Transient fluid dynamics from the Boltzmann equation

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Dissipative fluid dynamics

Conservation laws & tensor decompositions

$$\partial_\mu N^\mu = 0$$
$$\partial_\mu T^{\mu\nu} = 0$$
$$N^\mu = n u^\mu + n^\mu$$
$$T^{\mu\nu} = e u^\mu u^\nu - (p + \Pi)\Delta^{\mu\nu} + 2W^{(\mu}u^{\nu)} + \pi^{\mu\nu}$$

$n = u_\mu N^\mu$	LRF particle density
$n^\mu = \Delta^\mu_\alpha N^\alpha$	particle diffusion current
$e = u_\mu T^{\mu\nu} u_\nu$	LRF energy density
$W^\mu = \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta$	energy diffusion current
$p(e,n) + \Pi = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu}$	isotropic pressure (p_{eq} + *bulk*)
$\pi^{\mu\nu} = T^{\langle\mu\nu\rangle}$	shear stress tensor

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$
$$T^{\langle\mu\nu\rangle} = \left[\frac{1}{2}\left(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\nu_\alpha \Delta^\mu_\beta\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] T^{\alpha\beta}$$

Navier-Stokes theory

Consider only shear viscosity:

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$$
$$\sigma^{\mu\nu} = \nabla^{\langle\mu}u^{\nu\rangle}$$

- Shear viscous tensor directly proportional to the gradients of velocity
- Does not introduce any new dynamical variables (only e, n and u^{μ})
- In relativistic case acausal and unstable $\longrightarrow$ Not useful for numerical calculations

Transient fluid dynamics

Solution: transient fluid dynamics (e.g. Israel and Stewart)

$$\tau_\pi \frac{d}{d\tau}\pi^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \tau_\pi C\pi^{\mu\nu}\left(\nabla_\alpha u^\alpha\right) + \cdots$$

- $\pi^{\mu\nu}$ independent variable
- New transport coefficient τ_π = relaxation time
- $\tau_\pi \to 0$ NS-theory (NS is asymptotic solution of the transient theory)
- Stable and causal for range of transport coefficients τ_π and η
- Usually called "Second order theories"
- **Our goal is to derive all the terms and calculate coefficients for this type of theory**

Gradient expansion

- 2nd order gradient expansion (Burnett equations)

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + C_1 \frac{d}{d\tau}\sigma^{\mu\nu} + C_2 \omega_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} + C_3 \sigma_\lambda{}^{\langle\mu}\sigma^{\nu\rangle\lambda} + \cdots$$

- In the case of the Boltzmann equation can be derived via the Chapman-Enskog expansion
- Like the NS-theory: no new independent variables
- Acausal and unstable already in the non-relativistic case

Question: Can transient fluid dynamics be derived from the gradient expansion, i.e. is $C_1/2\eta = \tau_\pi$?

Gradient expansion II

Short answer: no

- Add blue terms in the gradient expansion

$$\tau_\pi \frac{d}{d\tau}\pi^{\mu\nu} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + C_1 \frac{d}{d\tau}\sigma^{\mu\nu} + \tau_\pi \frac{d}{d\tau}\left[2\eta\sigma^{\mu\nu}\right] + \cdots$$

- For any τ_π the asymptotic solution is the gradient expansion.
- **Second order gradient expansion $\neq$ transient fluid dynamics**

Power Counting

We power count with

- Knudsen number

$$\mathrm{Kn} = \frac{\ell_{\mathrm{micr}}}{L_{\mathrm{macr}}}$$

- e.g. $\frac{2\eta}{P_0}\sigma^{\mu\nu} = O(\mathrm{Kn})$
- inverse Reynolds number

$$\mathrm{R}_{\Pi}^{-1} \sim \frac{|\Pi|}{P_0},\ \mathrm{R}_{n}^{-1} \sim \frac{|n^{\mu}|}{n_0},\ \mathrm{R}_{\pi}^{-1} \sim \frac{|\pi^{\mu\nu}|}{P_0}$$

- Note that in general Knudsen number and inverse Reynolds number are not equivalent (only in the NS/Burnett-limit)
- Aim is to derive fluid dynamics with some definite order in Kn and R^{-1}

Boltzmann equation

Boltzmann Equation

- Evolution equation for the single particle distribution function $f_{\mathbf{k}}$

$$k^{\mu}\partial_{\mu} f_{\mathbf{k}} = C\left[f\right]$$

- Collision term for binary collisions ($\tilde{f}_{\mathbf{k}} = 1 - a f_{\mathbf{k}}$, $a = 0, \pm 1$)

$$C\left[f\right] = \frac{1}{\nu} \int dK' dP dP' \, W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} \left(f_{\mathbf{p}} f_{\mathbf{p}'} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p}'} \right)$$

Expansion of $f_{\mathbf{k}}$

- First write $f_{\mathbf{k}}$ in the form $f_{\mathbf{k}} = f_{0\mathbf{k}}\left(1 + \tilde{f}_{0\mathbf{k}}\phi_{\mathbf{k}}\right)$
- $f_{0\mathbf{k}} = (\exp(\beta_0 E_{\mathbf{k}} - \alpha_0) + a)^{-1}$ is eq. distribution ($E_{\mathbf{k}} = u \cdot k$)
- Expand $\phi_{\mathbf{k}}$ using orthogonal basis

$$\phi_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \lambda_{\mathbf{k}}^{\langle\mu_1\cdots\mu_\ell\rangle} k_{\langle\mu_1} \ldots k_{\mu_\ell\rangle} \ ,$$

$$\lambda_{\mathbf{k}}^{\langle\mu_1\cdots\mu_\ell\rangle} = \sum_{n=0}^{N_\ell} c_{(n)}^{\langle\mu_1\cdots\mu_\ell\rangle} P_{\mathbf{k}}^{(n\ell)}$$

$$k^{\langle\mu_1} \ldots k^{\mu_m\rangle} \equiv \Delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m} k^{\nu_1} \ldots k^{\nu_m}$$

$$\int dK\ F_{\mathbf{k}} k^{\langle\mu_1} \ldots k^{\mu_m\rangle} k_{\langle\nu_1} \ldots k_{\nu_n\rangle} = \frac{m!\,\delta_{mn}}{(2m+1)!!} \Delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m} \int dK\ F_{\mathbf{k}} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^m$$

$$P_{\mathbf{k}}^{(n\ell)} = \sum_{r=0}^{n} a_{nr}^{(\ell)} (E_{\mathbf{k}})^r$$

$$\int dK \omega_\ell P_{\mathbf{k}}^{(n\ell)} P_{\mathbf{k}}^{(m\ell)} = \delta^{mn}, \qquad \omega_\ell \equiv \frac{W_\ell}{(2\ell+1)!!} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^\ell f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}$$

Expansion of $f_{\mathbf{k}}$ II

Orthogonality conditions imply that

$$c_{(n)}^{\langle\mu_1\cdots\mu_\ell\rangle} = \frac{W_\ell}{\ell!}\left\langle P_{\mathbf{k}}^{(n\ell)} k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle}\right\rangle_\delta \equiv \frac{W_\ell}{\ell!}\int dK\, P_{\mathbf{k}}^{(n\ell)} k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle}\delta f_{\mathbf{k}}$$

More convenient variable is

$$\rho_{(r)}^{\mu_1\cdots\mu_\ell} \equiv \left\langle (E_{\mathbf{k}})^r\, k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle}\right\rangle_\delta$$

Full expansion of $f_{\mathbf{k}}$ reads:

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}\sum_{\ell=0}^{\infty}\sum_{n=0}^{N_\ell}\mathcal{H}_{\mathbf{k}}^{(n\ell)}\rho_{(n)}^{\mu_1\cdots\mu_\ell}k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle}$$

with

$$\mathcal{H}_{\mathbf{k}}^{(n\ell)} \equiv \frac{W_\ell}{\ell!}\sum_{m=n}^{N_\ell} a_{mn}^{(\ell)} P_{\mathbf{k}}^{(m\ell)}$$

General equations of motion

Landau matching conditions $e = e_0$ and $n = n_0$ and the definition of the velocity (Landau frame) field imply that $\rho_{(1)} = \rho_{(2)} = \rho^{\mu}_{(1)} = 0$. By definition:

$$\rho_{(0)} = -\frac{3}{m^2}\,\Pi\,, \quad \rho^{\mu}_{(0)} = n^{\mu}\,, \quad \rho^{\mu\nu}_{(0)} = \pi^{\mu\nu}$$

Equations of motion:

$$\dot{\rho}^{\langle\mu_1\cdots\mu_\ell\rangle}_{(r)} = \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\frac{d}{d\tau}\int dK\,(E_{\mathbf{k}})^r\,k^{\langle\nu_1}\ldots k^{\nu_\ell\rangle}\delta f_{\mathbf{k}}$$

where $\dot{A} \equiv u^{\mu}\partial_{\mu}A \equiv dA/d\tau$ and $\dot{\rho}^{\langle\mu_1\cdots\mu_\ell\rangle}_{(r)} \equiv \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\ \dot{\rho}^{\nu_1\cdots\nu_\ell}_{(r)}$. By using the Boltzmann equation in the form

$$\delta\dot{f}_{\mathbf{k}} = -\dot{f}_{0\mathbf{k}} - E^{-1}_{\mathbf{k}}k_{\nu}\nabla^{\nu}f_{0\mathbf{k}} - E^{-1}_{\mathbf{k}}k_{\nu}\nabla^{\nu}\delta f_{\mathbf{k}} + E^{-1}_{\mathbf{k}}C\,[f]$$

where $\nabla_{\mu} = \Delta^{\nu}_{\mu}\partial_{\nu}$, one can obtain the *exact* equations for the comoving derivatives of $\rho^{\mu_1\cdots\mu_l}_{(r)}$

General equations of motion

$$
\begin{aligned}
\dot{\rho}_{(r)} - C_{(r-1)} &= \beta_{\zeta}^{(r)}\theta - \frac{G_{2r}}{D_{20}}\Pi\theta + \frac{G_{2r}}{D_{20}}\pi^{\mu\nu}\sigma_{\mu\nu} + \frac{G_{3r}}{D_{20}}\partial_\mu n^\mu + (r-1)\,\rho_{(r-2)}^{\mu\nu}\sigma_{\mu\nu} + r\rho_{(r-1)}^{\mu}\dot{u}_\mu - \nabla_\nu\rho_{(r-1)}^{\nu} \\
&- \frac{1}{3}\left[(r+2)\,\rho_{(r)} - (r-1)\,m^2\rho_{(r-2)}\right]\theta \\
\dot{\rho}_{(r)}^{\langle\mu\rangle} - C_{(r-1)}^{\langle\mu\rangle} &= \beta_{\kappa}^{(r)}\nabla^\mu\alpha_0 + \rho_{\nu(r)}\omega^{\mu\nu} + \frac{1}{3}\left[(r-1)\,m^2\rho_{(r-2)}^{\mu} - (r+3)\,\rho_{(r)}^{\mu}\right]\theta - \Delta_{\alpha}^{\mu}\nabla_\nu\rho_{(r-1)}^{\alpha\nu} + r\rho_{(r-1)}^{\mu\lambda}\dot{u}_\lambda \\
&+ \frac{1}{5}\left[(2r-2)\,m^2\rho_{(r-2)}^{\nu} - (2r+3)\,\rho_{(r)}^{\nu}\right]\sigma_{\nu}^{\mu} + \frac{1}{3}\left[m^2 r\rho_{(r-1)} - (r+3)\,\rho_{(r+1)}\right]\dot{u}^\mu \\
&+ \frac{\beta_0 J_{r+2,1}}{\varepsilon_0 + P_0}\left(\Pi\dot{u}^\mu - \nabla^\mu\Pi + \Delta_{\nu}^{\mu}\partial_\lambda\pi^{\nu\lambda}\right) - \frac{1}{3}\nabla^\mu\left(m^2\rho_{(r-1)} - \rho_{(r+1)}\right) + (r-1)\,\rho_{(r-2)}^{\mu\nu\lambda}\sigma_{\nu\lambda} \\
\dot{\rho}_{(r)}^{\langle\mu\nu\rangle} - C_{(r-1)}^{\langle\mu\nu\rangle} &= 2\beta_{\eta}^{(r)}\sigma^{\mu\nu} - \frac{2}{7}\left[(2r+5)\,\rho_{(r)}^{\rho\langle\mu} - m^2 2\,(r-1)\,\rho_{(r-2)}^{\rho\langle\mu}\right]\sigma_{\rho}^{\nu\rangle} + 2\rho_{(r)\lambda}^{\langle\mu}\omega^{\nu\rangle\lambda} \\
&+ \frac{2}{15}\left[(r+4)\,\rho_{(r+2)} - (2r+3)\,m^2\rho_{(r)} + (r-1)\,m^4\rho_{(r-2)}\right]\sigma^{\mu\nu} + \frac{2}{5}\nabla^{\langle\mu}\left(\rho_{(r+1)}^{\nu\rangle} - m^2\rho_{(r-1)}^{\nu\rangle}\right) \\
&- \frac{2}{5}\left[(r+5)\,\rho_{(r+1)}^{\langle\mu} - rm^2\rho_{(r-1)}^{\langle\mu}\right]\dot{u}^{\nu\rangle} - \frac{1}{3}\left[(r+4)\,\rho_{(r)}^{\mu\nu} - m^2\,(r-1)\,\rho_{(r-2)}^{\mu\nu}\right]\theta \\
&+ (r-1)\,\rho_{(r-2)}^{\mu\nu\lambda\rho}\sigma_{\lambda\rho} - \Delta_{\alpha\beta}^{\mu\nu}\nabla_\lambda\rho_{(r-1)}^{\alpha\beta\lambda} + r\rho_{(r-1)}^{\mu\nu\sigma}\dot{u}_\sigma\,.
\end{aligned}
$$

Generalized collision terms

$$
C_{(r)}^{\langle\mu_1\cdots\mu_\ell\rangle} = \int dK\,(E_{\mathbf{k}})^r\,k^{\langle\mu_1}\ldots k^{\mu_\ell\rangle}\,C[f]\,.
$$

Collision term

The generalized collision temrs can be written as

$$C_{(r-1)}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\sum_{n=0}^{\infty} \mathcal{A}_\ell^{(rn)} \rho_{(n)}^{\mu_1\cdots\mu_\ell} + (\text{nonlinear terms})$$

Then the general equations of motion are of the form

$$\begin{aligned} \dot{\rho}_{(r)} + \sum_{n=0}^{\infty} \mathcal{A}_0^{(rn)} \rho_{(n)} &= \beta_\zeta^{(r)} \theta + \cdots, \\ \dot{\rho}_{(r)}^{\langle\mu\rangle} + \sum_{n=0}^{\infty} \mathcal{A}_1^{(rn)} \rho_{(n)}^{\mu} &= \beta_n^{(r)} \nabla^\mu \alpha_0 + \cdots \\ \dot{\rho}_{(r)}^{\langle\mu\nu\rangle} + \sum_{n=0}^{\infty} \mathcal{A}_2^{(rn)} \rho_{(n)}^{\mu\nu} &= 2\beta_\eta^{(r)} \sigma^{\mu\nu} + \cdots \end{aligned}$$

Even the linear part is coupled set of equations for all the moments $\rho_{(n)}^{\mu\cdots}$

How to reduce the infinite degrees of freedom to fluid dynamics?

14-moment approximation

- Truncate the expansion of $f_\mathbf{k}$ such that only 14 variables remain (Israel, Stewart 1978 and Denicol, Koide, Rischke 2010):

$$\phi_\mathbf{k} = \epsilon + \epsilon_\mu k^\mu + \epsilon_{\mu\nu} k^\mu k^\nu$$

- Every moment $\rho_{(n)}^{\mu\cdots}$ becomes proportional to the dissipative currents:
- $\rho_{(r)} \propto \Pi$, $\rho_{(r)}^{\mu} \propto n^\mu$, $\rho_{(r)}^{\mu\nu} \propto \pi^{\mu\nu}$
- Equations for dissipative currents automatically closed for any choice of r in the full eom, e.g.

$$\dot{\rho}_{(r)}^{\langle\mu\nu\rangle} + \sum_{n=0}^{\infty} \mathcal{A}_2^{(rn)} \rho_{(n)}^{\mu\nu} = 2\beta_\eta^{(r)} \sigma^{\mu\nu} + \cdots$$

- **problem 1:** Every choice of r will give different transport coefficients.
- **problem 2:** 14-moment truncation is not truncation in the Knudsen number
- In fact it neglects infinitely many terms first order in Kn (Navier-Stokes terms)

We need some other truncation procedure

(quasi-)Normal modes

Diagonalize $\mathcal{A}_\ell$:
eigenvalues

$$\Omega_\ell^{-1}\mathcal{A}_\ell\Omega_\ell = \mathrm{diag}\left(\chi_\ell^{(0)},\ldots,\chi_\ell^{(j)},\ldots\right)$$

eigenvectors:

$$X_{(i)}^{\mu_1\cdots\mu_\ell} \equiv \sum_{j=0}^{N_\ell}\left(\Omega_\ell^{-1}\right)^{ij}\rho_{(j)}^{\mu_1\cdots\mu_\ell}$$

EoM take the form

$$\begin{aligned}
\dot{X}_{(i)} + \chi_0^{(i)} X_{(i)} &= \beta_{\chi 0}^{(i)}\theta + (\text{higher order terms})\,,\\
\dot{X}_{(i)}^{\langle\mu\rangle} + \chi_1^{(i)} X_{(i)}^{\mu} &= \beta_{\chi 1}^{(i)}\nabla^\mu\alpha_0 + (\text{higher order terms})\,,\\
\dot{X}_{(i)}^{\langle\mu\nu\rangle} + \chi_2^{(i)} X_{(i)}^{\mu\nu} &= \beta_{\chi 2}^{(i)}\sigma^{\mu\nu} + (\text{higher order terms})
\end{aligned}$$

- $1/\chi_\ell^{(i)}$ = microscopic time-scales of the (linearized) Boltzmann equation.
- $X_{(i)}^{\mu_1\cdots\mu_\ell}$ = Normal modes of the (linearized) Boltzmann equation.

Reduction of the d.o.f.

Dynamical modes $i = 0$ (slowest relaxation time)

$$\begin{aligned}
\dot{X}_{(i)} + \chi_0^{(i)} X_{(i)} &= \beta_{\chi 0}^{(i)} \theta + (\text{higher order terms}) , \\
\dot{X}_{(i)}^{\langle\mu\rangle} + \chi_1^{(i)} X_{(i)}^{\mu} &= \beta_{\chi 1}^{(i)} \nabla^{\mu} \alpha_0 + (\text{higher order terms}) , \\
\dot{X}_{(i)}^{\langle\mu\nu\rangle} + \chi_2^{(i)} X_{(i)}^{\mu\nu} &= \beta_{\chi 2}^{(i)} \sigma^{\mu\nu} + (\text{higher order terms})
\end{aligned}$$

Asymptotic values $i > 0$

$$\begin{aligned}
X_{(i)} &\sim \frac{\beta_{\chi 0}^{(i)}}{\chi_0^{(r)}} \theta + (\text{higher order terms}) , \\
X_{(i)}^{\mu} &\sim \frac{\beta_{\chi 1}^{(i)}}{\chi_1^{(r)}} \nabla^{\mu} \alpha_0 + (\text{higher order terms}) , \\
X_{(i)}^{\mu\nu} &\sim \frac{\beta_{\chi 2}^{(i)}}{\chi_2^{(r)}} \sigma^{\mu\nu} + (\text{higher order terms})
\end{aligned}$$

Reduction

Previous relations can be inverted to give

$$\begin{aligned}
\left(m^2/3\right)\rho_{(i)} &\simeq -\Omega_0^{i0}\Pi - \left(\zeta^{(i)} - \Omega_0^{i0}\zeta^{(0)}\right)\theta = -\Omega_0^{i0}\Pi + \mathcal{O}(\mathrm{Kn}), \\
\rho_{(i)}^{\mu} &\simeq \Omega_1^{i0} n^{\mu} + \left(\kappa_n^{(i)} - \Omega_1^{i0}\kappa_n^{(0)}\right)\nabla^{\mu}\alpha_0 = \Omega_1^{i0} n^{\mu} + \mathcal{O}(\mathrm{Kn}), \\
\rho_{(i)}^{\mu\nu} &\simeq \Omega_2^{i0}\pi^{\mu\nu} + \left(\eta^{(i)} - \Omega_2^{i0}\eta^{(0)}\right)\sigma^{\mu\nu} = \Omega_2^{i0}\pi^{\mu\nu} + \mathcal{O}(\mathrm{Kn}), \\
\rho_{(i)}^{\mu\nu\lambda\ldots} &\simeq \mathcal{O}(\mathrm{R}_i^{-1}\mathrm{Kn}, \mathrm{Kn}^2)
\end{aligned}$$

$$\zeta^{(i)} = \frac{m^2}{3}\sum_{r=0,\neq 1,2}^{N_0} \tau_0^{ir}\beta_\zeta^{(r)}, \quad \kappa_n^{(i)} = \sum_{r=0,\neq 1}^{N_1} \tau_1^{ir}\beta_\kappa^{(r)}, \quad \eta^{(i)} = \sum_{r=0}^{N_2} \tau_2^{ir}\beta_\eta^{(r)}$$

These can be used in the full e.o.m. to give evolution equation for dissipative quantities

Equations of fluid dynamics

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\frac{\zeta}{\tau_\Pi}\theta + \mathcal{J} + \mathcal{R} + \mathcal{K},$$

$$\dot{n}^{\langle\mu\rangle} + \frac{n^\mu}{\tau_n} = \frac{\kappa_n}{\tau_n}\nabla^\mu\alpha_0 + \mathcal{J}^\mu + \mathcal{R}^\mu + \mathcal{K}^\mu \ ,$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\frac{\eta}{\tau_\pi}\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu} + \mathcal{K}^{\mu\nu}$$

Terms of order Knudsen×Reynolds:

$$\mathcal{J} = -\ell_{\Pi n}\nabla\cdot n - \tau_{\Pi n} n\cdot F - \delta_{\Pi\Pi}\Pi\theta - \lambda_{\Pi n} n\cdot I + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu},$$

$$\mathcal{J}^\mu = -n_\nu\omega^{\nu\mu} - \delta_{nn}n^\mu\theta - \ell_{n\Pi}\nabla^\mu\Pi + \ell_{n\pi}\Delta^{\mu\nu}\nabla_\lambda\pi^\lambda_\nu + \tau_{n\Pi}\Pi F^\mu - \tau_{n\pi}\pi^{\mu\nu}F_\nu$$
$$- \lambda_{nn}n_\nu\sigma^{\mu\nu} + \lambda_{n\Pi}\Pi I^\mu - \lambda_{n\pi}\pi^{\mu\nu}I_\nu,$$

$$\mathcal{J}^{\mu\nu} = 2\pi^{\langle\mu}_\alpha\,\omega^{\nu\rangle\alpha} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\langle\mu}_\alpha\,\sigma^{\nu\rangle\alpha} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi n}n^{\langle\mu}F^{\nu\rangle}$$
$$+ \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \lambda_{\pi n}n^{\langle\mu}I^{\nu\rangle}$$

where

$$I^\mu = \nabla^\mu\alpha_0, \qquad F^\mu = \nabla^\mu P_0$$

Rn^2 and Kn^2

Terms of order Reynolds×Reynolds:

$$\mathcal{R} = \varphi_1 \Pi^2 + \varphi_2 n \cdot n + \varphi_3 \pi_{\mu\nu}\pi^{\mu\nu},$$
$$\mathcal{R}^\mu = \varphi_4 n_\nu \pi^{\mu\nu} + \varphi_5 \Pi n^\mu,$$
$$\mathcal{R}^{\mu\nu} = \varphi_6 \Pi \pi^{\mu\nu} + \varphi_7 \pi^{\lambda\langle\mu}\pi^{\nu\rangle}_{\lambda} + \varphi_8 n^{\langle\mu} n^{\nu\rangle}$$

Terms of order Knudsen×Knudsen:

$$\mathcal{K} = \omega_{\mu\nu}\omega^{\mu\nu} + \zeta_1 \sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta_2 \theta^2 + \zeta_3 I \cdot I + \zeta_4 F \cdot F + \zeta_5 I \cdot F + \zeta_6 \nabla \cdot I + \zeta_7 \nabla \cdot F,$$
$$\mathcal{K}^\mu = \kappa_1 \sigma^{\mu\nu} I_\nu + \kappa_2 \sigma^{\mu\nu} F_\nu + \kappa_3 I^\mu \theta + \kappa_4 F^\mu \theta + \kappa_5 \omega^{\mu\nu} I_\nu + \kappa_6 \Delta^\mu_\alpha \partial_\nu \sigma^{\alpha\nu} + \kappa_7 \nabla^\mu \theta,$$
$$\mathcal{K}^{\mu\nu} = \eta_1 \omega_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_2 \theta \sigma^{\mu\nu} + \eta_3 \sigma^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda + \eta_4 \sigma^{\langle\mu}_\lambda \omega^{\nu\rangle\lambda}$$
$$+ \eta_5 I^{\langle\mu} I^{\nu\rangle} + \eta_6 F^{\langle\mu} F^{\nu\rangle} + \eta_7 I^{\langle\mu} F^{\nu\rangle} + \eta_8 \nabla^{\langle\mu} I^{\nu\rangle} + \eta_9 \nabla^{\langle\mu} F^{\nu\rangle}$$

Transport coefficients

$$\tau_\Pi = \frac{1}{\chi_0^{(0)}}, \quad \tau_n = \frac{1}{\chi_1^{(0)}}, \quad \tau_\pi = \frac{1}{\chi_2^{(0)}},$$

$$\zeta = \frac{m^2}{3} \sum_{r=0,\neq 1,2}^{N_0} \tau_0^{0r} \beta_\zeta^{(r)}, \quad \kappa_n = \sum_{r=0,\neq 1}^{N_1} \tau_1^{0r} \beta_\kappa^{(r)}, \quad \eta = \sum_{r=0}^{N_2} \tau_2^{0r} \beta_\eta^{(r)}$$

In practice we have to use only finite number of terms in the expansion of $f_{\mathbf{k}}$ ...
Check for convergence

number of moments	η	τ_π	κ_n	τ_n
14	$4/(3\sigma\beta_0)$	$5/3\lambda_{mfp}$	$3/(16\sigma)$	$9/4\lambda_{mfp}$
23	$14/(11\sigma\beta_0)$	$2\lambda_{mfp}$	$21/(128\sigma)$	$2.59\lambda_{mfp}$
32	$1.268/(\sigma\beta_0)$	$2\lambda_{mfp}$	$0.1605/\sigma$	$2.57\lambda_{mfp}$
41	$1.267/(\sigma\beta_0)$	$2\lambda_{mfp}$	$0.1596/\sigma$	$2.57\lambda_{mfp}$

Table: Transport coefficients for classical gas with constant cross-section in the ultrarelativistic limit, in the 14, 23, 32 and 41-moment approximation

Note that:

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}_{\mathbf{k}}^{(n\ell)} \rho_{(n)}^{\mu_1\cdots\mu_\ell} k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle}$$

with

$$\mathcal{H}_{\mathbf{k}}^{(n\ell)} \equiv \frac{W_\ell}{\ell!} \sum_{m=n}^{N_\ell} a_{mn}^{(\ell)} P_{\mathbf{k}}^{(m\ell)}$$

Conclusions

- We have derived transient fluid dynamics using the method of moments
- We truncate equation of motion not expansion of $f_{\mathbf{k}}$
- 14-moment approximation is incomplete
- δf is infinite series