Higher flow harmonics from viscous hydrodynamics with fluctuating initial conditions

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Relativistic hydrodynamics

Hydrodynamics: conservation laws.
Works for small mean free path (compared to the system size).

$$\partial_\mu T^{\mu\nu} = 0$$

Generally:

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - P g^{\mu\nu} + \pi^{\mu\nu}.$$

with $T^{\mu\nu}u_\nu = \epsilon u^\mu$ and $u_\mu u^\mu = 1$. So $\pi^{\mu\nu}u_\nu = 0$.

Physics: Microscopic physics is thermal $\Rightarrow$ averages out.

Equation of state: $P = P(\epsilon)$

(from e.g. lattice QCD / hadron gas model)

New algorithm

We use a new (Kurganov-Tadmor) algorithm

- low numerical viscosity
- deals well with shocks

A. Kurganov, E. Tadmor, Journal of Computational Physics **160**, 241-282 (2000)

Finite volume method for Conservation Laws

How to solve the hyperbolic equations?

Example: 1-D current

- Average over a cell centered at x_j to turn

$$\partial_0 \rho = -\partial_x J$$

  into

$$\frac{d}{dt}\bar{\rho}_j = -\frac{1}{\Delta x}\left(J(x_{j+1/2}) - J(x_{j-1/2})\right)$$

  Exact so far. But J depends on $\rho(x_{j\pm 1/2})$, not $\bar{\rho}_j$.

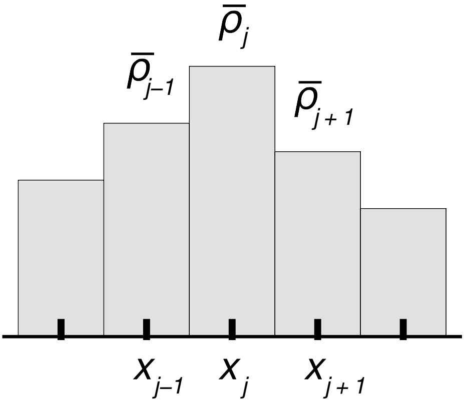
- Goal: Derive an approximate equation

$$\frac{d}{dt}\bar{\rho}_j = -\frac{1}{\Delta x}\left(H(x_{j+1/2}) - H(x_{j-1/2})\right)$$

  where H is constructed using $\bar{\rho}_j$, not $\rho(x_{j\pm 1/2})$.

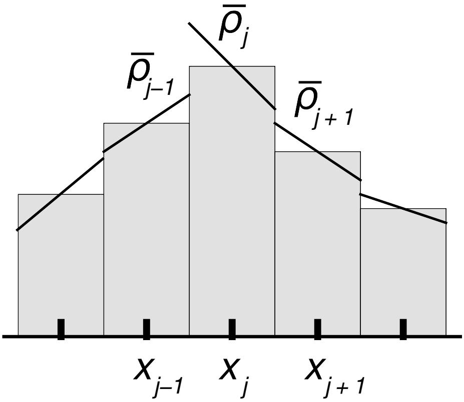
Finite volume method for Conservation Laws

- Main problem: How to get $J(x_{j\pm1/2})$ from $\bar{\rho}_j$?



Finite volume method for Conservation Laws

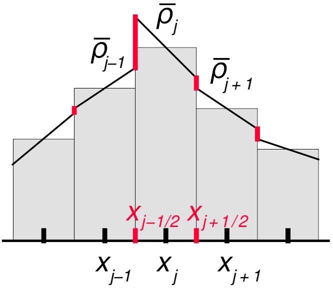
- Main problem: How to get $J(x_{j\pm 1/2})$ from $\bar{\rho}_j$?



- Do a piecewise linear reconstruction of the local values.

Kurganov-Tadmor

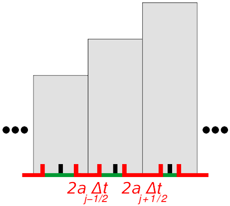
- **Trouble**: $x_{j\pm1/2}$ are where the boundaries are.
  Solve Riemann problems **or** avoid evaluating J at $x_{j\pm1/2}$



- **One idea**: Use staggered grid with cells centered at $x_{j-1/2}$ and $x_{j+1/2}$. So J is evaluated at x_j and x_{j+1}. In the next step use original grid, since discontinuities are now at x_j and x_{j+1}. Repeat (Nessyahu and Tadmor).
- **Better idea**: Use the propagation speed (Kurganov and Tadmor).

Kurganov-Tadmor

- Avoid evaluating J at discontinuity
- Divide intervals into $2a\Delta t$ around the discontinuity and the rest
- a: Maximum propagation speed



- Evaluate currents at edges of new cells
  (using charge conserving linear extrapolations)
- Project $a\Delta t$ intervals and $\Delta x - a\Delta t$ intervals onto the original Δx grid.

A. Kurganov, E. Tadmor, Journal of Computational Physics **160**, 241-282 (2000)

Kurganov-Tadmor

- In the end take $\Delta t \to 0$ limit to get

$$\frac{d}{dt}\bar{\rho}_j = -\frac{H_{j+1/2} - H_{j-1/2}}{\Delta x}$$

where

$$H_{j\pm1/2} = \frac{J(x_{j\pm1/2,+}) + J(x_{j\pm1/2,-})}{2} - \frac{a_{j\pm1/2}}{2}\left(\bar{\rho}_{j\pm1/2,+} - \bar{\rho}_{j\pm1/2,-}\right)$$

where

$$x^n_{j+1/2,\pm} = x_{j+1/2} \pm a^n_{j+1/2}\Delta t$$

and

$$\bar{\rho}_{j+1/2,+} = \bar{\rho}_{j+1} - \frac{\Delta x}{2}(\rho_x)_{j+1} \qquad \bar{\rho}_{j+1/2,-} = \bar{\rho}_j + \frac{\Delta x}{2}(\rho_x)_j$$

with the *minmod* flux limiter for the derivative $(\rho_x)_j$.

MUSIC

MUScl for Ion Collisions:

B. Schenke, S. Jeon, and C. Gale Phys. Rev. **C82**, 014903 (2010), arXiv:1004.1408

MUSCL = Monotonic Upstream Centered Scheme for Conservation Laws

- Solve KT

$$\frac{d}{dt}\bar{\rho}_j = -\frac{H_{j+1/2} - H_{j-1/2}}{\Delta x}$$

  with a second order Runge-Kutta scheme
- In 3+1 dimensions and $\tau - \eta$ coordinates
- Cooper-Frye freeze-out
  with sophisticated freeze-out surface construction
- Includes different equations of state including s95p-v1 from Huovinen and Petreczky (Lattice-QCD)
  P. Huovinen and P. Petreczky, Nucl. Phys. **A837**, 26-53 (2010)

Equations to be solved - ideal case

$$\partial_\mu T^{\mu\nu} = 0$$

In τ-η coordinates the relevant equations to be solved can be written as

$$\partial_\tau(\tau T^{\tau\tau}) + \partial_v(\tau T^{v\tau}) + \partial_{\eta_s}(T^{\eta_s\tau}) + T^{\eta_s\eta_s} = 0$$

and

$$\partial_\tau(\tau T^{\tau\eta_s}) + \partial_{\eta_s}(T^{\eta_s\eta_s}) + \partial_v(\tau T^{v\eta_s}) + T^{\tau\eta_s} = 0$$

and

$$\partial_\tau(\tau T^{\tau v}) + \partial_{\eta_s}(T^{\eta_s v}) + \partial_w(\tau T^{wv}) = 0$$

terms in red are expressed by the Kurganov-Tadmor term "$\frac{H_{j+1/2} - H_{j-1/2}}{\Delta x}$"

terms in green come from the coordinate transformation and are treated as external sources

Maximum propagation speed

Max speed in the k direction is the max eigenvalue of the Jacobian

$$\mathcal{J}^k_{ab} = \frac{\partial J^k_a}{\partial J^\tau_b}$$

a, b are five currents (net baryon, energy, and momentum).

Including viscosity

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu} + \pi^{\mu\nu}$$

$\pi^{\mu\nu}$ = shear part with $\pi^{\mu\nu}u_\nu = 0$ and $\pi^\mu_\mu = 0$.

We use the second order Israel-Stewart formalism, where $\pi^{\mu\nu}$ satisfies

$$\begin{aligned}\pi^{\mu\nu} =& \eta(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla_\alpha u^\alpha) \\ & - \frac{4}{3}\tau_\pi \pi^{\mu\nu}\partial_\alpha u^\alpha - \tau_\pi \Delta^\mu_\alpha \Delta^\nu_\beta u^\sigma \partial_\sigma \pi^{\alpha\beta}\,,\end{aligned}$$

with $\partial_\mu = \nabla_\mu + u_\mu(u_\alpha \partial^\alpha)$ and $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$.

The relaxation time τ_π is set to $\frac{3\eta}{\epsilon+\mathcal{P}}$.

Equations to be solved - viscous case

In τ-η coordinates the relevant equations to be solved can be written explicitly as

$$\partial_\tau(\tau T^{\tau\tau}) + \partial_v(\tau T^{v\tau}) + \partial_{\eta_s}(T^{\eta_s\tau}) + T^{\eta_s\eta_s}$$
$$+\partial_\tau(\tau\pi^{\tau\tau}) + \partial_v(\tau\pi^{v\tau}) + \partial_{\eta_s}(\pi^{\eta_s\tau}) + \pi^{\eta_s\eta_s} = 0$$

$$\partial_\tau(\tau T^{\tau\eta_s}) + \partial_v(\tau T^{v\eta_s}) + \partial_{\eta_s}(T^{\eta_s\eta_s}) + T^{\tau\eta_s}$$
$$+\partial_\tau(\tau\pi^{\tau\eta_s}) + \partial_v(\tau\pi^{v\eta_s}) + \partial_{\eta_s}(\pi^{\eta_s\eta_s}) + \pi^{\tau\eta_s} = 0$$

$$\partial_\tau(\tau T^{\tau v}) + \partial_w(\tau T^{wv}) + \partial_{\eta_s}(T^{\eta_s v})$$
$$+\partial_\tau(\tau\pi^{\tau v}) + \partial_w(\tau\pi^{wv}) + \partial_{\eta_s}(\pi^{\eta_s v}) = 0$$

Kurganov-Tadmor current
treated as sources

Equations to be solved - viscous case

In addition, we have to solve the equation for $\pi^{\mu\nu}$:

$$
\begin{aligned}
\partial_c(u^c\pi^{ab}) = & -\frac{1}{2\tau}u^\tau\pi^{ab} + \frac{1}{\tau}\Delta^{a\eta}u^\eta\pi^{b\tau} - \frac{1}{\tau}\Delta^{a\tau}u^\eta\pi^{b\eta} \\
& - g_{cf}\pi^{cb}u^a Du^f - \frac{\pi^{ab}}{2\tau_\pi} - \frac{1}{6}\pi^{ab}\partial_c u^c \\
& + \frac{\eta}{\tau_\pi}\Bigg(-\frac{1}{\tau}\Delta^{a\eta}g^{b\eta}u^\tau + \frac{1}{\tau}\Delta^{a\eta}g^{b\tau}u^\tau \\
& \qquad + g^{ac}\partial_c u^b - u^a Du^b - \frac{1}{3}\Delta^{ab}\partial_c u^c\Bigg) \\
& + (a \leftrightarrow b)\,,
\end{aligned}
$$

Cooper-Frye - ideal case

We compute the thermal particle spectra using the **Cooper-Frye formula**

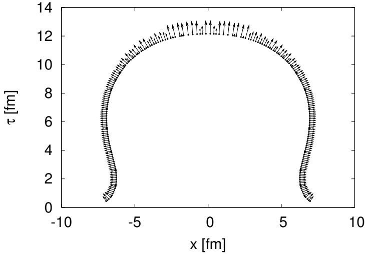
$$E\frac{dN}{dyp_Tdp_Td\phi_p} = g_i \int_\Sigma f(u^\mu p_\mu)p^\alpha d^3\Sigma_\alpha \,,$$

with

$$f(u^\mu p_\mu) = \frac{1}{(2\pi)^3}\frac{1}{\exp\left[(u^\mu p_\mu - \mu_i)/T_{\rm FO}\right] \pm 1} \,,$$

and the freeze-out surface Σ.

Σ is constructed geometrically, extending the triangulation algorithm for 2+1D hydro to 3+1 dimensions. That means, every surface element is built from tetrahedra.



Including viscosity

Viscous correction to the equilibrium distribution functions:

$$f \rightarrow f + \delta f$$

with

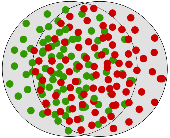
$$\delta f = f_0(1 \pm f_0) p^\alpha p^\beta \pi_{\alpha\beta} \frac{1}{2(\epsilon + \mathcal{P})T^2}$$

The choice $\delta f \sim p^2$ is not unique.
Ambiguity in δf leads to large uncertainty.

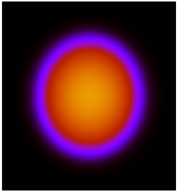
see Dusling, Moore, and Teaney, Phys.Rev.C81:034907 (2010)

Event-by-event hydrodynamics

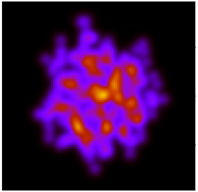
average, then evolve



evolve, then average

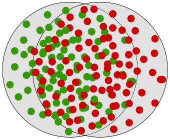


initial energy density

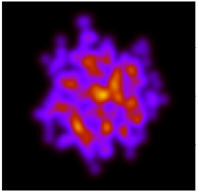


initial energy density

Event-by-event hydrodynamics



evolve, then average

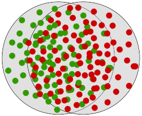


initial energy density

Event-by-event!

Initialization:

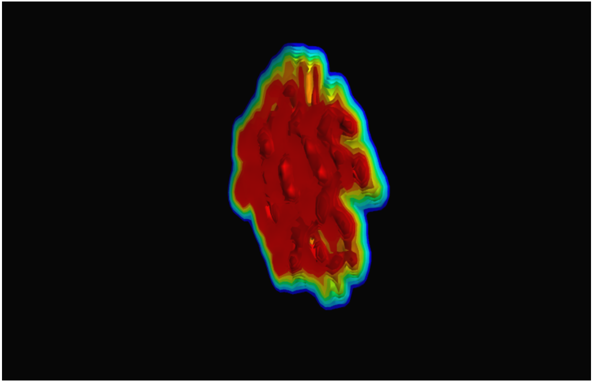
- Sample Woods-Saxon distributions to determine all nucleon positions
- Overlap those distributions using impact parameter b

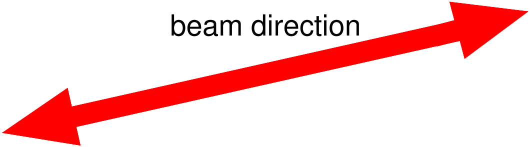


b is sampled from $P(b)db = 2bdb/(b_{\mathrm{max}}^2 - b_{\mathrm{min}}^2)$

- Nucleon-nucleon collision occurs if distance is $< \sqrt{\sigma_{NN}/\pi}$
- At position of collision add 2D-Gaussian energy density distribution with width σ_0.
  For now we use $\sigma_0 = 0.4\,\mathrm{fm}$.

Single event initial conditions

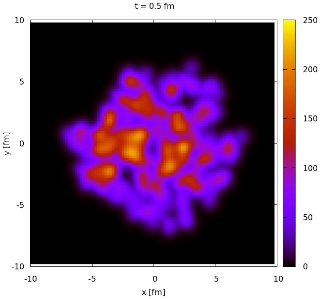




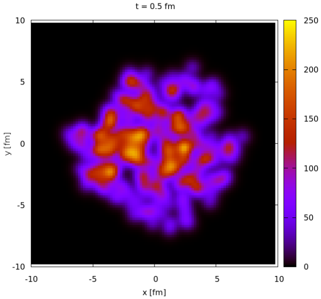
Viscosity in event-by-event simulations

(scale in 1/fm^4):

ideal



$\eta/s = 0.16$



B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

Viscosity in event-by-event simulations

(scale in $1/\text{fm}^4$):

ideal

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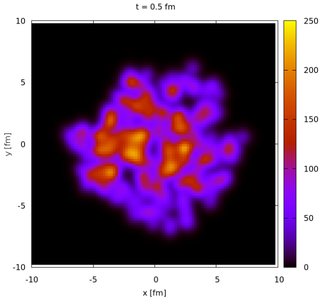
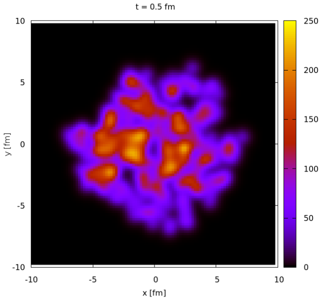




(the pulsating comes from adjusting the energy density scale of the plot)

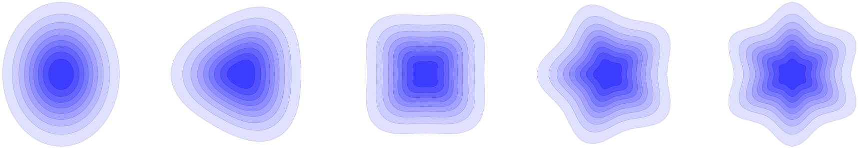
B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

Higher harmonics

Anisotropic flow can be characterized by a Fourier decomposition of the azimuthal particle distributions.

$$\frac{dN}{d\phi} = \frac{N}{2\pi}\left(1 + \sum_n (2v_n \cos(n\phi))\right)$$

When including fluctuations, all moments appear:



$n = 2$ $n = 3$ $n = 4$ $n = 5$ $n = 6$

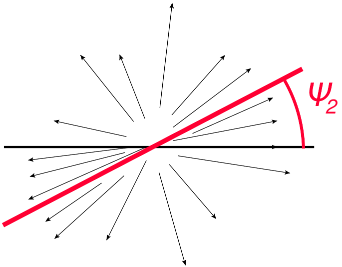
also v_1 and $n > 6$.

Event plane

To get non-zero odd moments, we rotate the event plane in each event.

Event plane is defined by the angle:

$$\psi_n = \frac{1}{n} \arctan \frac{\langle p_T \sin(n\phi) \rangle}{\langle p_T \cos(n\phi) \rangle}$$



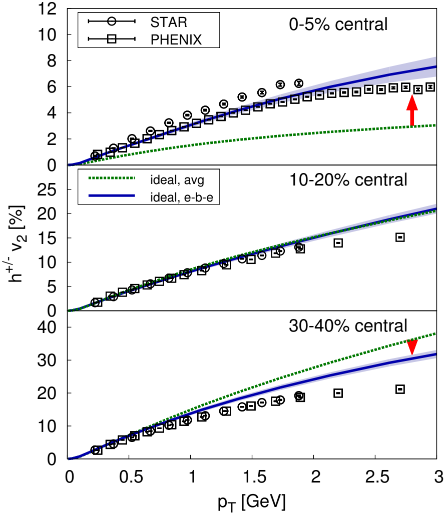
using particle momenta.

A.Poskanzer and S.Voloshin, Phys.Rev.C58:1671-1678 (1998)

$$v_n = \langle \cos(n(\phi - \psi_n)) \rangle$$

... different angle for every flow coefficient.

Flow results from e-b-e viscous MUSIC

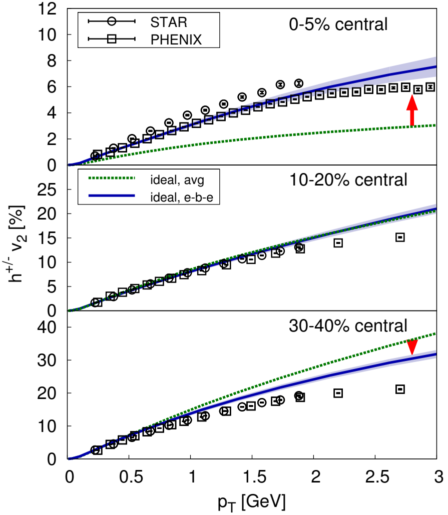


Most central collisions:
fluctuations increase elliptic flow

Larger centralities:
fluctuations decrease elliptic flow

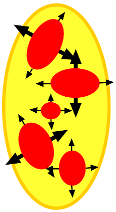
Experimental data: J. Adams et al. (STAR), Phys.Rev.C72, 014904 (2005)
A. Adare et al. (PHENIX), Phys.Rev.Lett.105, 062301 (2010)

Flow results from e-b-e viscous MUSIC



Why?

- Measure with respect to the event-plane: Maximizes flow.
- Competing effect: Substructure of hotspots



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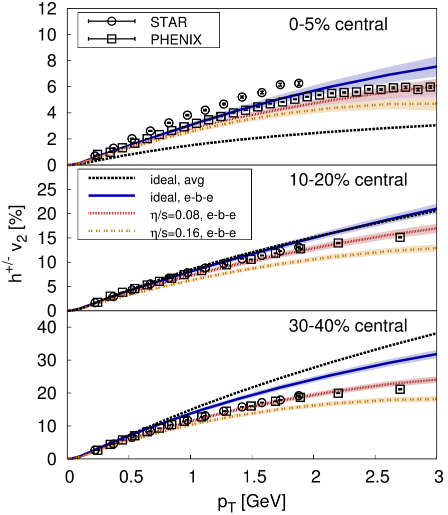

B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

Now add shear viscosity.

$\eta/s = 0.08$ and

$\eta/s = 0.16$

Experimental data: J. Adams et al. (STAR), Phys.Rev.C72, 014904 (2005)
A. Adare et al. (PHENIX), Phys.Rev.Lett.105, 062301 (2010)

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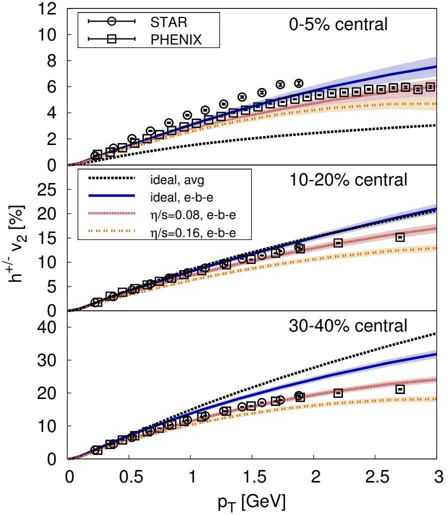
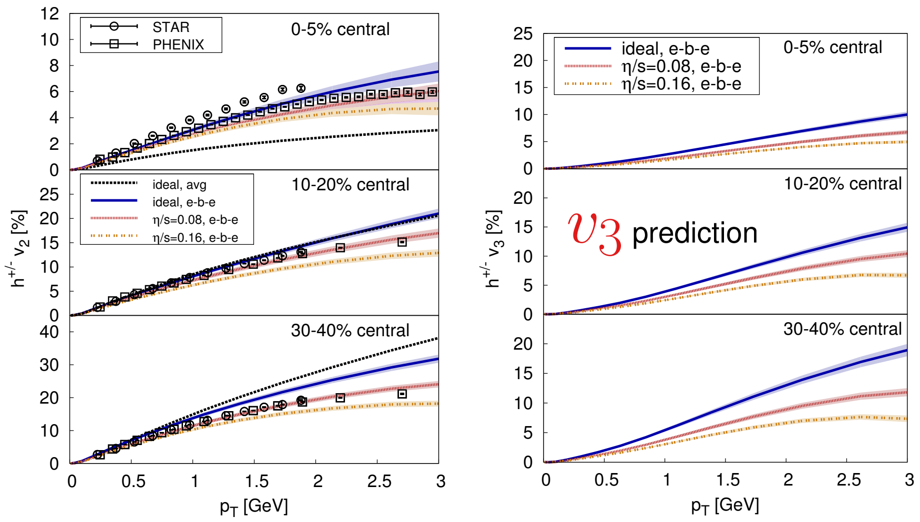


B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

- event-by-event fluctuations important!
- Viscosity is **very low.** The lower bound of viscosity/entropy density conjectured from AdS/CFT duality is $\eta/s = 1/4\pi \approx 0.08$ (red curves)

Experimental data: J. Adams et al. (STAR), Phys.Rev.C72, 014904 (2005)
A. Adare et al. (PHENIX), Phys.Rev.Lett.105, 062301 (2010)

Flow results from e-b-e viscous MUSIC



B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

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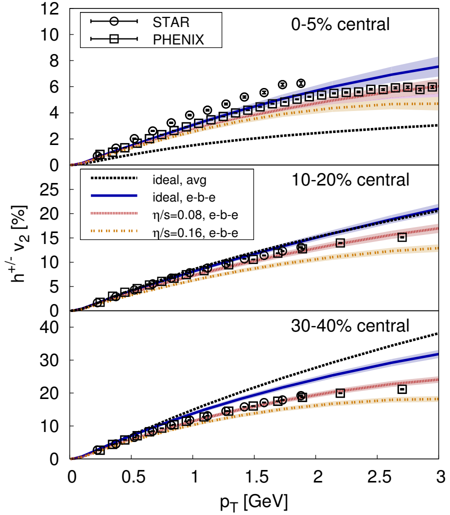
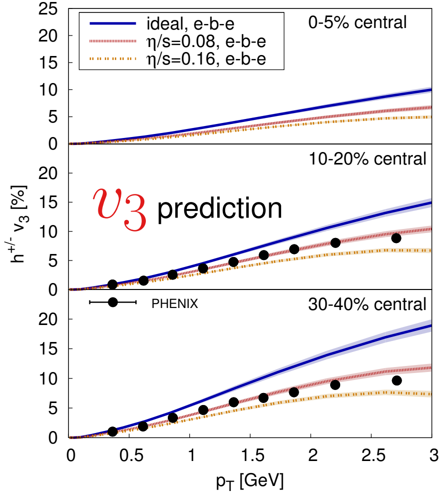
Flow results from e-b-e viscous MUSIC

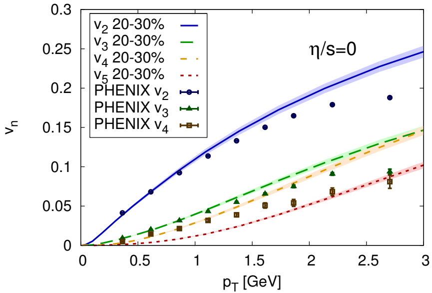


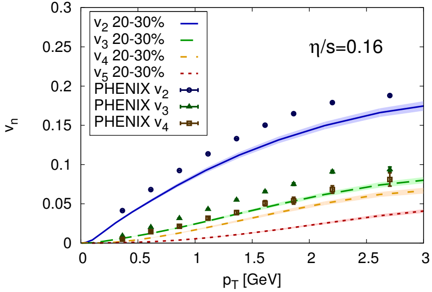


B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

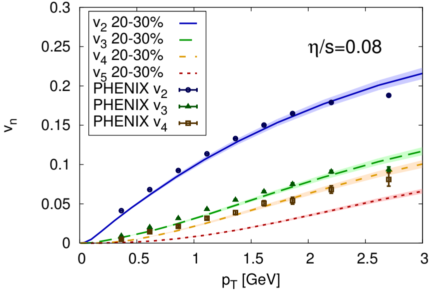
Experimental data: J. Adams et al. (STAR), Phys.Rev.C72, 014904 (2005)
A. Adare et al. (PHENIX), Phys.Rev.Lett.105, 062301 (2010) , A. Adare et al. (PHENIX), arXiv:1105.3928 (2011)

Using higher harmonics to determine η/s





PRELIMINARY

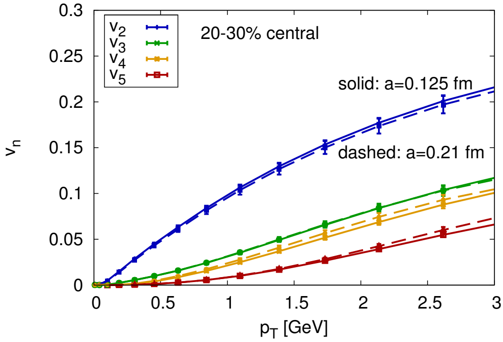


This is promising.
Need systematic study of all v_n as function of initial conditions, granularity, η/s, ...

Experimental data: PHENIX, arXiv:1105.3928

Sensitivity to the lattice spacing

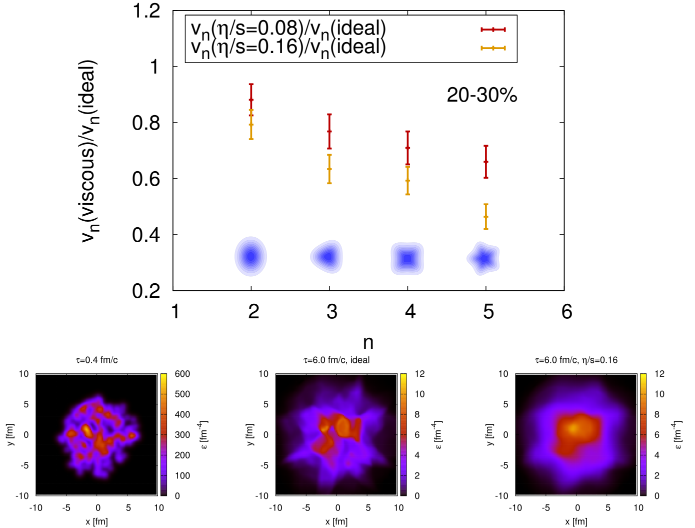
Higher v_n are driven by fine spatial structures. Are we resolving these?



With finite viscosity (here $\eta/s = 0.08$) we are ok up to at least v_5!

Dependence on η/s

Higher Fourier coefficients are suppressed more by viscosity.



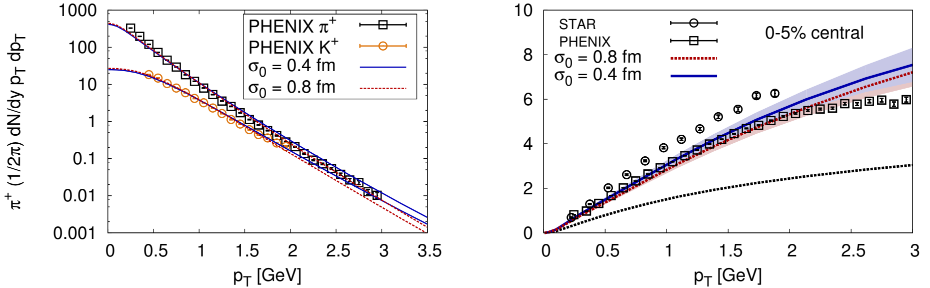
Summary

- 3+1d relativistic event-by-event viscous hydrodynamic simulation
- Kurganov-Tadmor algorithm has very low numerical viscosity, deals well with shocks.
- Viscous event-by-event simulations are necessary (and feasible).
- Event-by-event hydro allows to compute all flow harmonics.
- Need to study higher harmonics and their fluctuations to determine initial conditions and η/s separately.

BACKUP

A little more detail

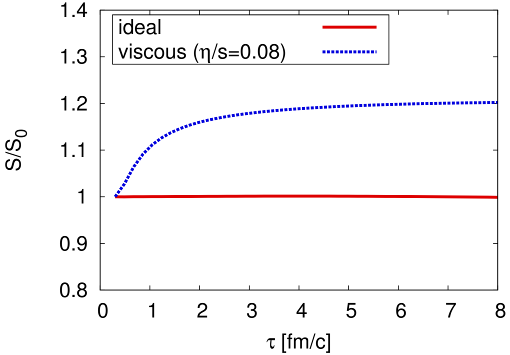
Dependence on the width σ_0, used to initialize the lumpy energy density distribution.



Qualitatively: Harder spectra and larger v_2 for smaller σ_0.
Quantitatively: Small effect - within statistical error bars for $p_T < 2\,\mathrm{GeV}$.

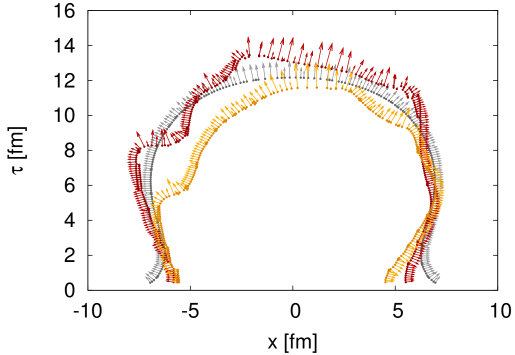
Entropy production

Ideal hydro: $\partial_\mu S^\mu = 0$ with $S^\mu = s u^\mu \Rightarrow S = \int_V d\mathbf{x}_\perp \tau d\eta s u^\tau = \text{const.}$
Viscous hydro: physical entropy production



Freeze-out surface in a single event

The freeze-out finder works great in a single event:



B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

Issues

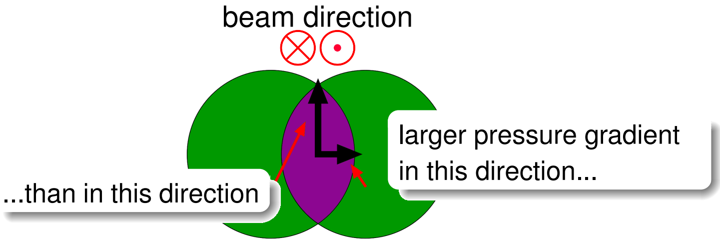
What if $\pi^{\mu\nu}$, the viscous correction to $T^{\mu\nu}$, becomes larger than $T^{\mu\nu}$?
It does happen. Mostly at edges where the energy density is low.

Options:

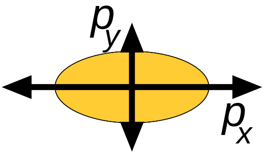
- Revert to previous value of $\pi^{\mu\nu}$:
  Will stop evolution of $\pi^{\mu\nu} \rightarrow$ Overestimates $\pi^{\mu\nu}$ at freeze-out.
- Set $\pi^{\mu\nu}$ to zero:
  Numerically unstable. Underestimates viscous contribution.
- Return to previous value and reduce it by 5%:
  Best tested method. Smoothly decreases $\pi^{\mu\nu}$. Numerically stable.

Another observable: Flow

Non-central collision



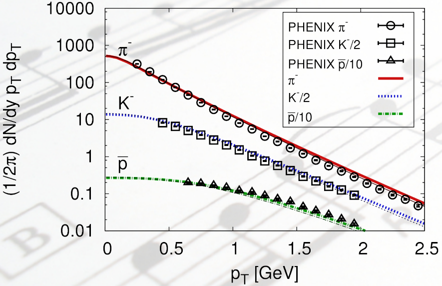
Particle distribution in **momentum space** will be anisotropic.
Quantify using a Fourier decomposition:



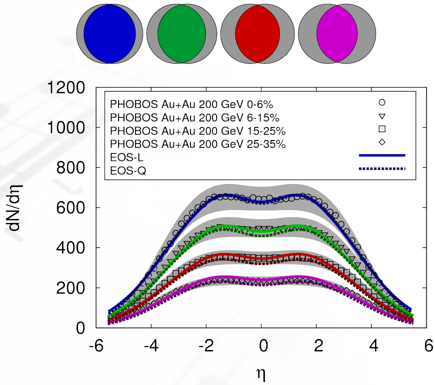
$$\frac{dN}{d\phi} = \frac{N}{2\pi}\left(1 + \sum_n (2v_n \cos(n\phi))\right) \Rightarrow v_2 \text{ characterizes elliptic flow}$$

Results from ideal MUSIC

Include resonances up to $2\,\mathrm{GeV}$ and compute their decays to get the Particle spectra



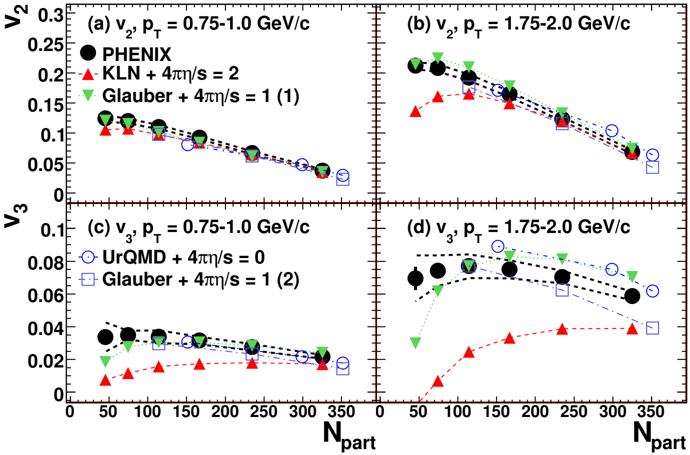
transverse momentum



pseudo-rapidity

B. Schenke, S. Jeon, and C. Gale Phys. Rev. **C82**, 014903 (2010)

Flow results from e-b-e viscous MUSIC



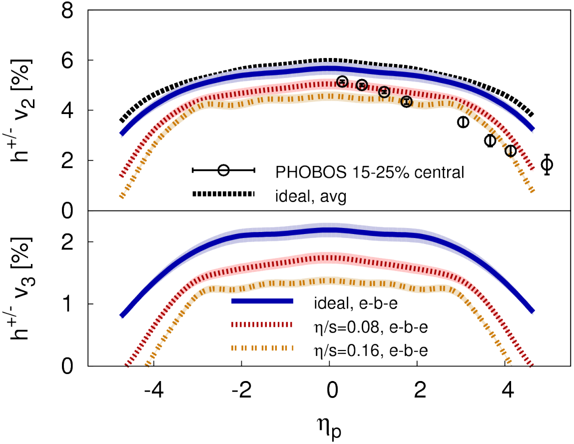
blue squares are MUSIC

B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)
Experimental data: A. Adare et al. (PHENIX), arXiv:1105.3928 (2011)

Flow results from e-b-e viscous MUSIC

Event-by-event fluctuations and viscosity improve agreement with experimental data.

v_2 and v_3 as functions of pseudo-rapidity η_p

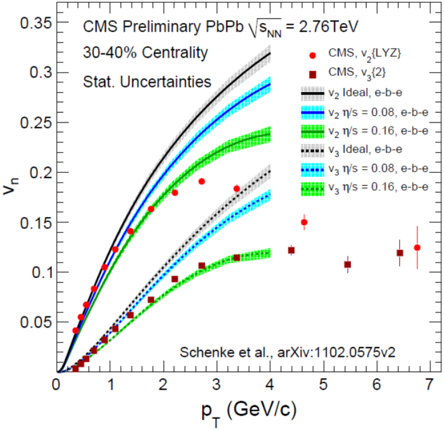


B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

Experimental data: B.B. Back et al. (PHOBOS), Phys.Rev.C72:051901 (2005)

LHC $v_2(p_T)$ and $v_3(p_T)$

v_3 was a prediction!



No perfect agreement yet. But not a lot of tuning was done for LHC.

Experimental data: The CMS collaboration 2011

B. Schenke, S. Jeon, C. Gale, Phys. Lett. B702, 59-63 (2011)

Numerical viscosity

- In the Lax scheme, the time derivative is expressed by

$$\frac{d}{dt}\rho = \frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} - \frac{\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n}{2\Delta t}$$

  to deal with instability of the simplest discretization (FTCS).
  In "continuum form" the last term reads $\frac{(\Delta x)^2}{2\Delta t}\partial_x^2\rho$, introducing numerical viscosity

- In KT we have

$$\frac{d}{dt}\bar{\rho}_j(t) = -\frac{\left[J\left(\bar{\rho}^+_{j+1/2}(t)\right) + J\left(\bar{\rho}^-_{j+1/2}(t)\right)\right] - \left[J\left(\bar{\rho}^+_{j-1/2}(t)\right) + J\left(\bar{\rho}^-_{j-1/2}(t)\right)\right]}{2\Delta x}$$
$$+\frac{1}{2\Delta x}\left\{a_{j+1/2}(t)\left[\bar{\rho}^+_{j+1/2}(t) - \bar{\rho}^-_{j+1/2}(t)\right] - a_{j-1/2}(t)\left[\bar{\rho}^+_{j-1/2}(t) - \bar{\rho}^-_{j-1/2}(t)\right]\right\}$$

  Taylor expanding $\bar{\rho}_{j\pm1}$ in $\bar{\rho}^{\pm}_{j\pm1/2}$ around $\bar{\rho}_j$ shows that numerical viscosity is of order $(\Delta x)^3$
  No $1/\Delta t$ term!

Maximum propagation speed

Max speed in the k direction is the max eigenvalue of the Jacobian

$$\mathcal{J}^k_{ab} = \frac{\partial J^k_a}{\partial J^\tau_b}$$

a, b are five currents (net baryon, energy, and momentum).
For $k = x, y$:

- Two eigenvalues are u^k/u^τ.
- Two eigenvalues are $\lambda^\pm_k = \frac{A \pm \sqrt{B}}{D}$,

  with $A = u^\tau u^k (1 - v_s^2)$, $B = [u_\tau^2 - u_k^2 - (u_\tau^2 - u_k^2 - 1)v_s^2]v_s^2$, $D = u_\tau^2 - (u_\tau^2 - 1)v_s^2$

For $k = \eta_s$:

- Same eigenvalues but scaled by $1/\tau$

Largest eigenvalue is $\lambda^\pm_k = \frac{|A| + \sqrt{B}}{D}$ (times $1/\tau$ if $k = \eta_s$).
That's the maximum local propagation speed.

Minmod flux limiter

The minmod flux limiter takes care of switching between a second order derivative where the function is smooth and a first order derivative where there are large gradients.

The derivative is

$$(\rho_x)_j = \text{minmod}\left(\theta\frac{\bar{\rho}_{j+1}-\bar{\rho}_j}{\Delta x}, \frac{\bar{\rho}_{j+1}-\bar{\rho}_{j-1}}{2\Delta x}, \theta\frac{\bar{\rho}_j-\bar{\rho}_{j-1}}{\Delta x}\right),$$

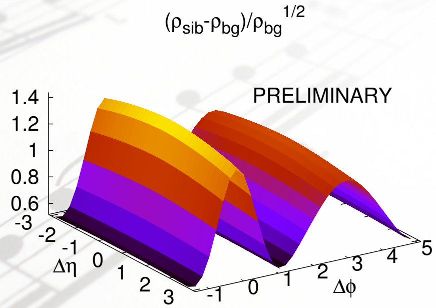
with

$$\text{minmod}(x_1, x_2, \cdots) = \begin{cases} \text{min}_i\{x_i\}, & \text{if } x_i > 0\ \forall i \\ \text{max}_i\{x_i\}, & \text{if } x_i < 0\ \forall i \\ 0, & \text{otherwise} \end{cases}$$

$\theta \in [1, 2]$. We use $\theta = 1.1$.

Correlations: away-side ridge

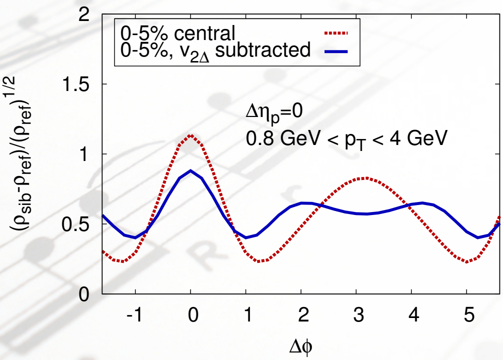
Untriggered $\Delta\eta - \Delta\phi$ correlations from viscous hydro.



Correlations: Fourier decomposition

Fourier decomposition, subtract the elliptic flow component:

as in B. Alver and G. Roland, Phys.Rev.C81:054905 (2010)



PRELIMINARY