Real-Time Methods for Critical Dynamics

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NA7-Hf-QGP | Oct 6, 2021

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DFG

CRC-TR 211 Strong-interaction matter under extreme conditions

HGS-HIRe for FAIR Helmholtz Graduate School for Hadron and Ion Research

References

- DS, S. Schlichting, L. von Smekal.
  Spectral functions and dynamic critical behavior of relativistic Z_2 theories.
  NuclPhysB 960 115165, arXiv:2007.03374
- DS, S. Schlichting, L. von Smekal.
  Critical dynamics of relativistic diffusion.
  TBP, arXiv:2110.01696
- J. V. Roth, DS, L. Sieke, L. von Smekal.
  Real-time methods for spectral functions.
  TBP

Phase transitions

Phase transitions

- First-order phase transition: discontinuity in first derivative of thermodynamic potential (Ehrenfest)
- solid $\rightarrow$ liquid $\rightarrow$ gaseous, e.g. water, CO_2

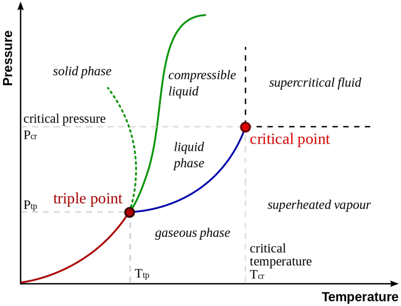

Figure: Image by Lanju Fotografie (CC0), Typical phase diagram, Maksim (GFDL)

- Continuous transition: thermodynamic potential analytic


- Second order phase transition: discontinuity in second derivatives
  - E.g. in binary mixtures, ferromagnets, . . .

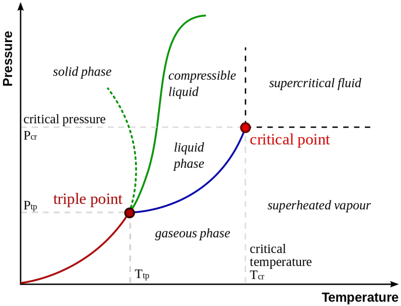

Figure: Images by Sorin Gheorghita and Dan Dennis (CC0), Typical phase diagram, Maksim (GFDL)

Critical point – observables

- Competing processes minimize free energy
  $F = U - TS$
- Strong fluctuations $\Rightarrow$
  divergent correlation length ξ, scale invariance
  $$\langle \phi(x)\phi(x') \rangle \sim e^{-|x-x'|/\xi}$$

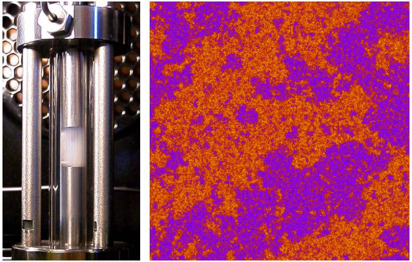

Figure:
(1): Critical opalescence of ethane
Dr. Sven Horstmann (CC3.0)
(2): Field configuration at the critical point

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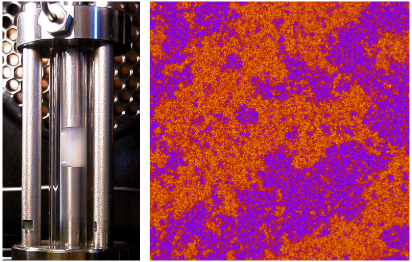
- Observables $\rightarrow$ power laws with model-dependent amplitudes, universal exponents

$$\langle O(T) \rangle = a|\tau|^{\sigma}, \quad \tau \equiv \frac{T - T_c}{T_c}$$



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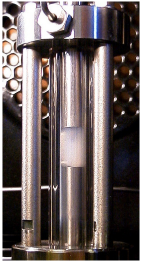
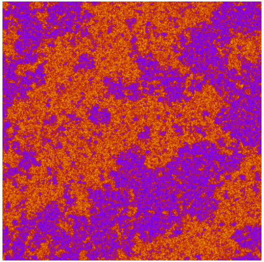

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Universality

Microscopic details irrelevant $\Rightarrow$ classical physics

- Universality class of QCD CEP: Z_2 Ising
  - chiral $SU(2)_A$ symmetry sponaneously broken
  - order parameter $\langle \bar{\psi}\psi \rangle$

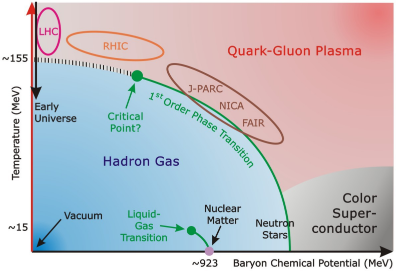

Figure: Semi-Quantitatives Phasendiagramm der QCD

Chiral endpoint of QCD

- Universality class of QCD CEP: Z_2 Ising
  - chiral $SU(2)_A$ symmetry sponaneously broken
  - order parameter $\langle\bar{\psi}\psi\rangle$
- Landau-Ginzburg-Wilson

$$\mathcal{P}[\phi] \sim e^{-\beta\mathfrak{H}[\phi]},$$

$$\mathfrak{H}[\phi] = \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 - J\phi$$

- $J = 0$: Z_2 symmetry $\phi \to -\phi$
  - $m^2 < 0$: spontaneous symmetry-breaking at $T < T_c$
  - order parameter $M \equiv \langle\phi\rangle$

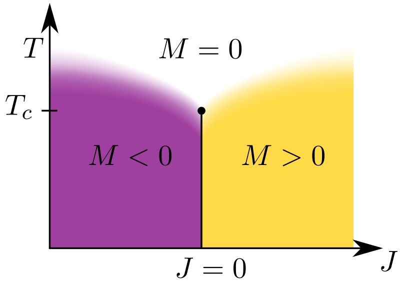

Figure: Semi-Quantitatives Phasendiagramm der LGW-Theorie

QCD vs. Model

- Observing QCD matter in heavy-ion collisions
  - transient process
  - equilibrium not guaranteed

⇒ Dynamics relevant for determining the CEP

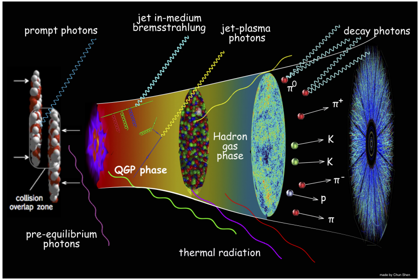

Figure: Visualization of a relativistic heavy-ion collision (Chun Shen)

Universality

Study dynamics of LGW theory to make predictions about QCD

Non-equilibrium field theory

Real-time QFT

- Von-Neumann equation

$$\partial_t \hat{\rho}(t) = -\mathrm{i}\left[\hat{H}(t),\, \hat{\rho}(t)\right],$$

formally solved by evolution operator

$$\hat{U}_{t,t'} = \mathbb{T}\exp\left(-\mathrm{i}\int_{t'}^{t}\hat{H}(t)\mathrm{d}t\right), \quad \hat{\rho}(t) = \hat{U}_{t,-\infty}\hat{\rho}(-\infty)\hat{U}_{-\infty,t},$$

- Expectation values of observables:

$$\langle\hat{O}(t)\rangle \equiv \frac{\mathrm{Tr}\left\{\hat{O}\hat{\rho}(t)\right\}}{\mathrm{Tr}\left\{\hat{\rho}(t)\right\}} = \frac{\mathrm{Tr}\left\{\hat{U}_{-\infty,t}\hat{O}\hat{U}_{t,-\infty}\hat{\rho}(-\infty)\right\}}{\mathrm{Tr}\left\{\hat{\rho}(t)\right\}}$$

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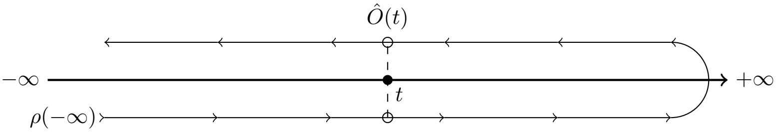
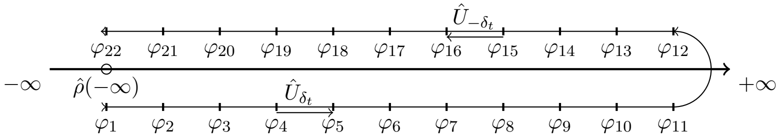


- Closed time path:

$$\langle \hat{O}(t) \rangle = \frac{\mathrm{Tr}\left\{ \hat{U}_{-\infty,\infty} \hat{U}_{\infty,t} \hat{O} \hat{U}_{t,-\infty} \hat{\rho}(-\infty) \right\}}{\mathrm{Tr}\left\{ \hat{\rho}(-\infty) \right\}}$$

  - Continue evolution of state ρ up to $t \to \infty$
  - Insert hermitian operator $\hat{O}$ at time t on branches
  - Generating function: $\hat{H} \to \hat{H}_V^{\pm} \equiv \hat{H} \pm \hat{O} V(t)$

$$Z[V] \equiv \frac{\mathrm{Tr}\left\{ \hat{U}_{\mathcal{C}}[V] \hat{\rho}(-\infty) \right\}}{\mathrm{Tr}\left\{ \hat{\rho}(-\infty) \right\}} \quad \Rightarrow \quad \langle O(t) \rangle = \frac{\mathrm{i}}{2} \left. \frac{\delta Z}{\delta V(t)} \right|_{V \equiv 0}$$



- Write partition function as path integral
  - Discretize evolution along time contour
  - Insert unity in coherent state basis: $\hat{\mathbb{1}} = \int \mathrm{d}\left[\bar{\varphi}, \varphi\right] e^{-|\varphi|^2} \left|\varphi\right\rangle \left\langle\varphi\right|$
  - Evaluate infinitesimal evolution operators $\hat{U}_{\pm\delta_t}$

$$Z = \frac{1}{\mathrm{Tr}\left\{\hat{\rho}(t_0)\right\}} \int \prod_{j=1}^{2N} \mathrm{d}\left[\bar{\varphi}_j, \varphi_j\right] \exp\left(\mathrm{i} \sum_{j,j'=1}^{2N} \bar{\varphi}_j G^{-1}_{jj'} \varphi_{j'}\right)$$

- Take formal continuum limit

$$Z = \int \mathbf{D}\left[\bar{\varphi}, \varphi\right] e^{\mathrm{i}S[\bar{\varphi},\varphi]}, \quad S\left[\bar{\varphi}, \varphi\right] = \int_{\mathcal{C}} \mathrm{d}t\mathrm{d}t' \bar{\varphi}(t) \hat{G}^{-1}(t,t') \varphi(t')$$

Closed time path

$$Z=\int \mathbf{D}\left[\bar{\varphi},\varphi\right]e^{\mathrm{i}S[\bar{\varphi},\varphi]},\quad S\left[\bar{\varphi},\varphi\right]=\int_{\mathcal{C}}\mathrm{d}t\mathrm{d}t'\bar{\varphi}(t)\hat{G}^{-1}(t,t')\varphi(t')$$

- Rewrite action to get rid of contour integral, split fields into parts evaluated on forward ($\varphi^{+}(t)$) and backward ($\varphi^{-}(t)$) branch

$$S\left[\bar{\varphi},\varphi\right]=\int_{-\infty}^{\infty}\mathrm{d}t\mathrm{d}t'\begin{pmatrix}\bar{\varphi}^{+}(t) & \bar{\varphi}^{-}(t)\end{pmatrix}\hat{G}^{-1}(t,t')\begin{pmatrix}\varphi^{+}(t')\\ \varphi^{-}(t')\end{pmatrix}$$

where

$$\hat{G}(t,t')=\begin{pmatrix}G^{\mathbb{T}}(t,t') & G^{>}(t,t')\\ G^{<}(t,t') & G^{\tilde{\mathbb{T}}}(t,t')\end{pmatrix}=-\mathrm{i}\begin{pmatrix}\langle\varphi^{+}(t)\bar{\varphi}^{+}(t')\rangle & \langle\varphi^{-}(t)\bar{\varphi}^{+}(t')\rangle\\ \langle\varphi^{+}(t)\bar{\varphi}^{-}(t')\rangle & \langle\varphi^{-}(t)\bar{\varphi}^{-}(t')\rangle\end{pmatrix}$$

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- Correlators are not linearly independent:

$$G^{\mathbb{T}}(t,t') - G^{>}(t,t') - G^{<}(t,t') + G^{\tilde{\mathbb{T}}}(t,t') = 0$$

$\Rightarrow$ Change of coordinates: *Keldysh rotation* $\varphi^{\mathsf{c}/\mathsf{q}} \equiv \frac{1}{\sqrt{2}}\left(\varphi^{+} \pm \varphi^{-}\right)$

$$\Rightarrow \quad \hat{G}(t,t') = \begin{pmatrix} G^{K}(t,t') & G^{R}(t,t') \\ G^{A}(t,t') & 0 \end{pmatrix} = -\mathrm{i} \begin{pmatrix} \langle \varphi^{\mathsf{c}}(t)\bar{\varphi}^{\mathsf{c}}(t')\rangle & \langle \varphi^{\mathsf{c}}(t)\bar{\varphi}^{\mathsf{q}}(t')\rangle \\ \langle \varphi^{\mathsf{q}}(t)\bar{\varphi}^{\mathsf{c}}(t')\rangle & \langle \varphi^{\mathsf{q}}(t)\bar{\varphi}^{\mathsf{q}}(t')\rangle \end{pmatrix}$$

Example: Harmonic Oscillator

- Feynman Lagrangian action: $S[\phi] = \int_{\mathcal{C}} \mathrm{d}t \left[\frac{1}{2}\dot{\phi}^2 - \frac{\omega_0^2}{2}\phi^2\right] \equiv \int_{\mathcal{C}} \mathrm{d}t \left[\frac{1}{2}\dot{\phi}^2 - V(\phi)\right]$

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- Matrix representation of Keldysh action for $V(\phi) = \omega_0^2\phi^2/2$:

$$\left[G_0^{-1}\right]^R = (\mathrm{i}\partial_t + \mathrm{i}\varepsilon)^2 - \omega_0^2$$

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Example: Harmonic Oscillator

- Keldysh-rotated propagators:

$$G_0^R(t,t') = -\mathrm{i}\theta(t-t')e^{-\omega_0|t-t'|} \qquad \rightarrow \quad G_0^R(\omega) = \frac{1}{(\omega+\mathrm{i}\varepsilon)^2-\omega_0^2}$$

Example: Harmonic Oscillator

- Keldysh-rotated propagators:

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- G_0^K component in continuous notation is regularization:

$$G_0^K(\omega) = \coth\frac{\beta\omega}{2}\left[G_0^R(\omega) - G_0^A(\omega)\right]$$

in equilibrium, with $\hat{\rho}(-\infty) \sim e^{-\beta\hat{H}} \rightarrow$ FDR

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Causality

Poles of retarded propagator are in the lower half-plane,
$(G^R)^\dagger = G^A$, and $G^{\tilde{K}} = 0$

Heat bath – Caldeira-Leggett

- Couple bosonic particle in potential to infinitely many harmonic oscillators:

$$S[\phi, \{\varphi_s\}] = S[\phi] + \frac{1}{2}\int \mathrm{d}t \sum_s \left((\vec{\varphi})^T \hat{G}_{0,s}^{-1} \vec{\varphi} + g_s (\vec{\phi})^T \hat{\sigma}_1 \vec{\varphi} \right)$$

where $\vec{\varphi} = (\varphi^{\mathsf{c}}, \varphi^{\mathsf{q}})^T$

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- Bath oscillators enter the action quadratically $\Rightarrow$ can be integrated out by completing the square

$$\Delta \hat{G}^{-1}(t - t') = -\hat{\sigma}_1 \left[\sum_s g_s^2 \hat{G}_{0,s}(t - t') \right] \hat{\sigma}_1$$

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- Introduce bath spectral density $J(\omega) = \pi \sum (g_s^2/\omega_s)\delta(\omega-\omega_s)$ to get for propagators

$$[\Delta G^{-1}(\omega)]^{R(A)} = \int_0^\infty \frac{\mathrm{d}\omega'}{2\pi} \frac{2\omega' J(\omega')}{(\omega')^2 - (\omega \pm \mathrm{i}\varepsilon)^2}$$

Heat bath – Caldeira-Leggett

- Assume ohmic bath $J_\Lambda(\omega) = 2\gamma\omega\Theta(\Lambda - \omega)$ to arrive at Caldeira-Leggett model:

$$[\Delta G^{-1}(\omega)]^{R(A)} = \text{const} \pm 2\mathrm{i}\gamma\omega \quad \to \quad \mp 2\gamma\delta(t-t')\partial_{t'}$$

$$[\Delta G^{-1}(\omega)]^{K} = 4\mathrm{i}\gamma\omega\coth\frac{\beta\omega}{2} \quad \to \quad 4\mathrm{i}\gamma\left[(2T+C)\delta(t-t') - \frac{\pi T^2}{\sinh^2\left(\pi T\left(t-t'\right)\right)}\right]$$

- Keldysh action:

$$S[\phi^{\mathsf{c}},\phi^{\mathsf{q}}] = \int_{-\infty}^{+\infty}\mathrm{d}t\Bigg[-2\phi^{\mathsf{q}}\left(\partial_t^2 + \gamma\partial_t\right)\phi^{\mathsf{c}} - V(\phi^{\mathsf{c}}+\phi^{\mathsf{q}}) + V(\phi^{\mathsf{c}}-\phi^{\mathsf{q}})$$
$$+ 2\mathrm{i}\gamma\left(2T(\phi^{\mathsf{q}})^2 + \frac{\pi T^2}{2}\int_{-\infty}^{+\infty}\mathrm{d}t'\frac{\left(\phi^{\mathsf{q}}(t)-\phi^{\mathsf{q}}(t')\right)^2}{\sinh^2\left(\pi T(t-t')\right)}\right)\Bigg]$$

Classical approximat2on

- Go to limit $\hbar \to 0$: Expand around $\phi^{\mathsf{q}} = 0$, keeping track of units

$$S[\phi^{\mathsf{c}}, \phi^{\mathsf{q}}] = \int_{-\infty}^{\infty} \mathrm{d}t \left[-2\phi^{\mathsf{q}} \left(\ddot{\phi}^{\mathsf{c}} + \gamma \dot{\phi}^{\mathsf{c}} + V'(\phi^{\mathsf{c}}) \right) + 4\mathrm{i}\gamma T (\phi^{\mathsf{q}})^2 \right] \tag{1}$$

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$$e^{-4\gamma T \int \mathrm{d}t (\phi^{\mathsf{q}})^2} = \int \mathcal{D}[\xi(t)] e^{-\int \mathrm{d}t \left[\frac{1}{4\gamma T} \xi^2 - 2\mathrm{i}\xi\phi^{\mathsf{q}} \right]} \tag{2}$$

Classical approximat2on

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- Resulting action is linear in ϕ^{q} $\Rightarrow$ Keldysh PI becomes delta functional

$$\int \mathcal{D}[\phi^{\mathsf{c}}, \phi^{\mathsf{q}}, \xi] e^{-\frac{1}{4\gamma T} \int \mathrm{d}t \xi^2} e^{-2\mathrm{i} \int \mathrm{d}t \phi^{\mathsf{q}} \left(\ddot{\phi}^{\mathsf{c}} + \gamma \dot{\phi}^{\mathsf{c}} + V'(\phi^{\mathsf{c}}) - \xi \right)} \tag{3}$$

Classical approximat2on

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$$\int \mathcal{D}[\phi^{\mathsf{c}}, \xi] e^{-\frac{1}{4\gamma T} \int \mathrm{d}t \xi^2} \delta \underbrace{\left(\ddot{\phi}^{\mathsf{c}} + \gamma \dot{\phi}^{\mathsf{c}} + V'(\phi^{\mathsf{c}}) - \xi \right)}_{\text{Langevin equation}} \tag{3}$$

A word on interactions

- Keldysh-rotating $V(\phi) \propto \phi^4$ potential term yields

$$V(\phi^{\mathsf{c}} + \phi^{\mathsf{q}}) - V(\phi^{\mathsf{c}} - \phi^{\mathsf{q}}) \propto \underbrace{\phi^{\mathsf{q}}(\phi^{\mathsf{c}})^3}_{\text{class. vertex}} + \underbrace{(\phi^{\mathsf{q}})^3\phi^{\mathsf{c}}}_{\text{quant. vertex}}$$

A word on interactions

- Keldysh-rotating $V(\phi) \propto \phi^4$ potential term yields

$$V(\phi^{\mathsf{c}} + \phi^{\mathsf{q}}) - V(\phi^{\mathsf{c}} - \phi^{\mathsf{q}}) \propto \underbrace{\phi^{\mathsf{q}}(\phi^{\mathsf{c}})^3}_{\text{class. vertex}} + \underbrace{(\phi^{\mathsf{q}})^3\phi^{\mathsf{c}}}_{\text{quant. vertex}}$$

- Quantum vertex dropped for classical approximation

Dynamical Models

Dynamical models

- Self-interacting scalar field
- Equation of motion

$$\ddot{\phi}(\vec{x},t) = -\frac{\delta \mathfrak{H}}{\delta \phi(\vec{x},t)}$$

  - Hamiltonian dynamics

Dynamical models

- Self-interacting scalar field
- Equation of motion A

$$\ddot{\phi}(\vec{x},t) = -\frac{\delta \mathfrak{H}}{\delta\phi(\vec{x},t)} - \gamma\dot{\phi} + \sqrt{2\gamma T}\eta(\vec{x},t)$$

$$\langle \eta(\vec{x},t)\rangle = 0, \quad \langle \eta(\vec{x}',t')\eta(\vec{x},t)\rangle = \delta^d(\vec{x}'-\vec{x})\delta(t'-t)$$

  - Langevin-dynamics
  - energy conservation in the limit $\gamma \to 0$ (Model C)

Dynamical models

- Self-interacting scalar field
- Equation of motion B

$$\ddot{\phi}(\vec{x},t) = \mu \vec{\nabla}^2 \frac{\delta \mathfrak{H}}{\delta \phi(\vec{x},t)} - \gamma \dot{\phi} + \sqrt{2\gamma\mu T}\eta(\vec{x},t)$$

$$\langle \eta(\vec{x},t)\rangle = 0, \quad \langle \eta(\vec{x}',t')\eta(\vec{x},t)\rangle = -\vec{\nabla}^2 \delta^d(\vec{x}'-\vec{x})\delta(t'-t)$$

  - diffusive dynamics
  - energy conservation in the limit $\gamma \to 0$ (Model D)
  - order parameter $M = \int \mathrm{d}^d x \phi(\vec{x},t)$ conserved
- Models A-D have same equilibrium state

Dynamical models

- Self-interacting scalar field
- Equation of motion B

$$\ddot{\phi}(\vec{x},t) = \mu \vec{\nabla}^2 \frac{\delta \mathfrak{H}}{\delta \phi(\vec{x},t)} - \gamma \dot{\phi} + \sqrt{2\gamma\mu T}\eta(\vec{x},t)$$

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  - diffusive dynamics
  - energy conservation in the limit $\gamma \to 0$ (Model D)
  - order parameter $M = \int \mathrm{d}^d x \phi(\vec{x},t)$ conserved
- Models A-D have same equilibrium state

Discretization on spatial lattice

Integrate ordinary differential equations via leapfrog method

Results

WARNING

The following footage may potentially trigger seizures for people with photosensitive epilepsy.

Spectral function

- Decomposition of the two-point correlation function

$$G^{\mathbb{T}}(t,t') = F(t,t') - \frac{i}{2}\rho(t,t')\,\mathrm{sgn}(t-t') \tag{4}$$

- Spectral function ρ, statistical two-point function F
- Thermal equilibrium: $F(\omega) = (n_B(\omega) + 1/2)\rho(\omega)$ (FDR)
- Classical approximation: $n_B(\omega) + 1/2 \to T/\omega$

$\Rightarrow$ Obtain spectral function from derivative of statististical function

$$\rho(t-t') = -\frac{1}{T}\partial_t F(t-t')$$

- Non-critical spectral functions dominated by quasi-particle structure
- Breit-Wigner with dispersion

$$\omega_c^2 = \begin{cases} m^2(T) + p^2, & \text{A/C} \\ \mu p^2(m^2(T) + p^2), & \text{B/D} \end{cases}$$

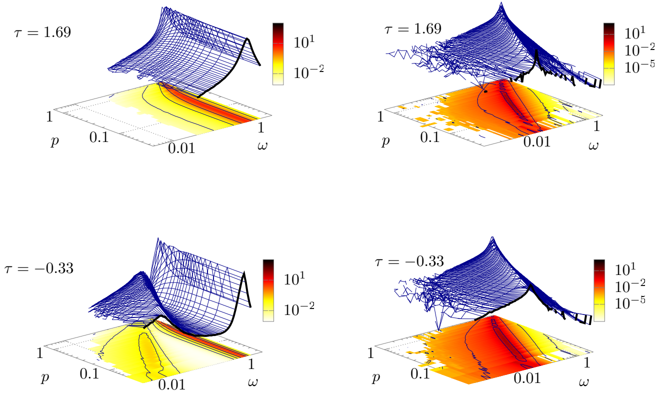


Figure: Spectral functions of Model C (left) and Model D (right)

Spectral function – results

- Non-critical spectral functions dominated by quasi-particle structure
- Breit-Wigner with dispersion

$$\omega_c^2 = \begin{cases} m^2(T) + p^2, & \text{A/C} \\ \mu p^2(m^2(T) + p^2), & \text{B/D} \end{cases}$$

- Langevin coupling γ increases decay width

$$\Gamma_p(\gamma) = \Gamma_p(0) + \gamma$$

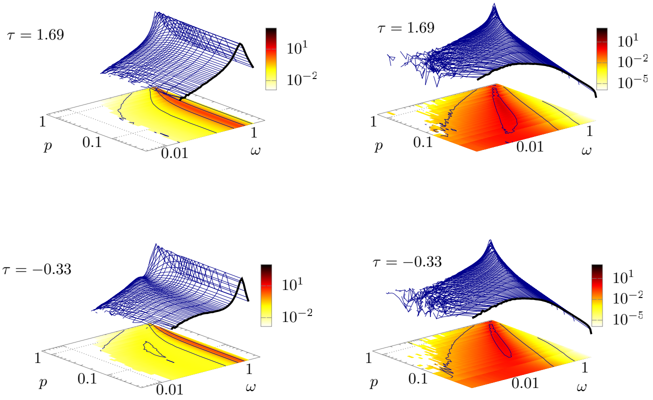


Figure: Spectral functions of Model A (left) and Model B (right)

Critical dynamics

WARNING

The following footage may potentially trigger seizures for people with photosensitive epilepsy.

Critical point – divergent correlation time

- Large clusters show less movement ("critical slowing-down")
- Quantified via multi-time correlation function

$$\langle \phi(t)\phi(t')\rangle \sim e^{-|t'-t|/\xi_t}$$

- Correlation time $\equiv$ critical time scale ξ_t
  - Correlation time diverges at critical point $\Rightarrow$ scale invariance extends to time direction
  - New critical exponent z

$$\xi_t \sim \xi^z$$

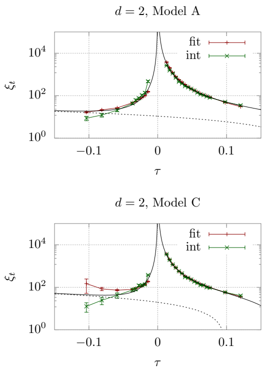


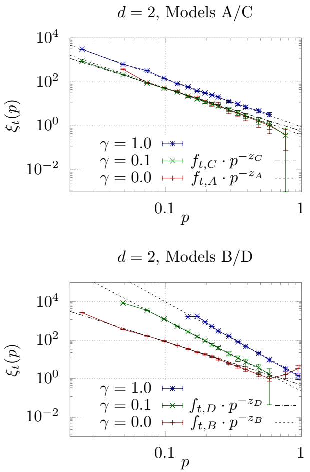
Figure: Divergent correlation time at the critical point

Critical dynamics – universality classes

- New universality classes depending on properties of dynamics
  - + conserved charges
  - + dynamic mode couplings
- LGW with equation of motion A/B: Model A-D
- QCD: Model H, more complex

Model	A	B	C	D
H	-	-	✓	✓
$\langle\phi\rangle$	-	✓	-	✓
z	$2 - c\eta$	$4 - \eta$	$2 + \alpha/\nu$	$(4 - \eta)$

Table: Dynamic universality classes ("Models") after Hohenberg and Halperin



Critical dynamics – spectral function

- Slow dynamics $\rightarrow$ large IR contribution
- Correlation time diverges $\Rightarrow$ scale-invariance

$$\rho(\omega, p, \tau) = s^{2-\eta}\rho(s^z\omega, sp, s^{1/\nu}\tau)$$

$$\Rightarrow \quad \rho(\omega, p, 0) = p^{-(2-\eta)}\rho(\omega/p^z, 1, 0)$$

- IR part described by *universal* function $\rho(x, 1, 0)$

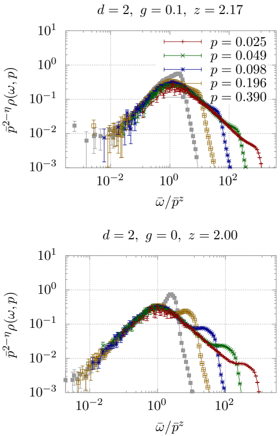


Figure: Spectral functions of Model A (upper) and Model C (lower)

- Model B: critical spectral function still has Breit-Wigner shape, overlap for $\omega < \Gamma$
- Modified dispersion relations

$$\omega_p^2 \sim p^{4-\eta}, \quad \Gamma_p(\gamma) \sim p^{z_\Gamma} + \gamma$$

$\Rightarrow$ Extract scaling function *analytically*

$$\frac{p^2 \omega \Gamma}{(\omega^2 - \omega_p^2)^2 + \Gamma_p^2 \omega^2} \overset{\omega \ll \Gamma_p}{\rightarrow} p^{-(2-\eta)} \rho_0 P(\omega/p^z, 1, 0)$$

$$P(x,1,0) = \frac{1}{x + \frac{1}{x}}$$

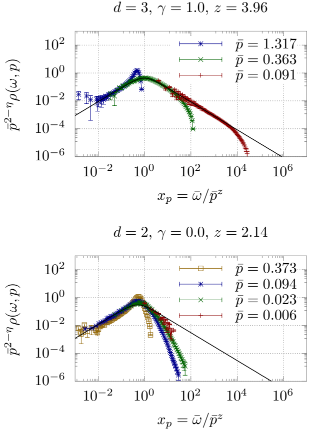


Figure: Spectral functions of Model B (upper) and Model D (lower)

Non-equilibrium scaling

WARNING

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Non-equilibrium dynamics

- ▶ Probing the QCD phase diagram with HIC

→ System evolves in real time

1. maximum compression
2. expansive cooling
3. phase transition
4. freeze-out

⇒ "Trajectories" in phase diagram

- ξ_t diverges at critical point
- system cannot equilibrate

- ▶ Emergence of non-equilibrium effects

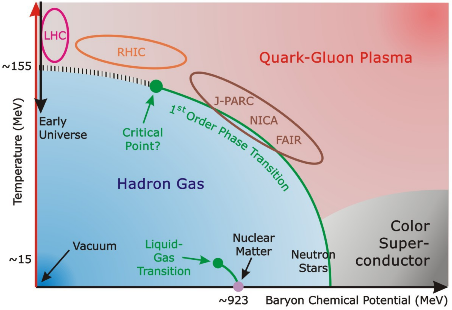

Figure: Semi-quantitative phase diagram of QCD

Non-equilibrium dynamics

- Probing the QCD phase diagram with HIC

→ System evolves in real time

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⇒ "Trajectories" in phase diagram

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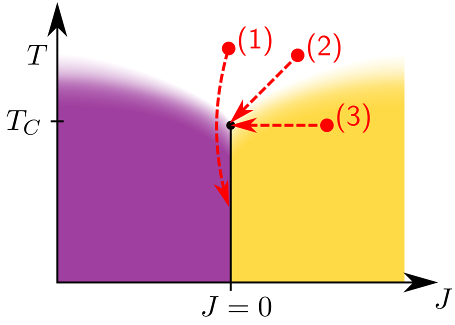


Figure: Qualitative LGW phase diagram

Non-equilibrium dynamics – critical quench

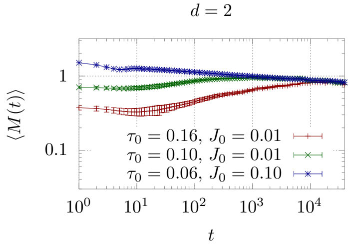
- Bachelor thesis by C. Kummer
- System thermalizes at T_0, J_0 for times $t < 0$
- Instant quench to $T = T_c$, $J = 0$ at $t = 0$

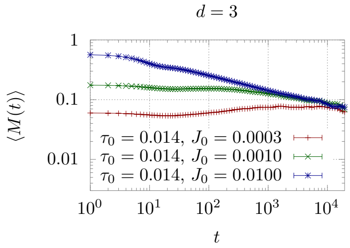
$\Rightarrow$ Scale-invariance, new exponent re-scales $m_0 \equiv M(t = 0)$

$$M(t, \tau, m_0) = s^{-\beta/\nu} M(s^{-z}t, s^{1/\nu}\tau, s^{x_0}m_0)$$

$$\Rightarrow \quad M(t, \tau = 0, m_0) = t^{-\beta/\nu z} M(1, 0, t^{x_0/z}m_0)$$

- Small $t^{x_0/z}m_0 \Rightarrow$ initial slip $M(t) \sim t^{\theta'}$
- Large $t^{x_0/z}m_0 \Rightarrow$ aging $M(t) \sim t^{-\beta/\nu z}$



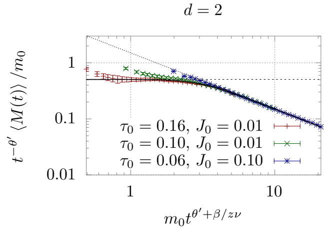


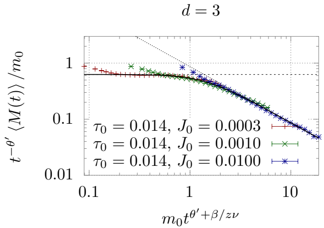
Non-equilibrium dynamics - universal functions

- Thermalization after critical quench described by universal function
$$t^{\beta/\nu z} M(t, \tau = 0, m_0) = M(1, 0, t^{x_0/z} m_0)$$

Universal functions

Exact description of $M(t)$



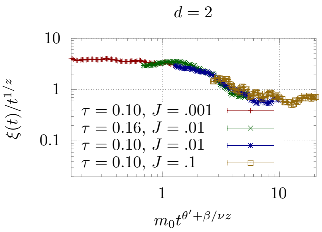


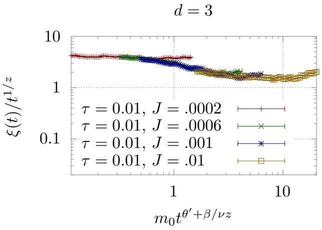
- Thermalization after critical quench described by universal function

$$t^{1/z}\xi(t,\tau=0,m_0)=\xi(1,0,t^{x_0/z}m_0)$$

Universal functions

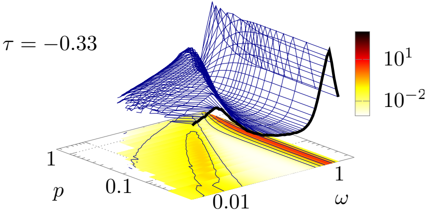
Exact description of $\xi_t(t)$

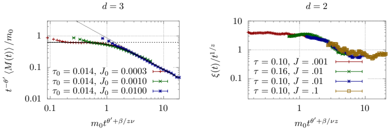




Conclusion & Outlook

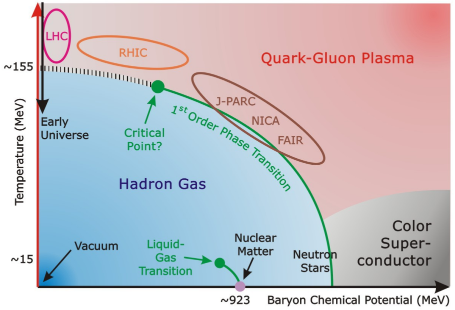
- Investigate non-equilibrium behaviour of QFTs with Schwinger-Keldysh formalism
- Classical-statistical simulations powerful tool for critical dynamics
    - Spectral functions
    - Dynamic critical exponent
    - Non-equilibrium scaling
    - Energy-momentum tensor





Outlook

- More classical-statistical simulations
  - Different dynamical models, leading to Model H
  - More non-equilibrium effects
    - realistic trajectories
    - dynamic phase ordering
    - Kibble-Zurek mechanism
- Alternatives to classical-statistical simulations:
  - Corrections to classical limit, e.g. Gaussian states (see Talk by Leon)
  - FRG on the Keldysh contour (see Talk by Johannes)



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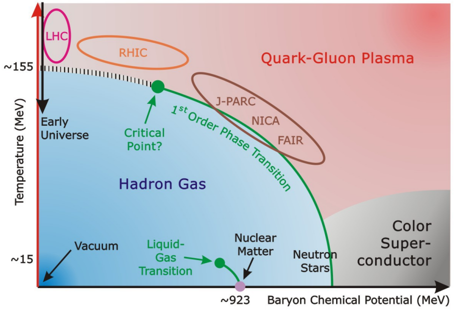

Thank you for your attention!

Appendix

Critical point – scale invariance

- Scale-invariant free energy: invariant under rescalings $x \to x/s$

$$F_{\text{sing}}(\tau, J, \ldots) = s^{d} F_{\text{sing}}\left(s^{y_1}\tau, s^{y_2}J, \ldots\right)$$

with reduced temperature $\tau \equiv \frac{T-T_c}{T_c}$, external field J

Critical point – scale invariance

- Scale-invariant free energy: invariant under rescalings $x \to x/s$

$$F_{\text{sing}}(\tau, J, \ldots) = s^{d} F_{\text{sing}}\left(s^{y_1}\tau, s^{y_2}J, \ldots\right)$$

with reduced temperature $\tau \equiv \frac{T-T_c}{T_c}$, external field J

- Derivatives of scale-invariant free energy are scale invariant, e.g. magnetization M with external field J

$$M(\tau, J) = \frac{\partial}{\partial J} F(\tau, J) = s^{y_2-d} M(s^{y_1}\tau, s^{y_2}J)$$

$$\Rightarrow M(\tau, J=0) = |\tau|^{\frac{d-y_2}{y_1}} M(\text{sgn}(\tau), 0)$$

Covariant representation

- Model A Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\nu\phi\partial^\nu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

- Equation of motion

$$0 = \partial_\nu\partial^\nu\phi + m^2\phi - \frac{\lambda}{6}\phi$$

Covariant representation

- Model A Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\nu\phi\partial^\nu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

- Dissipative equation of motion

$$0 = \partial_\nu\partial^\nu\phi + m^2\phi - \frac{\lambda}{6}\phi + \gamma u_\nu\partial^\nu\phi - \sqrt{2\gamma T}\eta$$

with heat-bath velocity u^ν

Covariant representation

- Model B Lagrangian

$$\mathcal{L} = \frac{\mu}{2}\nabla_\nu K \nabla^\nu K + K D_\tau \phi + \frac{1}{2}\nabla_\nu \phi \nabla^\nu \phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

- Equations of motion

$$0 = D_\tau K + \partial_\nu \nabla^\nu \phi + V'(\phi)$$
$$0 = D_\tau \phi - \mu \partial_\nu \nabla^\nu K$$

with heat-bath velocity u^ν,
longitudinal derivative $D_\tau = u_\nu \partial^\nu \to \partial_t$,
transversal projector $\Delta^{\nu\lambda} = g^{\nu\lambda} - u^\nu u^\lambda$,
transversal derivative $\nabla^\nu = \Delta^{\nu\lambda}\partial_\lambda \to -\vec{\nabla}$

Covariant representation

- Model B Lagrangian

$$\mathcal{L} = \frac{\mu}{2}\nabla_\nu K \nabla^\nu K + K D_\tau \phi + \frac{1}{2}\nabla_\nu \phi \nabla^\nu \phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

- Dissipative equations of motion

$$0 = D_\tau K + \partial_\nu \nabla^\nu \phi + V'(\phi)$$
$$0 = D_\tau \phi - \mu \partial_\nu \nabla^\nu K + \gamma D_\tau \phi - \sqrt{2\gamma T}\eta$$

with heat-bath velocity u^ν,
longitudinal derivative $D_\tau = u_\nu \partial^\nu \to \partial_t$,
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