Real-time methods for spectral functions I

Real-Time Functional Renormalization Group

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Based on
JR, D. Schweitzer, L. J. Sieke, L. von Smekal, *Real-time methods for spectral functions*, TBP,
JR, L. von Smekal, (In preparation)

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Why *real*-time?

Performing calculations directly in real-time (Minkowski space-time)

- avoids the need of an analytic continuation in comparison with the Matsubara formalism, and
- allows for treating phenomena arbitrarily far from equilibrium, e.g. many aspects of heavy ion collisions, which are very dynamic in nature.

Idea of the (Functional) Renormalization Group

- Suppose the effective action Γ of the theory is known at some momentum/energy scale k, which we denote by Γ_k, i.e. all fluctuations from modes $|\mathbf{p}| \gtrsim k$ have been taken into account.
- Realized by modifying the action with an *infrared cutoff* $\Delta S_k[\phi^c, \phi^q]$,

$$S \to S + \Delta S_k$$

  for which the term ΔS_k suppresses all modes with $|\mathbf{p}| < k$.
- Has the structure ($D = d + 1$ number of spacetime dimensions)

$$\Delta S_k[\phi] = \frac{1}{2} \int d^D x \int d^D x' \, \phi^T(x) R_k(x, x') \phi(x'), \qquad \phi^T = (\phi^c, \phi^q),$$

  with the 2×2-'regulator' matrix

$$R_k(p) = \begin{pmatrix} 0 & R_k^A(p) \\ R_k^R(p) & R_k^K(p) \end{pmatrix}.$$

  in momentum space.

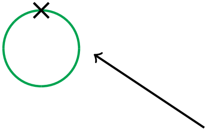
Idea of the (Functional) Renormalization Group

- Change the scale a bit $k \to k + dk$, arrive at 'flow' equation (Wetterich '93, Berges, Mesterházy '12)

$$\partial_k \Gamma_k[\phi^c, \phi^q] = -\frac{i}{2} \mathrm{tr}\left(\partial_k R_k \circ G_k\right), \quad G_k = -\left(\Gamma_k^{(2)} + R_k\right)^{-1}$$

- Has the form of a 1-loop integral,

$$\partial_k \Gamma_k = -\frac{i}{2}$$



Full propagator

but is exact.

- Have $\Gamma_k \xrightarrow{k \to \Lambda} S$, classical action.
  (Demonstrated via saddle-point approximation.)
- *Spectral function* given by $\rho(\omega) = 4i\, \mathrm{Im}\, G^R(\omega)$.

Causal Regulators

- Regulator changes analytic structure of the propagators,

$$G_k^R(\omega, \mathbf{p}) = -\frac{1}{\Gamma_k^{qc}(\omega, \mathbf{p}) + R_k^R(\omega, \mathbf{p})} \qquad \text{(retarded)}$$

$$G_k^A(\omega, \mathbf{p}) = -\frac{1}{\Gamma_k^{cq}(\omega, \mathbf{p}) + R_k^A(\omega, \mathbf{p})} \qquad \text{(advanced)}$$

- What are the consequences?
- Maybe everything fine for $k = 0$?

Causal Regulators?

Test:

- Observe property of Keldysh action:

$$S = \frac{1}{2} \int_p (\phi^c(-p), \phi^q(-p)) \begin{pmatrix} \mathbf{0} & \dots \\ \dots & \dots \end{pmatrix} \begin{pmatrix} \phi^c(p) \\ \phi^q(p) \end{pmatrix}$$

  follows from that if $\phi^+ = \phi^-$ the partition function is $Z = 1$, i.e. the action vanishes.
- Necessary condition for the correctness of the flow.

Find:

- Popular regulators like a sharp/exponential/algebraic/... cutoff produce such an unphysical component during the flow.
- Problem of causality is not trivial. (Duclut, Delamotte '18)
- An insufficient regulator indeed leads to an incorrect Keldysh action.

Causal Regulators

What can we do? (Start with the 0+1 dimensional case, i.e. quantum mechanics.)

The most simple regulator that we could write down has the form of a purely masslike shift, (Callan-Symanzik regulator)

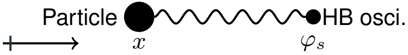
$$R_k^{R/A}(\omega) = -2k^2$$

- Trivially causal, since it induces only a mass-shift $m^2 \to m^2 + k^2$ in the propagators.
- Too simple?
- Flow no longer conformal with K. G. Wilson's idea of integrating out momentum/energy shells?

Heat Bath Regulators

Regulator motivated by physics: (Causality guaranteed!)

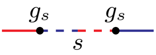
- Imagine ΔS_k is the result from integrating out an external 'heat bath'.
- The heat bath is modeled as an ensemble of independent harmonic oscillators, attached to the particle. (cf. talk by Dominik, Caldeira-Leggett model)



$$H' = \sum_s \left(\frac{\pi_s^2}{2} + \frac{\omega_s^2}{2} \left(\varphi_s - \frac{g_s}{\omega_s^2} x \right)^2 \right)$$

- Integrate out heat bath $\hat{=}$ Particle acquires self-energy $\Sigma^{R/A}$

$$\Sigma^R(\omega) = \sum_s$$

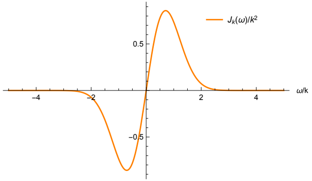


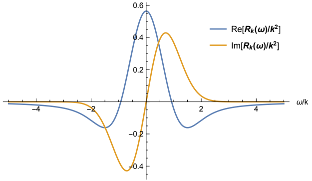
$$= -\int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega' J(\omega')}{(\omega + i\varepsilon)^2 - \omega'^2}$$

- Fully controlled by a spectral density $J(\omega) = \pi \sum_s \frac{g_s^2}{\omega_s} \delta(\omega - \omega_s)$
- Invert: $2\mathrm{Im}\,\Sigma^R(\omega) = J(\omega)$, but the self-energy Σ^R also has a non-vanishing real part.

Heat Bath Regulators

- Now make the spectral density k-dependent, $J(\omega) \to J_k(\omega)$ and choose it to *damp* infrared modes.
- The resulting self-energy is the regulator, $\Sigma^{R/A} \to R_k^{R/A}$.





$$m^2 \to m^2 - \Delta m_b^2(k)$$

Example:

$$J_k(\omega) = 2k\omega \exp\left\{-\omega^2/k^2\right\}$$

$\implies \phi(t) \sim e^{-kt}$ for $\omega \ll k$, damped

But: Heat bath induces *negative* (!) shift in the squared mass

Can be quantified by

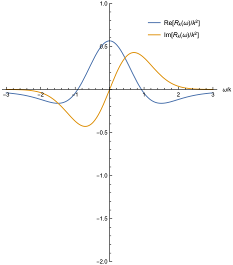
$$\Delta m_b^2(k) = \int_0^\infty \frac{d\omega}{2\pi} \frac{J_k(\omega)}{\omega} = \frac{k^2}{\sqrt{\pi}}$$

This makes the theory *unstable* and *acausal* for large enough values of k.

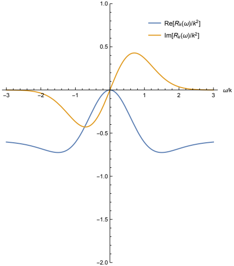
- Way out: We learned that a masslike shift is definitely causal.
- So: Just add a masslike 'counter-term' to compensate the shift in the squared mass!

Heat Bath Regulator

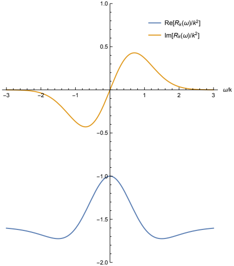
$$R_k^{R/A}(\omega) = -\int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega' J_k(\omega')}{(\omega \pm i\varepsilon)^2 - \omega'^2} - 2\alpha k^2$$



$\alpha = 0$

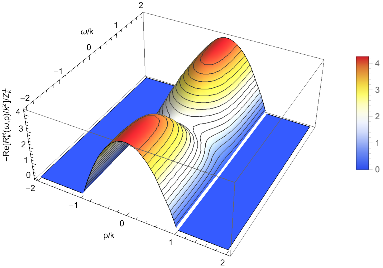


$\alpha = 1/\sqrt{4\pi}$,
(Balanced)

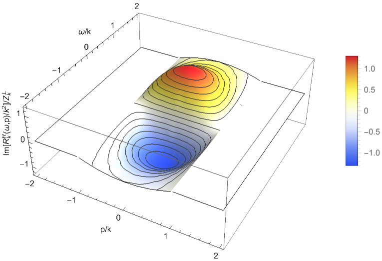


$\alpha = 1/\sqrt{4\pi} + 1/2$
(Balanced + regulated)

- What about a *field theory*?
- Arguably simplest ansatz: Imagine an independent bath of harmonic oscillators for every spatial momentum mode $\mathbf{p}$.
- New degree of freedom: Multiply the ω-dependent regulator with some function $r_k(\mathbf{p})$ that acts as a cutoff for $|\mathbf{p}| \gg k$, e.g. $r_k(\mathbf{p}) = (1 - \mathbf{p}^2/k^2)\Theta(k^2 - \mathbf{p}^2)$, ('Sharp' cutoff)



Real part (Mass shift)

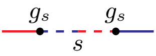


Imaginary part (Damping)

Heat Bath Regulators for a Field Theory

- And when we have no preferred frame of reference, e.g. no external medium? What about *Lorentz-invariance*?
- A regulator like above would break Lorentz-symmetry.
- Imagine the heat bath to be an ensemble of *Klein-Gordon fields* with a relativistic dispersion relation $\omega^2 = \mathbf{p}^2 + m_s^2$,
  $\rightsquigarrow$ Our field gains a self-energy

$$\Sigma_k^R = \sum_s$$



$$= -\int_0^\infty \frac{d\mu^2}{2\pi} \frac{J_k(\mu^2)}{(p^0 + i\varepsilon)^2 - \mathbf{p}^2 - \mu^2}$$

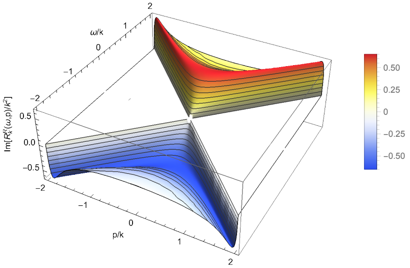
  with invariant spectral density $J(\mu^2) = 2\pi \sum_s g_s^2 \delta(\mu^2 - m_s^2)$.
- Reintroduce masslike counter-term $-2\alpha k^2$, and then:

Heat Bath Regulators for a Field Theory

General form of a Lorentz-invariant regulator:

$$R_k^{R/A}(p^2, \operatorname{sgn} p^0) = -\int_0^\infty \frac{d\mu^2}{2\pi} \frac{J_k(\mu^2)}{(p^0 + i\varepsilon)^2 - \mathbf{p}^2 - \mu^2} - 2\alpha k^2$$

Example:



$$J_k(\mu^2) = \frac{2k\mu}{(1 + \mu^2/k^2)^2}$$

- p^2 is a Lorentz-scalar.
- $\operatorname{sgn} p^0$ is also a Lorentz-scalar, but only if p is timelike and if we restrict the allowed Lorentz-transformations to the orthochronous subgroup $O^+(1, d)$.
- RG interpretation: Integrate out shells of constant invariant mass.

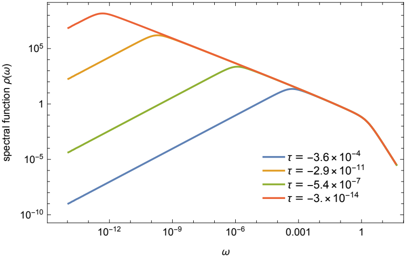
A Glance at Critical Dynamics - Model A

Spectral function (cf. talk by Dominik)

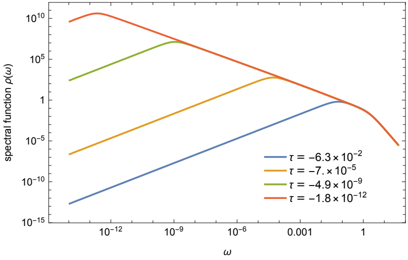
$$\rho(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \int d^d x\, i\langle[\phi(x), \phi(0)]\rangle \sim \omega^{-\sigma} \text{ at critical point}$$

- Scaling exponent $\sigma = (2 - \eta^{\perp})/z$
- Related to dynamical critical exponent z, defined by $\xi_t \sim \xi^z$

Results: (ϕ^4-theory with dissipative dynamics, 1-loop self-consistent truncation scheme)



$d = 2, z \approx 2.11$



$d = 3, z \approx 2.04$

Summary & Outlook

We have

- analyzed the influence of a non-causal regulator on the Keldysh action,
- constructed regulators in the real-time FRG that automatically take care of causality and Lorentz-invariance,
- calculated critical spectral functions using a 1-loop self-consistent truncation scheme in Model A.

For the future, we plan to

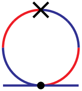
- include fermions ($\rightarrow$ Low-energy effective models of QCD in real-time),
- inspect the real-time dynamics of models B,C,D,...,H,...
- analyze non-equilibrium phenomena.

BACK UP

Causal Regulators?

Diagram(s) that correspond to the unphysical upper left (cc) component of the Keldysh action,

$$\partial_k \Gamma_k^{cc} = \frac{-i}{2} \Bigg[$$



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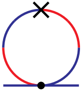


$$\Bigg]$$

$$= i\lambda_k \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \Big(G_k^R(\omega) \partial_k R_k^R(\omega) G_k^R(\omega) + G_k^A(\omega) \partial_k R_k^A(\omega) G_k^A(\omega) \Big)$$

$\overset{!}{=} 0$ for a flow that respects the causal structure of the action.

Propagators:

$$G_k^{R(A)}(\omega) = -\frac{1}{2} \frac{1}{\omega^2 \pm i\gamma\omega - m^2 + \frac{1}{2} R_k^{R(A)}(\omega)}$$

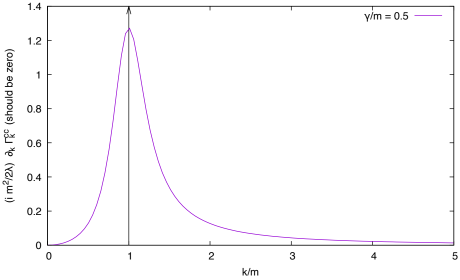
Causal Regulators?

- Well-known regulator from the Euclidean FRG (Litim '01)
- Regulator has the form

$$R_k^{R/A}(\omega) = 2(k^2 - \omega^2)\Theta(k^2 - \omega^2),$$

with a sharp cutoff at $\omega = k$.

- Result:



- Flow indeed generates an unphysical cc component in the action.
- Pole at $k = m$!

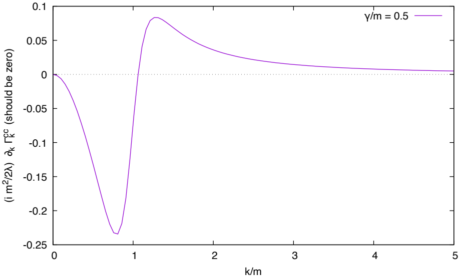
Causal Regulators?

- Is it the sign?
- Regulator now has the form

$$R_k^{R/A}(\omega) = -2(k^2 - \omega^2)\Theta(k^2 - \omega^2),$$

still with a sharp cutoff at $\omega = k$.

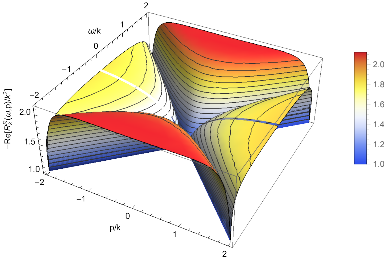
- Result:



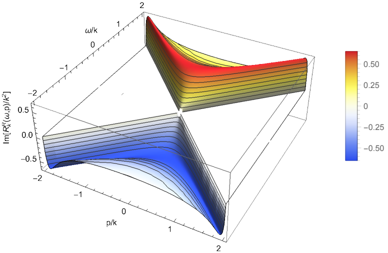
$+$ No more singularities in the flow.

$-$ Flow still generates an unphysical cc component in the action.

Lorentz-invariant causal regulator plots



Real part (Mass shift)



Imaginary part (Damping)

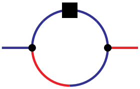
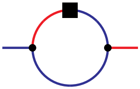
Critical Dynamics

Truncation: (1-loop self-consistent)

$$\Gamma_k = \int_p \phi^T(-p) \begin{pmatrix} 0 & Z_k^{\parallel}(\omega)\,\omega^2 - Z_k^{\perp} - m_k^2\,\mathbf{p}^2 - i\gamma_k(\omega)\omega \\ \text{c.c. of adv.} & 4i\gamma_k(\omega)T \end{pmatrix} \phi(p)$$
$$- \frac{2}{4!}\int_x \lambda_k \phi^c(x)\phi^c(x)\phi^c(x)\phi^q(x)$$

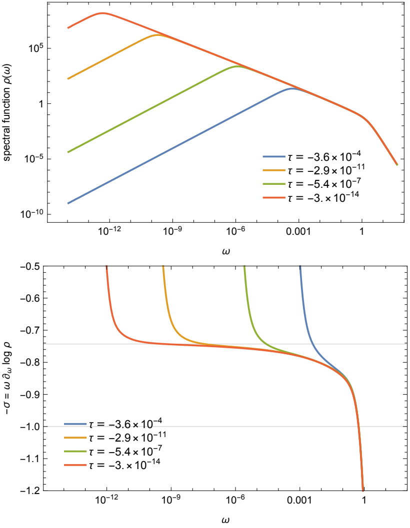
Flowing quantities: $Z_k^{\parallel}(\omega), \gamma_k(\omega)$ on grid, and $Z_k^{\perp}, m_k^2, \lambda_k$

1-Loop Flow Equation:

$$\partial_k \Delta\Gamma_k^{cq}(\omega) = -\frac{i}{2}\Big\{ 2\times$$

$$+\,2\times$$

$$\Big\}_{\mathbf{p}=0}$$

with $\frac{1}{2}\Delta\Gamma_k^{cq}(\omega) = Z_k^{\parallel}(\omega)\omega^2 - i\gamma_k(\omega)\omega$.

Critical Dynamics - Spectral functions of Model A



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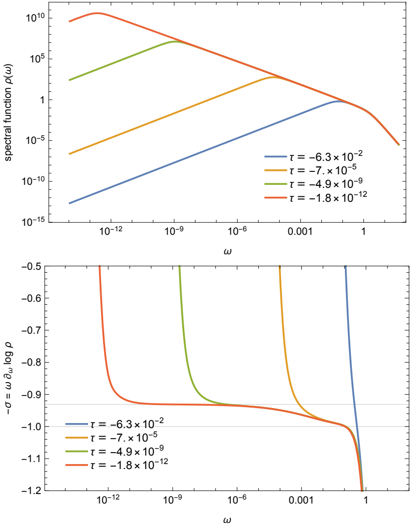

$d = 3, z \approx 2.04$