{p}) = s^{-2+\eta}\rho_{ab}(p_0, \mathbf{p})$$
$$F_{ab}(s^z p_0, s\mathbf{p}) = s^{-2-\kappa}F_{ab}(p_0, \mathbf{p})$$

Implies scaling of the single-particle momentum distribution:

$$n(s\mathbf{p}) = s^{z-2-\kappa}n(\mathbf{p})$$

$$= -\zeta$$

Local radial flux only

$$n_k \sim k^{-\zeta}$$

With kinetic (Boltzmann) eq.

$$T(k) \sim g \;/\; |1 + const. \times g\, k^{d-2} n(k)|$$

$$2 - d + \zeta$$

Scattering integral:

$$J(\mathbf{k},t) = g^2 \int \mathrm{d}^d p\, \mathrm{d}^d q\, \mathrm{d}^d r\, |T_{\mathbf{kpqr}}|^2 \delta(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r}) \times \delta(\omega_{\mathbf{k}} + \omega_{\mathbf{p}} - \omega_{\mathbf{q}} - \omega_{\mathbf{r}}) \times [(n_{\mathbf{k}}+1)(n_{\mathbf{p}}+1)n_{\mathbf{q}}n_{\mathbf{r}} - n_{\mathbf{k}}n_{\mathbf{p}}(n_{\mathbf{q}}+1)(n_{\mathbf{r}}+1)]$$

$$3d \;\; -d \;\; -2 \;\; -3\zeta$$

Radial flux density is k-independent, $Q(k) \equiv Q$, if:

$$d - 1 + 1 + 3d - d - 2 - 3\zeta + 2(2 - d + \zeta) = 0$$

(with d, -1, $+1$ and $-d$ struck out)

Local radial flux only

$n_k \sim k^{-\zeta}$

With kinetic (Boltzmann) eq.

$$T(k) \sim g \;/\; |1 + const. \times g\, k^{d-2} n(k)|$$

$2 - d + \zeta$

Scattering integral:

$$J(\mathbf{k},t) = g^2 \int \mathrm{d}^d p\, \mathrm{d}^d q\, \mathrm{d}^d r\, |T_{\mathbf{kpqr}}|^2 \delta(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r}) \times \delta(\omega_\mathbf{k}+\omega_\mathbf{p}-\omega_\mathbf{q}-\omega_\mathbf{r}) \times [(n_\mathbf{k}+1)(n_\mathbf{p}+1)n_\mathbf{q}n_\mathbf{r} - n_\mathbf{k}n_\mathbf{p}(n_\mathbf{q}+1)(n_\mathbf{r}+1)]$$

$3d \; -d$

-2

-3ζ

Radial flux density is k-independent, $Q(k) \equiv Q$, if:

$$d - 1 + 1 + 3d - d - 2 - 3\zeta + 2(2 - d + \zeta) = 0 \;\Rightarrow\; \zeta = d + 2$$

Scaling exponents
(in *d* dimensions)

C. Scheppach, J. Berges, T. Gasenzer PRA **81** (10) 033611

UV: $\zeta = d + (z - 2 + \eta)/3$ $\zeta = d - (2 - \eta)/3$

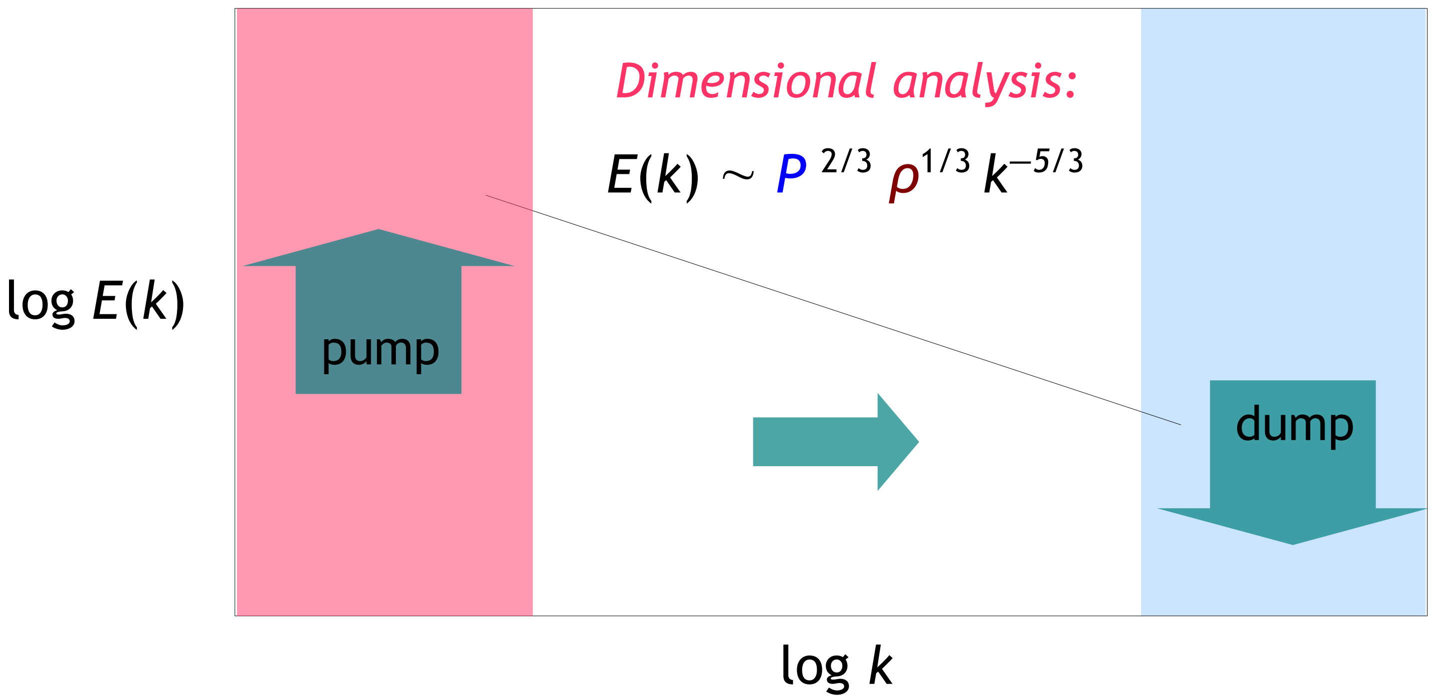
Constant $P(k) \equiv P$ Constant $Q(p)$

IR: $\zeta = d + z + 2 - \eta$ $\zeta = d + 2 - \eta$

$$n \sim k^{-\zeta}$$

Kolmogorov's theory of turbulence
(1941)

Scale invariant (self-similar) stationary transport:

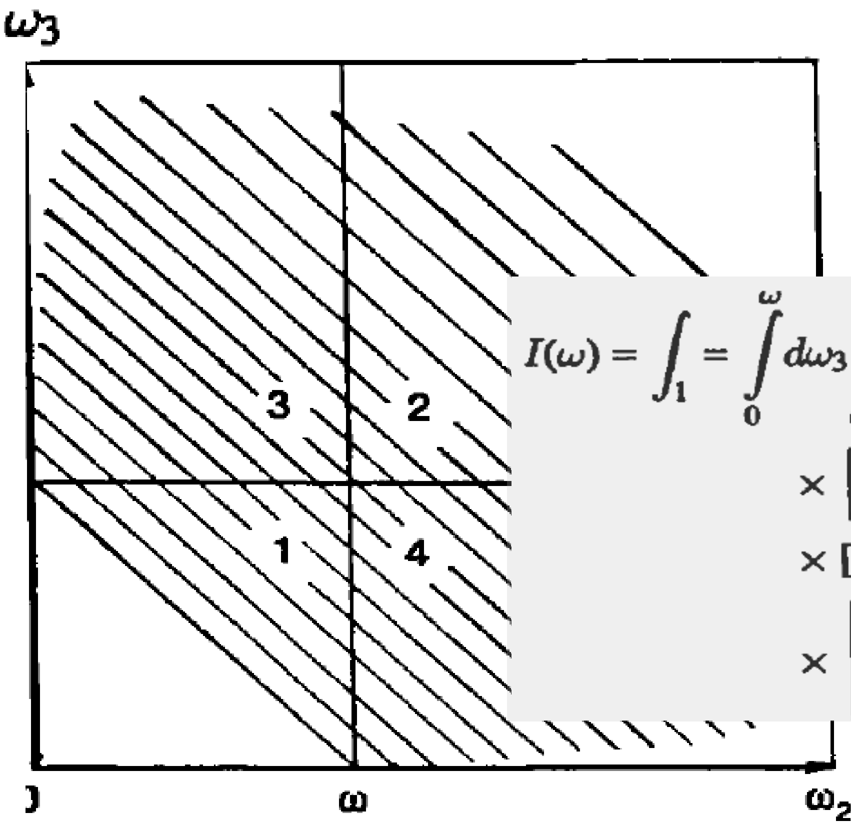


3D:	Radial energy density	E	[kg s^{-2}]
	Radial energy flux	P	[kg m^{-1} s^{-3}]
	Density	ρ	[kg m^{-3}]

To derive scaling

Familiarize the...

reader with a little miracle of the theory of wave turbulence theory, the so-called Zakharov transformations. They factorize the collision integral. As a result one



$$I(\omega) = \int_{\Omega} d\omega_2 d\omega_3 U(\omega, \omega_2+\omega_3-\omega, \omega_2, \omega_3) \times n(\omega_2) n(\omega_2+\omega_3-\omega) n(\omega) n(\omega_3) \times \left[n^{-1}(\omega) + n^{-1}(\omega_2+\omega_2-\omega) - n^{-1}(\omega_2) - n^{-1}(\omega_3)\right]$$

$$I(\omega) = \int_1 = \int_0^{\omega} d\omega_3 \int_{\omega-\omega_3}^{\omega} d\omega_2 U(\omega, \omega_2+\omega_3-\omega, \omega_2, \omega_3) \times \left[\omega^x + (\omega_2+\omega_3-\omega)^x - \omega_2^x - \omega_3^x)\right] \times [\omega(\omega_2+\omega_3-\omega)\omega_2\omega_3]^{-x} \times \left[1 + \left(\frac{\omega_2+\omega_3-\omega}{\omega}\right)^y - \left(\frac{\omega_2}{\omega}\right)^y - \left(\frac{\omega_3}{\omega}\right)^y\right] = 0$$

$$y = 3x - 3 - \left[\frac{2m+3d}{\alpha} - 4\right]$$

V.E. Zakharov, V.S. L'vov, G. Falkovich, *Kolmogorov Spectra of Turbulence* (Springer, Berlin, 1992)

Turbulence vs. Critical Phenomena

Universality from IR tuning	Universality from Microphys.
Spatial separation r	wave number k
Integral scale L	UV Cutoff Λ
Dissipation scale $k_\mathrm{d}=\eta^{-1}$	Correlation length ξ
Viscosity ν	Temperature $T - T_\mathrm{c}$
Intermittency exponent μ	Anomalous expt. η_c

Alternative: UV fixed point

Spatial separation r	Spatial separation r
Integral scale ξ_L	Correlation length ξ
Viscosity ν or dissip. Scale η	lattice spacing a

Real-time flow: Fully developed turbulence as an unstable fixed point.

Lewis Fry Richardson, FRS (1881-1953)

Big whirls have little whirls that feed on their velocity,
and little whirls have lesser whirls and so on to viscosity.

(L.F. Richardson, *The supply of energy from and to Atmospheric Eddies*, 1920)

Great fleas have little fleas upon their backs to bite 'em,
And little fleas have lesser fleas, and so ad infinitum.
And the great fleas themselves, in turn, have greater fleas to go on;
While these again have greater still, and greater still, and so on.

(Augustus de Morgan, *A Budget of Paradoxes, 1872, p. 370*)

So, naturalists observe, a flea
Has smaller fleas that on him prey;
And these have smaller still to bite 'em;
And so proceed ad infinitum.

(Jonathan Swift: *Poetry, a Rhapsody*, 1733)