Stochastic Variational Method and Viscous Hydrodynamics

A tentative approach

Stochastic Variational Method and Viscous Hydrodynamics

T. Koide (小出　知威)

T. Kodama (小玉　剛)

Federal Univ. of Rio de Janeiro

TK and TK, arXiv:1108.0124

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Variation Principle for (Ideal) Hydrodynamics

$$\delta I = \delta \int dt d\vec{r} L(\rho, \vec{v}) = 0$$

Once established,

a. Effective Hydrodynamics (can reduce DOF)
b. Optimization for Descretization (SPH)
Physically stable -> Event by event analysis

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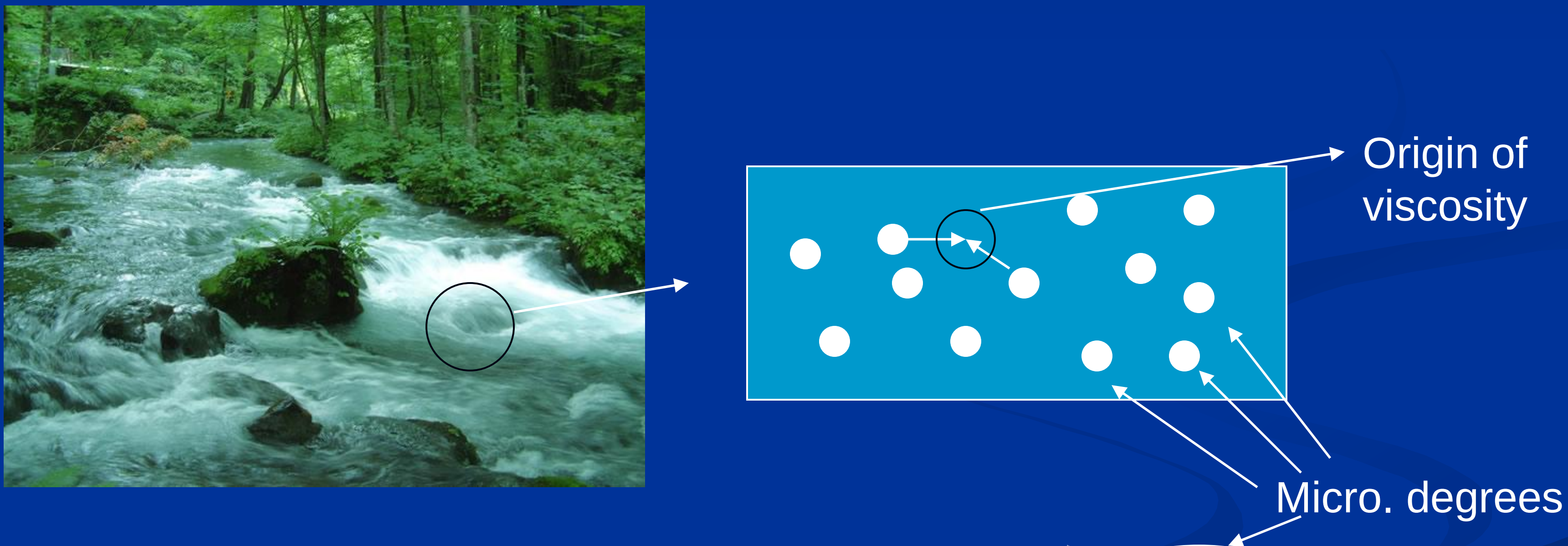
Nucl. Phys. A698, 639 (2002),
Phys. Rev. Lett. 93, 182301 (2004).
Phys. Rev. Lett. 97, 202302 (2006).
Phys. Rev. Lett. 101, 112301 (2008).
Phys. Rev. Lett. 103, 242301 (2009).

Limitation of Classical VP

How to deal with dissipation?

Dissipation: Effects of "invisible" microscopic degrees of freedom



$$I(\rho,\vec{v}) \longrightarrow I(\rho,\vec{v},q,\dot{q},\ldots)$$

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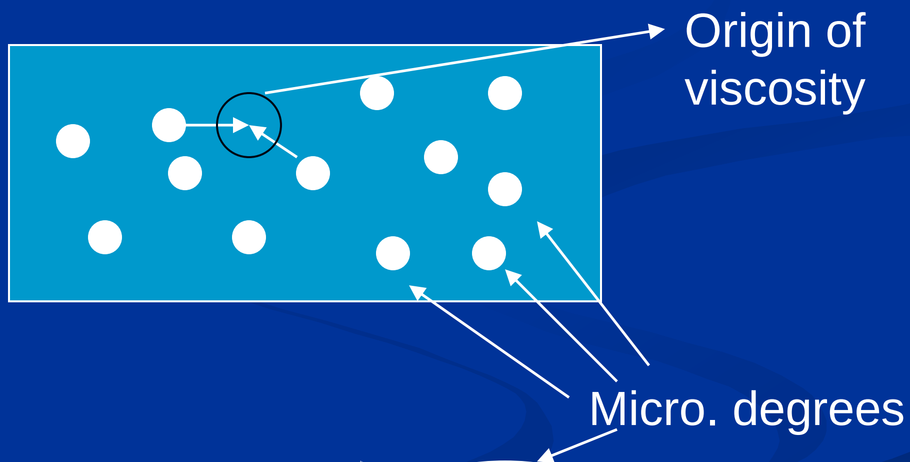


$$I(\rho,\vec{v}) \longrightarrow I(\rho,\vec{v},q,\dot{q},\ldots)$$

Hydrodynamics with noises ? : Csernai, Capusta,…

Phenomenological approach by Rayleigh Dissipative Function

The action contains a microscopic degree S,

$$L(x,\dot{x}) \longrightarrow L(x,\dot{x},S)$$

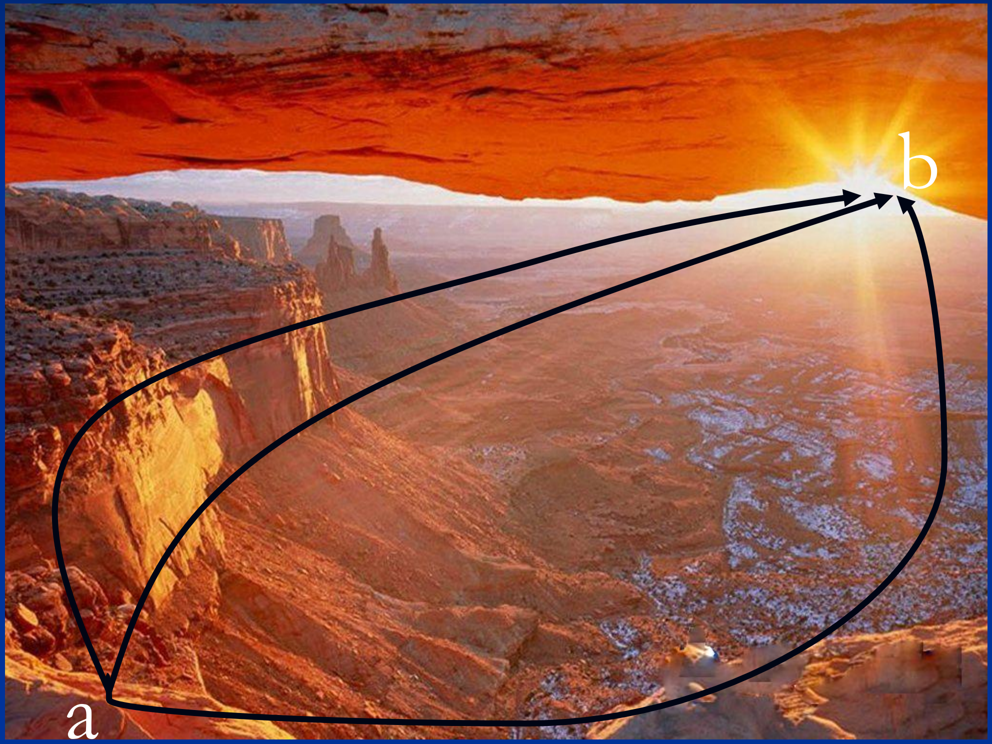
Variation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \Xi \quad \text{where} \quad \Xi(x,\dot{x},S) = \frac{1}{\dot{x}}\frac{\partial L}{\partial S}\frac{dS}{dt}$$

The form of Ξ is tuned so as to derive a dissipative equation which we wish to derive.

Hydrodynamics with noises ? : Csernai, Capusta,…

Variational Principle with noise?



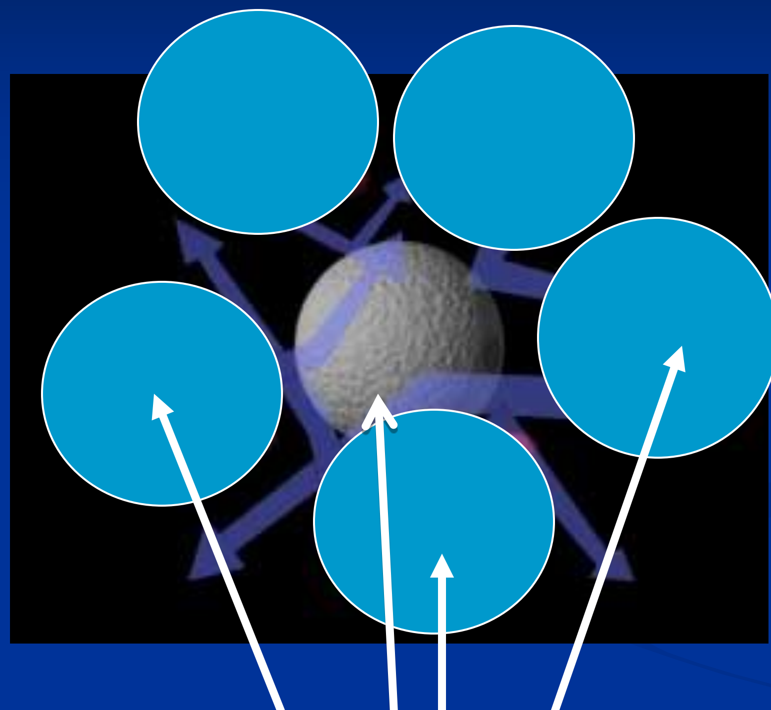
How to deal with stochastic variables, when the action DOE contains effects of noises?

$$I = \int_a^b dt L(X, DX)$$

With X: Stochastic Variables

- If possible, a wide applicability expected...

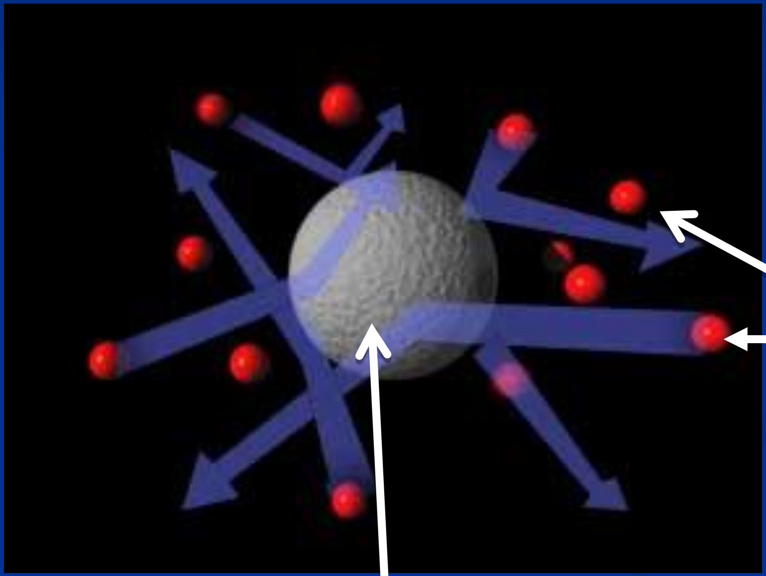
Can be approximated by noise ?



Fluid element

Undeterminancy of fluid elements

Can be approximated by noise ?



Molecules (inside other fluid elements)

Fluid element

The effect of molecules can be absorbed into that of noise as the Brownian Motion !?

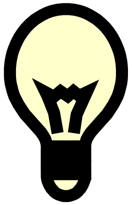
Stochastic Variational Method

Instead modifying the action, the effect of microscopic degrees of freedom is represented as noise.

classical V.P. $$\frac{d}{dt}X(t)=V(t)$$

↓

stochastic V.P. $$\frac{d}{dt}X(t)=V(t)+\xi(t)$$



Yasue, J. Funct. Anal, 41, 327 (‘81), Guerra&Morato, Phys. Rev. D27, 1774 (‘83), Nelson, “Quantum Fluctuations” (‘85).

Noise changes classical path

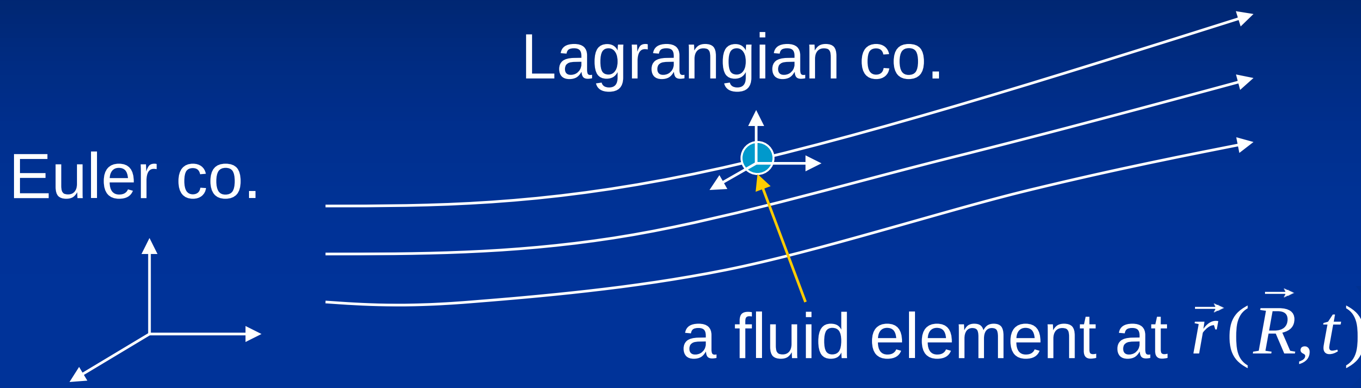
noise (molecules)

Brownian particle

Cannot walk straight !

Path by CVP

Euler and Lagrange Coordinates



Euler		Langange
$\rho(\vec{r},t)$ mass density	→	$\rho = \det\left\lvert\frac{\partial\vec{r}}{\partial\vec{R}}\right\rvert\rho_0 \quad \rho_0 = \rho(\vec{R},t_a)$
$\vec{v}(\vec{r},t)$ velocity	→	$\frac{\partial\vec{r}(\vec{R},t)}{\partial t}$

Initial position ($\vec{R}$)

Action for Hydrodynamics (Non-relativistic)

The action is given by kinetic energy and potential energy.

Euler

$$K = \frac{\rho}{2}\vec{v}^2 \qquad V = \varepsilon(\rho, S)$$

Lagrange

$$K = \frac{\rho_0}{2}\left(\frac{\partial \vec{r}}{\partial t}\right)^2 \qquad V = \frac{\rho_0}{\rho}\varepsilon(\rho, S)$$

Action

$$I_{cla} = \int_a^b dt \int d^3R\,(K - V) = \int_a^b dt \int d^3R \left(\frac{\rho_0}{2}\left(\frac{\partial \vec{r}}{\partial t}\right)^2 - \frac{\rho_0}{\rho}\varepsilon(\rho, S) \right)$$

Classical variational method

Variations

As the variation, we consider only $r \to r + \delta r$

$$v \to v + d\delta r / dt$$

$$\rho \to \rho - \frac{\rho}{J} \sum_{ij} \frac{\partial J}{\partial(\partial r^i / \partial R^j)} \frac{\partial \delta r^i}{\partial R^j}$$

$$S \to S$$ ⟵ Constraint from entropy conservation

$$\delta I = 0$$

= pressure, P

$$\rho \left[\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right] \vec{v} - \nabla \left\{ \frac{d}{d(1/\rho)} \left(\frac{\varepsilon}{\rho} \right) \right\} = 0$$

Variations

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$$S \to S$$ ← Constraint from entropy conservation

$\delta I = 0$

Euler equation

= pressure, P

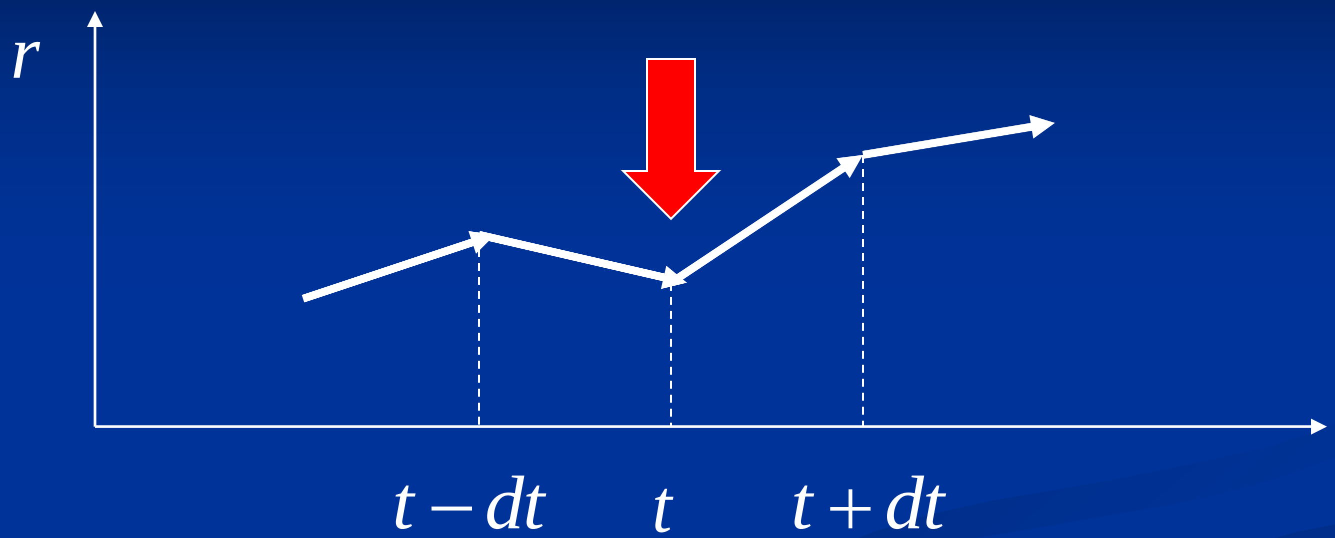
$$\rho \left[\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right] \vec{v} - \nabla P = 0$$

Stochastic variational method

The Lagrangian coordinates of fluid elements are stochastic variables ->

Derivatives are discontinuous !

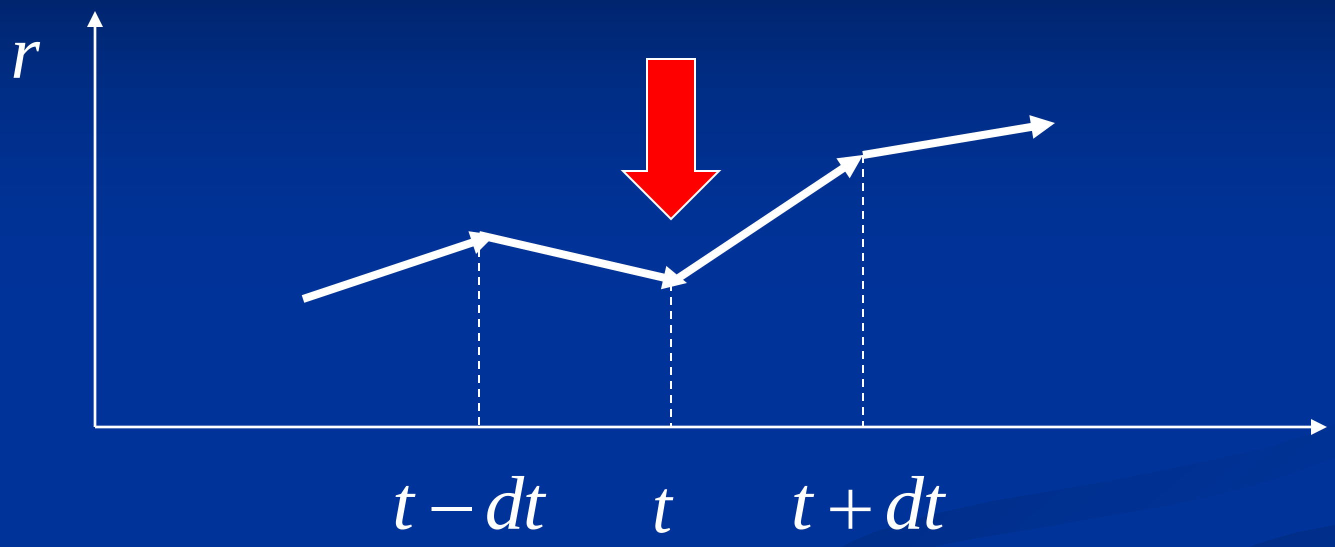
How to define velocity ?



$$\vec{v} = \lim_{dt \to 0+} \frac{\vec{r}(\vec{R}, t + dt) - \vec{r}(\vec{R}, t)}{dt}$$ ⇐ Forward SDE

$$\tilde{\vec{v}} = \lim_{dt \to 0+} \frac{\vec{r}(\vec{R}, t) - \vec{r}(\vec{R}, t - dt)}{dt}$$

How to define velocity ?



$$\vec{v} = \lim_{dt\to 0+} \frac{\vec{r}(\vec{R},t+dt)-\vec{r}(\vec{R},t)}{dt}$$

Forward SDE

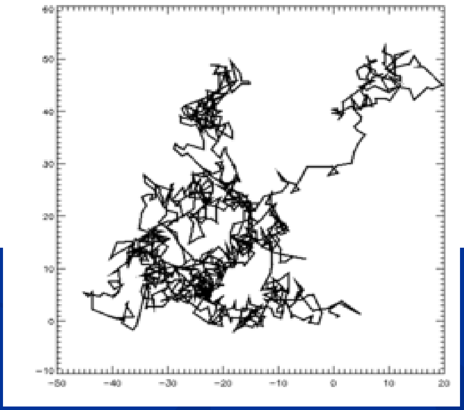
$$\tilde{\vec{v}} = \lim_{dt\to 0+} \frac{\vec{r}(\vec{R},t)-\vec{r}(\vec{R},t-dt)}{dt}$$

Backward SDE

Forward SDE

Forward Stochastic Differential Equation $\left(dt > 0\right)$

$$d\vec{r} = \vec{u}(\vec{r}(\vec{R},t),t)dt + \sqrt{2\nu} \cdot d\vec{W}(t)$$



Gaussian White Noise

$$\left\langle d\vec{W}(t) \right\rangle = 0 \qquad \left\langle dW^i(t) dW^j(t) \right\rangle = \delta^{ij} dt$$

Backward SDE

Backward Stochastic Differential Equation $\left(dt < 0\right)$

$$d\vec{r} = \vec{\tilde{u}}(\vec{r}(\vec{R},t),t)dt + \sqrt{2\nu} \cdot d\vec{\tilde{W}}(t)$$

Gaussian White Noise

$$\left\langle d\vec{\tilde{W}}(t) \right\rangle = 0 \qquad \left\langle d\tilde{W}^i(t) d\tilde{W}^j(t) \right\rangle = \delta^{ij} \, | \, dt \, |$$

Backward SDE

Backward Stochastic Differential Equation $(dt < 0)$

$$d\vec{r} = \vec{\tilde{u}}(\vec{r}(\vec{R},t),t)dt + \sqrt{2\nu} \cdot d\vec{\tilde{W}}(t)$$

To describe the backward process, $\vec{\tilde{u}}$ is not independent of $\vec{u}$.

Gaussian White Noise

$$\left\langle d\vec{\tilde{W}}(t) \right\rangle = 0 \qquad \left\langle d\tilde{W}^i(t) d\tilde{W}^j(t) \right\rangle = \delta^{ij} \, | \, dt \, |$$

Consistency Condition

Fokker-Plank equation (Forward)

$$\partial_t \rho = -\nabla\left(\vec{u} - \nu\nabla\right)\rho$$

Fokker-Plank equation (Backward)

$$\partial_t \rho = -\nabla\left(\vec{\tilde{u}} + \nu\nabla\right)\rho$$

The two equation must be equivalent.

$\Rightarrow$

$$\vec{\tilde{u}} = \vec{u} + 2\nu\nabla \ln \rho$$

Two Fluid Velocities

The velocities $\vec{u}$ and $\vec{\tilde{u}}$ are not parallel to the mass current.
The mass velocity parallel to the mass current is given by

$$\partial_t \rho = -\nabla \left(\vec{u} - \nu \nabla \right) \rho = -\nabla (\rho \vec{v}_m)$$

where $$\vec{v}_m = \frac{\vec{u} + \vec{\tilde{u}}}{2} = \vec{u} - \nu \nabla \ln \rho$$

$\vec{u}$: the diffusion velocity

$\vec{v}_m$: the mass velocity

Partial Integration Formula

Because of the two definitions of velocities,
we introduce two different time derivative operators

Mean forward derivative $D\vec{r} = \vec{u}$

Mean backward derivative $\tilde{D}\vec{r} = \tilde{\vec{u}}$

stochastic partial integration formula

$$\int_a^b dt E\left[(DX)\cdot Y\right]$$

$$= E\left[X(b)Y(b) - X(a)Y(a)\right] - \int_a^b dt E\left[X\cdot(\tilde{D}Y)\right]$$

Stochastic Representation of Action

$$I_{cla} = \int_a^b dt \int d^3R \left(\frac{\rho_0}{2} \left(\frac{\partial \vec{r}}{\partial t} \right)^2 - \frac{\rho_0}{\rho} \varepsilon(\rho, S) \right)$$

We have to replace $\vec{v}$ by $D\vec{r}$ and/or $\tilde{D}\vec{r}$.

$$\vec{v}^{\,2} \Rightarrow \begin{cases} 1) \quad D\vec{r} \cdot D\vec{r} \\ 2) \quad \tilde{D}\vec{r} \cdot \tilde{D}\vec{r} \\ 3) \quad \dfrac{D\vec{r} \cdot D\vec{r} + \tilde{D}\vec{r} \cdot \tilde{D}\vec{r}}{2} \end{cases}$$

Inverse of 1)

Gross-Pitaevskii Eq.

We apply 1)

$$\Rightarrow \quad I_{sto} = \int_a^b dt \int d^3R \left(\frac{\rho_0}{2} (D\vec{r}) \cdot (D\vec{r}) - \frac{\rho_0}{\rho} \varepsilon(\rho, S) \right)$$

Stochastic Variation for Kinetic Term

As the variation, we consider only $r \to r + \delta r$

$$\delta \int_a^b dt \int d^3R \frac{\rho_0}{2} (D\vec{r}) \cdot (D\vec{r}) = \int_a^b dt \int d^3R \rho_0 (D\vec{r}) \cdot (D\delta\vec{r})$$

$$= \int_a^b dt \int d^3R \rho_0 \vec{u} \cdot (D\delta\vec{r})$$

$$= -\int_a^b dt \int d^3R \rho_0 \tilde{D}\vec{u} \cdot \delta\vec{r}$$

From the Ito formula $\tilde{D}\vec{u} = \left(\partial_t + \vec{\tilde{u}} \cdot \nabla - \nu\Delta\right)\vec{u}$

Variation of Action

The variation for the potential term is same as the classical VP. Thus we have

$$\delta I = -\int_a^b dt \int d^3 R \rho_0 \left[\left(\left(\partial_t + \vec{\tilde{u}} \cdot \nabla - \nu \Delta \right) \vec{u} + \frac{1}{\rho} \nabla P \right) \delta \vec{r} + \frac{T}{\rho} \delta S \right]$$

from kinetic term   from potential term

Now entropy is not a conserved quantity and $\delta S \neq 0$.

Variation of Entropy

If there is non-quasi-static changes of fluids, entropy is not conserved. This entropy change will be expressed as a function of λ, which characterizes the difference of time scales,

$$\lambda = \frac{\tau_{\min}}{\tau_{hyd}} = \frac{\tau_{\min}}{\dot{\rho}/\rho}$$

In $\lambda = 0$, the process becomes quasi-static and $\delta S = 0$. Thus

$$\delta S = \delta\left(a_1\lambda + a_2\lambda^2 + \cdots\right)$$

Lowest order truncation

$$\delta S = \delta\left(g(\rho)\dot{\rho}\right)$$

Hydrodynamics

$$\delta I = -\int_a^b dt \int d^3 R \rho_0 \left[\left(\left(\partial_t + \vec{\tilde{u}} \cdot \nabla - \nu \Delta \right) \vec{u} + \frac{1}{\rho} \nabla P \right) \delta \vec{r} + \frac{T}{\rho} \delta S \right]$$

substitution

Hydrodynamics

$$\delta I = -\int_a^b dt \int d^3R \rho_0 \left[\left(\left(\partial_t + \vec{\tilde{u}} \cdot \nabla - \nu \Delta \right) \vec{u} + \frac{1}{\rho} \nabla P \right) \delta \vec{r} + \frac{T}{\rho} \delta S \right]$$

substitution

$$\rho \left(\partial_t + \vec{v}_m \cdot \nabla \right) \vec{u} + \nabla \left(P - \mu \nabla \cdot \vec{v}_m \right) - \sum_j \partial_j \left(\eta \partial_j \vec{u} \right) = 0$$

Shear viscosity coefficient $\eta \equiv \rho \nu$

Second coefficient of viscosity $\mu \equiv T \rho g(\rho)$

The contribution from δS effectively changes pressure by $\mu \nabla \cdot \vec{v}_m$.

$\vec{u}$ should be replaced with $\vec{v}_m$ using $\vec{v}_m = \vec{u} - \nu \nabla \ln \rho$.

Bulk viscosity coefficient

$$\varsigma = \mu + \frac{2}{3}\eta$$

$$\rho\left(\partial_t + \vec{v}_m \cdot \nabla\right)\vec{v}_m + \sum_j \partial_j \left[\left(P - \varsigma \nabla \cdot \vec{v}_m\right)\delta_{ij} - \eta e_{ij}^m\right]$$

$$-\sum_j \partial_j \left(\eta \partial_j \frac{\eta}{\rho} \nabla \ln \rho\right) = 0$$

$$e_{ij}^m = \partial_j v_m^i + \partial_i v_m^j - \frac{2}{3}(\nabla \cdot \vec{v}_m)\delta_{ij}$$

$\vec{u}$ should be replaced with $\vec{v}_m$ using $\vec{v}_m = \vec{u} - \nu \nabla \ln \rho$.

Bulk viscosity coefficient

$$\varsigma = \mu + \frac{2}{3}\eta$$

Navier-Stokes Equation

$$\rho\left(\partial_t + \vec{v}_m \cdot \nabla\right)\vec{v}_m + \sum_j \partial_j \left[\left(P - \varsigma \nabla \cdot \vec{v}_m\right)\delta_{ij} - \eta e^m_{ij}\right]$$

$$-\sum_j \partial_j \left(\eta \partial_j \frac{\eta}{\rho} \nabla \ln \rho\right) = 0$$

TK, Kodama, arXiv:1105.6256

The last term is higher order and should be neglected.

$$e^m_{ij} = \partial_j v^i_m + \partial_i v^j_m - \frac{2}{3}(\nabla \cdot \vec{v}_m)\delta_{ij}$$

Results of SVM for NS

- The most of viscous terms of NS is obtained from the kinetic term as noise.
- Differences of interaction among constituent molecules of various fluids affect only the form of the potential term.
- The potential term changes only the definition of pressure.

Thus NS is naturally obtained from the framework of SVM !

Generalized Hydrodynamics

$$\rho\left(\partial_t + \vec{v}_m \cdot \nabla\right)\vec{v}_m + \sum_j \partial_j \left[\left(P - \varsigma \nabla \cdot \vec{v}_m\right)\delta_{ij} - \eta e^m_{ij}\right] - \sum_j \partial_j \left(\eta \partial_j \frac{\eta}{\rho} \nabla \ln \rho\right) = 0$$

When the higher order correction is considered,

$$\rho\left(\partial_t + \vec{v}_m \cdot \nabla\right)\vec{u} + \nabla\left(P - \mu \nabla \cdot \vec{v}_m\right) - \sum_j \partial_j \left(\eta \partial_j \vec{u}\right) = 0$$

Generalized hydrodynamics can be expressed with two fluid velocities.

→ Consistent with Brenner's hydrodynamics.

H.Brenner, PRE70 (2004), Int.J.Eng.Sci 47(2011)

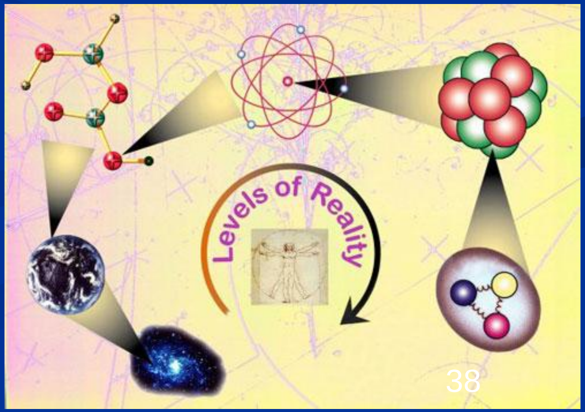
Idealized Case (no potential $\varepsilon = 0$)

This term is higher order in NS

SVM leads to

$$\rho \frac{d}{dt}\vec{v}_m^{\,i} - \sum_j \partial_j \left(\nu\rho\partial_i \vec{v}_m^{\,j} + \nu\rho\partial_j \vec{v}_m^{\,i} \right) - \sum_j \partial_j \left(\nu\rho\partial_j \nu\partial_i \ln\rho \right) = 0$$

In macro scale where the time dependence of $\vec{v}_m$ is negligible,



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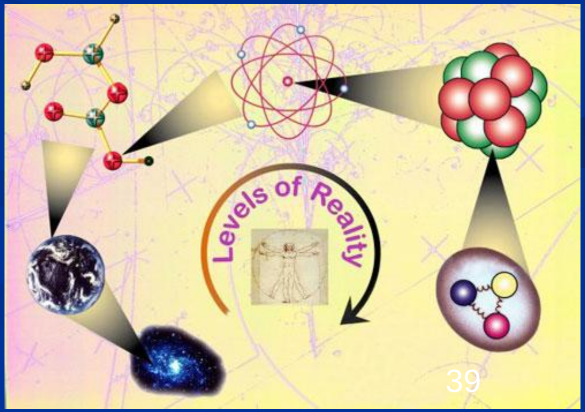
$$\rho \frac{d}{dt}\vec{v}_m^{\,i} - \sum_j \partial_j \left(\nu\rho\partial_i \vec{v}_m^{\,j} + \nu\rho\partial_j \vec{v}_m^{\,i} \right) - \sum_j \partial_j \left(\nu\rho\partial_j \nu\partial_i \ln\rho \right) = 0$$

In macro scale where the time dependence of $\vec{v}_m$ is negligible,

$$\vec{v}_m = -\frac{\nu}{2}\nabla \ln \rho$$



Generalized Diffusion Equation

From FP equation,

$$\frac{d}{dt}\rho = -\rho \nabla \cdot \vec{v}_m \quad + \quad \vec{v}_m = -\frac{\nu}{2}\nabla \ln \rho \quad \Rightarrow \quad \partial_t \rho = \frac{\nu}{2}\Delta\rho$$

Diffusion equation

Generalized Diffusion Equation

From FP equation,

$$\frac{d}{dt}\rho = -\rho\nabla\cdot\vec{v}_m \quad + \quad \vec{v}_m = -\frac{\nu}{2}\nabla\ln\rho \quad \Rightarrow \quad \partial_t\rho = \frac{\nu}{2}\Delta\rho$$

Diffusion equation

The equation obtained in SVM describes the generalized diffusion processes.

$$\frac{d}{dt}\rho = -\rho\nabla\cdot\vec{v}_m$$

$$\rho\frac{d}{dt}\vec{v}^{\,i}_{m} - \sum_j \partial_j\left(\nu\rho\partial_i\vec{v}^{\,j}_{m} + \nu\rho\partial_j\vec{v}^{\,i}_{m}\right) - \sum_j \partial_j\left(\nu\rho\partial_j\nu\partial_i\ln\rho\right) = 0$$

Generalized Diffusion Equation

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$$\frac{d}{dt}\rho = -\rho\nabla\cdot\vec{v}_m \quad + \quad \vec{v}_m = -\frac{\nu}{2}\nabla\ln\rho \quad \Rightarrow \quad \partial_t\rho = \frac{\nu}{2}\Delta\rho$$

Diffusion equation

The equation obtained in SVM describes the generalized diffusion processes.

Maybe important even for NS ?

$$\frac{d}{dt}\rho = -\rho\nabla\cdot\vec{v}_m$$

$$\rho\frac{d}{dt}\vec{v}^{\,i}_m - \sum_j \partial_j\left(\nu\rho\partial_i\vec{v}^{\,j}_m + \nu\rho\partial_j\vec{v}^{\,i}_m\right) - \sum_j \partial_j\left(\nu\rho\partial_j\nu\partial_i\ln\rho\right) = 0$$

Successful Applications of SVM

- Incompressible NS equation

Nakagomi, Yasue, Zambrini, Marra, Kanno, Cipriano, Cruzeiro, Shamarova,Arnaudon,....

- Compressible NS equation

TK&TK

- Diffusion phenomena

Hasegawa, Misawa, Jaekel, TK&TK ...

} classical

- Schroedinger equation

Yasue, Zambrini, Nelson, Davidson, Guerra, Morata, Nagasawa, Tanaka,…

- Gross-Pitaevskii equation

Loffred, Morato, TK&TK

} quantum

Concluding Remarks

- The NS equation can be derived from the action of the ideal fluid by SVM.
- Shear Viscosity in NS comes from noises.
- The higher order correction to NS is important in discussing generalization of the diffusion eq.
- The generalized hydro. can be expressed with two velocities. This is similar to Brenner’s idea.
- Diffusion and NS are macroscopic equations of different coarse-grained scales.

Future Perspective (further check of SVM)

- Importance of the higher order correction term (turbulence, glass transition,…)
- Magneto hydrodynamics
- Generalization of the white noise
- Field theory
- Relativistic systems

  relativistic Brownian motion: J. Dunkel and P. H¨anggi, PR471, 1 (2009),
  TK&TK, PRE83, 061111 (2011).

Future Perspective

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- Magnet
- Genera
- Field th
- Relativi
  relativistic

Спасибо!

Elena, Jörg,
Marcus, Igor

Danke Schön!

Merci!

Ευχαριστώ!

Спасибо!

Danke Schön!

Elena, Jörg, Marcus, Igor

Merci!

Ευχαριστώ!

Thank you!

ありがとう!

OBRIGADO!

Grazie!

謝謝!

Gracias !

Another Reduction to Diffusion Eq.

$$\rho \frac{d}{dt}\vec{v}_m^{\,i} - \sum_j \partial_j \left(\nu\rho\partial_i \vec{v}_m^{\,j} + \nu\rho\partial_j \vec{v}_m^{\,i} \right) - \sum_j \partial_j \left(\nu\rho\partial_j \nu\partial_i \ln\rho \right) = 0$$

If we assume $\vec{v}_m = -\nu\nabla\ln\rho \cdots\cdots (1)$

(Note: the coefficient is different from before)

$$\partial_t \rho = \nu\Delta\rho \cdots\cdots (2)$$

If we chose the initial condition satisfying (1), dynamics is described by the diffusion equation (2).

Noether Theorem

We consider the following linear transform,

$$\vec{r}(t) \longrightarrow G(\alpha)\vec{r}(t) \qquad G(0) = I$$

When the Lagrangian is invariant, we obtain

stochastic Neother theorem

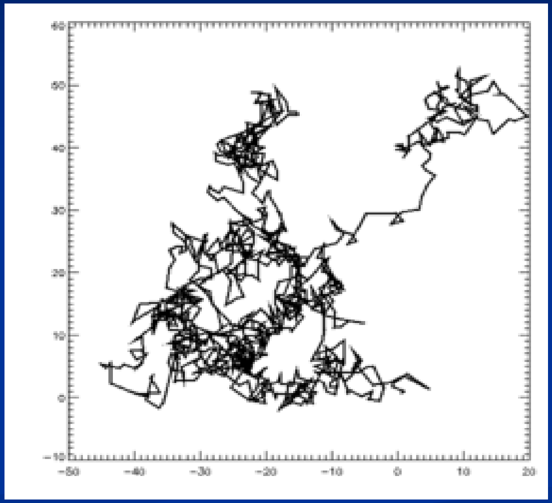
$$\frac{d}{dt} E\left[\left\{ \frac{\partial L}{\partial D\vec{r}(t)} + \frac{\partial L}{\partial \tilde{D}\vec{r}(t)} \right\} \cdot \left. \frac{dG(\alpha)}{d\alpha} \right|_{\alpha=0} \vec{r}(t) \right] = 0.$$

Classical NT

$$\frac{d}{dt} \left[\frac{\partial L}{\partial \dot{\vec{r}}(t)} \cdot \left. \frac{dG(\alpha)}{d\alpha} \right|_{\alpha=0} \vec{r}(t) \right] = 0.$$

History of Brownian Motion

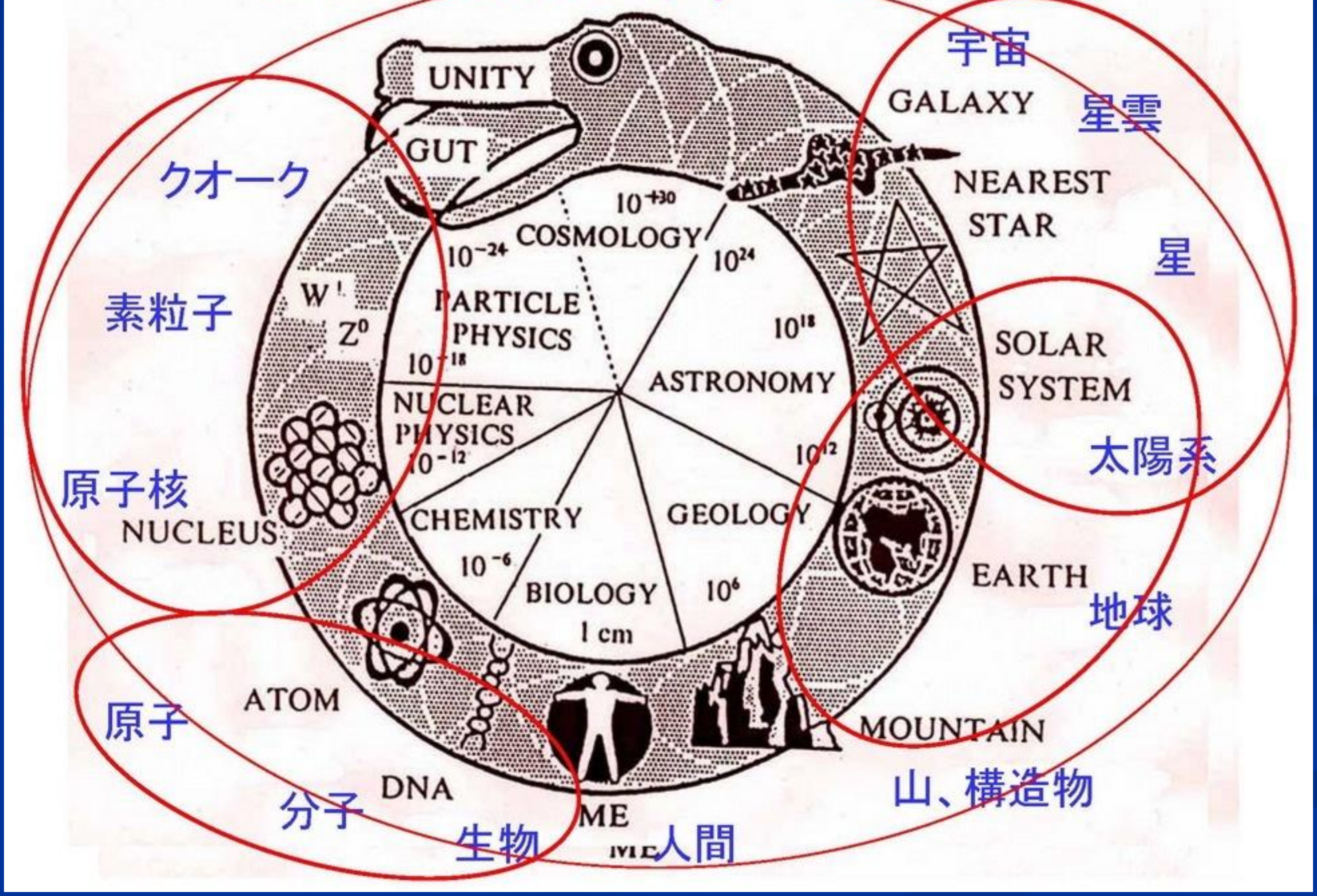
1882	the discovery of the Brownian motion	R. Brown
1905	the fluctuation-dissipation theorem	A. Einstein
1908	Avogadro's number	J. B. Perrin
	Langevin equation	P. Langevin
1940~	Mathematical formulation	K. Ito
		K. Yoshida
		N. Wiener
		P. Levy



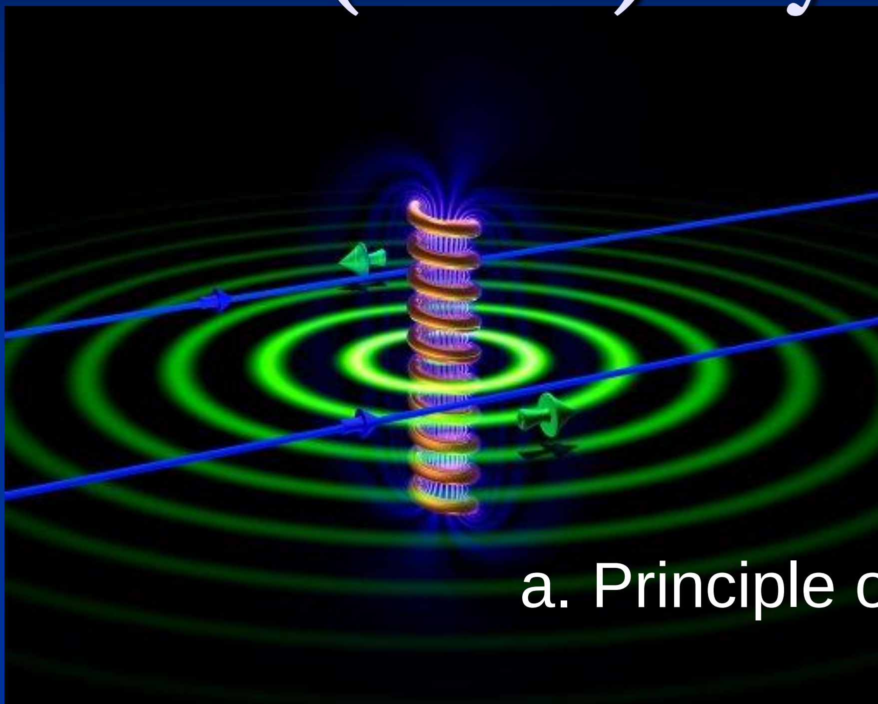
However, the theory of the relativistic Brownian motion has not yet established.

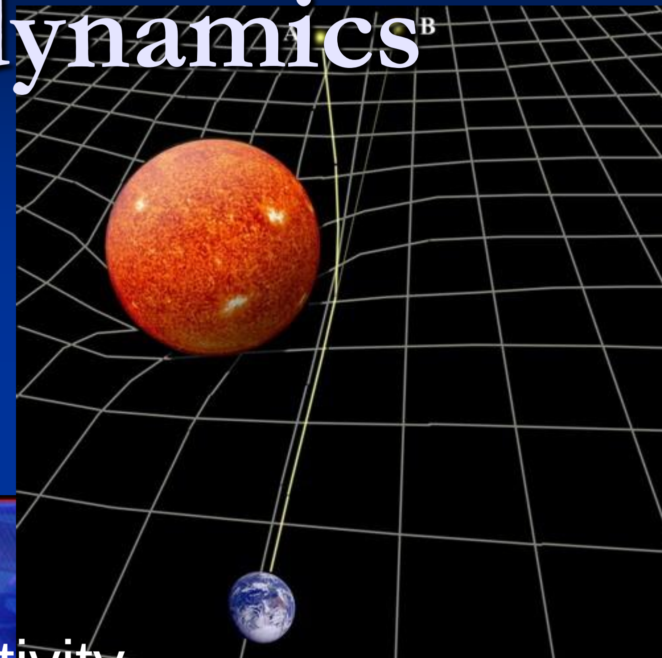
Hakin (1965)
Ben-Ya'acov (1981)
Debbasch,Mallick&Rivet (1997)
Oron&Horwitz (2003)
Dubkel&Haenggi (2005)

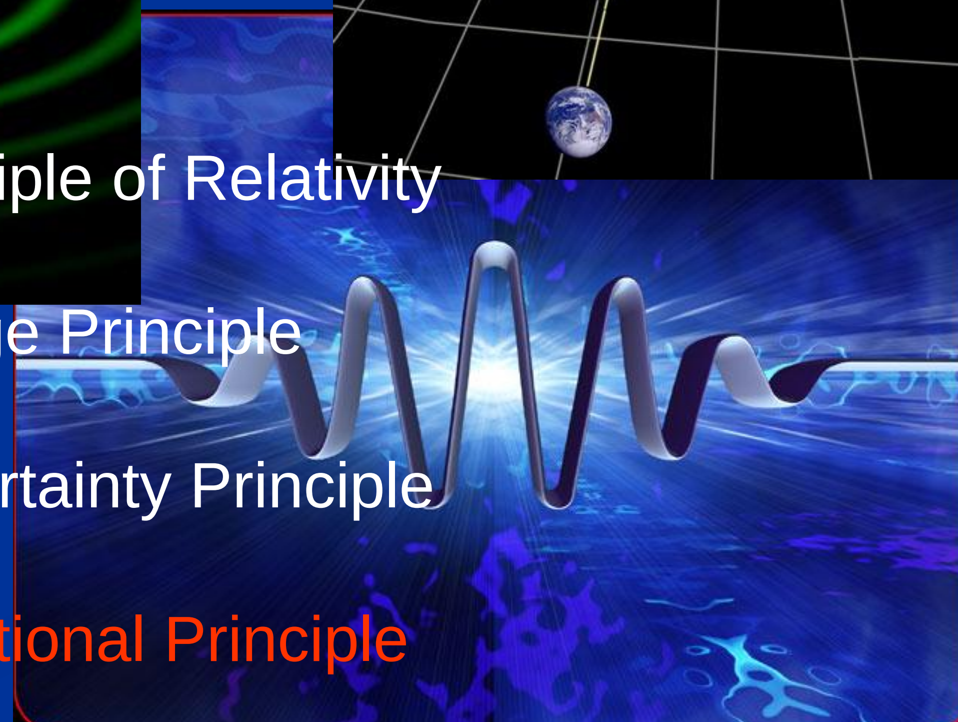
自然の階層性(by Glashow)



Variation Principle for (Ideal) Hydrodynamics







a. Principle of Relativity

b. Gauge Principle

c. Uncertainty Principle

d. Variational Principle