GOETHE UNIVERSITÄT FRANKFURT AM MAIN

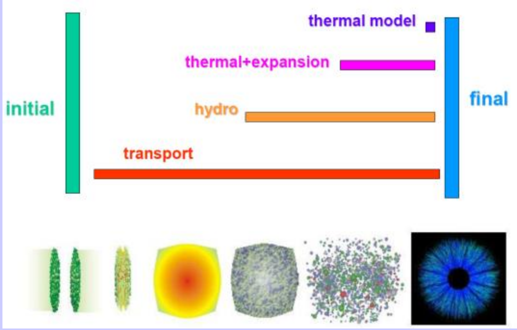
# Lecture

# Models for heavy-ion collisions: (Part I): transport models - Vlasov EoM

**SS2024: ‚Dynamical models for relativistic heavy-ion collisions‘**

# Basic models for heavy-ion collisions

Reminder: Lecture 1




- **Statistical models:**

**basic assumption:** system is described by a (grand) canonical ensemble of non-interacting fermions and bosons in thermal and chemical equilibrium = thermal hadron gas at freeze-out with common T and $\mu_B$

[ - : no dynamical information]

- **Hydrodynamical models:**

**basic assumption**: conservation laws + equation of state (EoS); assumption of local thermal and chemical equilibrium
- Interactions are ‚hidden' in properties of the fluid described by transport coefficients (shear and bulk viscosity $\eta$, $\zeta$, ..), which is 'input' for the hydro models

[ - : simplified dynamics]

- **Microscopic transport models:**

based on transport theory of relativistic quantum many-body systems
- Explicitly account for the interactions of all degrees of freedom (hadrons and partons) in terms of cross sections and potentials
- Provide a unique dynamical description of strongly interaction matter in- and out-off equilibrium:
- In-equilibrium: transport coefficients are calculated in a box – controled by lQCD
- Nonequilibrium dynamics – controled by HIC

Actual solutions: Monte Carlo simulations

[+ : full dynamics | - : very complicated]

# Dynamical description of heavy-ion collisions

**The goal:** to study the properties of strongly interacting matter under extreme conditions from a microscopic point of view

**Realization:** dynamical many-body transport approaches

**Plan:**

1) Dynamical transport models (nonrelativistic formulation): from the Schrödinger equation to Vlasov equation of motion ➔ BUU EoM

2) Density-matrix formalism: Correlation dynamics

3) Quantum field theory ➔ Kadanoff-Baym dynamics
➔ generalized off-shell transport equations

4) Transport models for HIC

# 1. From the Schrödinger equation to the Vlasov equation of motion

# Quantum mechanical description of the many-body system

Dynamics of heavy-ion collisions is a many-body problem!

**Schrödinger equation** for the system of **N particles** in three dimensions:

$$i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t) = H(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t)\, \Psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t)$$

*nonrelativistic formulation*

**Hartree-Fock approximation:**

- many-body wave function → $\Psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t) = A \prod_{i=1}^{N} \psi_i(\vec{r}_i, t)$

  antisym. product of single-particle wave functions

- many-body Hamiltonian → single-particle Hartree-Fock Hamiltonian

$$H(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t) = \sum_{i=1}^{N} T(\vec{r}_i) + V(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t)$$

kinetic term N-body potential

$$T(\vec{r}) = -\frac{\hbar}{2m} \vec{\nabla}_r^2$$

*(approximation)* $\approx \sum_{i=1}^{N} T(\vec{r}_i) + \sum_{i<j}^{N} V_{ij}(|\vec{r}_i - \vec{r}_j|, t) \approx \sum_{i=1}^{N} h_i(\vec{r}_i, t)$

kinetic term 2-body potential

# Hartree-Fock equation

**Time-dependent Hartree-Fock equation for a single particle *i*:**

$$i\hbar \frac{\partial}{\partial t}\psi_i(\vec{r},t) = \hat{h}\psi_i(\vec{r},t)$$

**Single-particle Hartree-Fock Hamiltonian operator:** $\hat{h} = \hat{T} + \hat{U}_H - \hat{U}_F$

- **Hartree term:** $\hat{U}_H = \sum_{i(occ)} \int d^3r' \psi_i^*(\vec{r}',t) V(\vec{r}-\vec{r}')\psi_i(\vec{r}',t)$ $\quad \hat{T} = -\frac{\hbar}{2m}\vec{\nabla}_r^2$

  **self-generated local mean-field potential**

- **Fock term:** $\hat{U}_F = \sum_{i<N} \psi_i^*(\vec{r}',t) V(\vec{r},\vec{r}',t)\psi_i(\vec{r},t)$

  **non-local mean-field exchange potential (quantum statistics)**

➔ **Equation-of-motion (EoM): propagation of particles in the self-generated mean-field:**

$$i\hbar \frac{\partial}{\partial t}\psi_i(\vec{r},t) = \left(T(\vec{r},t) + U_H(\vec{r},t)\right)\psi_i(\vec{r},t) - \int d^3r' \; U_F(\vec{r},\vec{r}',t)\psi_i(\vec{r}',t)$$

*local potential*

**We'll neglected the exchange (Fock) term**

**Note: TDHF approximation describes only the interactions of particles with the time-dependent mean-field !**

**In order to describe the collisions between the individual(!) particles, one has to go beyond the mean-field level ! (see Part 2: Correlation dynamics)**

# Single particle density matrix

- Introduce the **single particle density matrix:**

$$\rho(\vec{r},\vec{r}',t) \equiv \sum_{\beta_{occ}} \psi^*_\beta(\vec{r}',t)\psi_\beta(\vec{r},t)$$

Thus, the single-particle Hartree-Fock Hamiltonian operator can be written as

$$h(\vec{r},t) = T(\vec{r}) + \sum_{\beta_{occ}} \int d^3r' \, V(\vec{r}-\vec{r}',t)\rho(\vec{r}',\vec{r}',t) = T(\vec{r}) + U(\vec{r},t)$$

local potential

- Consider equation:

$$i\hbar \frac{\partial}{\partial t}\psi_\alpha(\vec{r},t) = h(\vec{r},t)\psi_\alpha(\vec{r},t) \quad (1)$$

$\psi^*_\alpha(\vec{r}',t) * (1)$:

$$\psi^*_\alpha(\vec{r}',t) i\hbar \frac{\partial}{\partial t}\psi_\alpha(\vec{r},t) = \psi^*_\alpha(\vec{r}',t)h(\vec{r},t)\psi_\alpha(\vec{r},t) \quad (2)$$

$(1)^+|_{for\ \vec{r}'} * \psi_\alpha(\vec{r},t)$:

$$-i\hbar\left[\frac{\partial}{\partial t}\psi^*_\alpha(\vec{r}',t)\right]\psi_\alpha(\vec{r},t) = h(\vec{r}',t)\psi^*_\alpha(\vec{r}',t)\psi_\alpha(\vec{r},t) \quad (3)$$

→ $\sum_\alpha ((2)-(3))$:

# Wigner transform of the density matrix

$$i\hbar \frac{\partial}{\partial t}\rho(\vec{r},\vec{r}',t) = [h(\vec{r},t) - h(\vec{r}',t)]\rho(\vec{r},\vec{r}',t) \quad (4)$$

$$\rho(\vec{r},\vec{r}',t) \equiv \sum_{\beta_{occ}} \psi^*_\beta(\vec{r}',t)\psi_\beta(\vec{r},t)$$

**The single-particle Hartree-Fock Hamiltonian:**

$$h(\vec{r},t) = T(\vec{r}) + \int d^3r'\, V(\vec{r}-\vec{r}',t)\rho(\vec{r}',\vec{r}',t)$$

$$= T(\vec{r}) + U(\vec{r},t)$$

**kinetic term + potential (local) term**

**➔EoM:**

$$\frac{\partial}{\partial t}\rho(\vec{r},\vec{r}',t) + \frac{i}{\hbar}\left[\frac{\hbar^2}{2m}\vec{\nabla}^2_r + U(\vec{r},t) - \frac{\hbar^2}{2m}\vec{\nabla}^2_{r'} - U(\vec{r}',t)\right]\rho(\vec{r},\vec{r}',t) = 0 \quad (5)$$

**Rewrite (5) using *x* instead of *r***

$$\frac{\partial}{\partial t}\rho(\vec{x},\vec{x}',t) + \frac{i}{\hbar}\left[\frac{\hbar^2}{2m}\vec{\nabla}^2_x + U(\vec{x},t) - \frac{\hbar^2}{2m}\vec{\nabla}^2_{x'} - U(\vec{x}',t)\right]\rho(\vec{x},\vec{x}',t) = 0$$

# Wigner transform of the density matrix

➔ **EoM:**

$$\frac{\partial}{\partial t}\rho(\vec{x},\vec{x}',t)+\frac{i}{\hbar}\left[\frac{\hbar^2}{2m}\vec{\nabla}_x^2+U(\vec{x},t)-\frac{\hbar^2}{2m}\vec{\nabla}_{x'}^2-U(\vec{x}',t)\right]\rho(\vec{x},\vec{x}',t)=0$$

- **Instead of considering the density matrix $\rho$, let's find the equation of motion for its Fourier transform, i.e. the Wigner transform of the density matrix:**

$$f(\vec{r},\vec{p},t)=\int d^3s\ \exp\left(-\frac{i}{\hbar}\vec{p}\vec{s}\right)\ \rho\left(\vec{r}+\frac{\vec{s}}{2},\vec{r}-\frac{\vec{s}}{2},t\right)$$

$$old:\ \vec{x}\qquad \vec{x}'$$

**New variables:**

$$\vec{r}=\frac{\vec{x}+\vec{x}'}{2},\qquad \vec{s}=\vec{x}-\vec{x}'$$

$f(\vec{r},\vec{p},t)$ **is the single-particle phase-space distribution function**

**Density in coordinate space:** $\rho(\vec{r},t)=\frac{1}{(2\pi\hbar)^3}\int d^3p\ f(\vec{r},\vec{p},t)$

**Density in momentum space:** $g(\vec{p},t)=\int d^3r\ f(\vec{r},\vec{p},t)$

# Uncertainty principle

$f(\vec{r}, \vec{p}, t)$ ⇨

**Is it consistent with Quantum Mechanics?**

**What about uncertainty principle?** $\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$

**Consider the case when disturbance varies only over macroscopic distances:** $\Delta x \sim \lambda$,

**where λ is a wave length of the particle:** $\lambda \geq \frac{\hbar}{2\Delta p}$

➔ **We can specify the momentum of the particle with microscopic accuracy** $\Delta p$ **!**

# Wigner transformation + Taylor expansion

$$\frac{\partial}{\partial t}\rho(\vec r+\frac{\vec s}{2},\vec r-\frac{\vec s}{2},t) \;+\frac{i}{\hbar}\left[\frac{\hbar^2}{2m}\vec\nabla^2_{\vec r+\frac{\vec s}{2}}+U(\vec r+\frac{\vec s}{2},t)-\frac{\hbar^2}{2m}\vec\nabla^2_{\vec r-\frac{\vec s}{2}}-U(\vec r-\frac{\vec s}{2},t)\right]\rho(\vec r+\frac{\vec s}{2},\vec r-\frac{\vec s}{2},t)=0 \qquad (1)$$

- Make Wigner transformation of eq.(1)

$$\int d^3s\;\exp\left(-\frac{i}{\hbar}\vec p\vec s\right)\frac{\partial}{\partial t}\rho\left(\vec r+\frac{\vec s}{2},\vec r-\frac{\vec s}{2},t\right)$$
$$+\frac{i}{2m}\frac{\hbar^2}{\hbar}\int d^3s\;\exp\left(-\frac{i}{\hbar}\vec p\vec s\right)\left[\vec\nabla^2_{\vec r+\frac{\vec s}{2}}-\vec\nabla^2_{\vec r-\frac{\vec s}{2}}\right]\rho\left(\vec r+\frac{\vec s}{2},\vec r-\frac{\vec s}{2},t\right) \qquad (2)$$
$$+\frac{i}{\hbar}\int d^3s\;\exp\left(-\frac{i}{\hbar}\vec p\vec s\right)\left[U(\vec r+\frac{\vec s}{2},t)-U(\vec r-\frac{\vec s}{2},t)\right]\rho\left(\vec r+\frac{\vec s}{2},\vec r-\frac{\vec s}{2},t\right)=0$$

- Use that $\vec\nabla^2_{\vec r+\frac{\vec s}{2}}-\vec\nabla^2_{\vec r-\frac{\vec s}{2}}=2\vec\nabla_{\vec r}\cdot\vec\nabla_{\vec s}$ (see Task 1) (3)

- Consider $U(\vec r+\frac{\vec s}{2},t)-U(\vec r-\frac{\vec s}{2},t)$

Make Taylor expansion around $r$; $s\to 0$

(4) $$=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}\vec s\cdot\vec\nabla_{\vec r}\right)^n U|_{s=0}-\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{2}\vec s\cdot\vec\nabla_{\vec r}\right)^n U|_{s=0}=2\sum_{odd}\left(\frac{1}{2}\vec s\cdot\vec\nabla_{\vec r}\right)^n U|_{s=0}\frac{1}{n!}$$

terms even in $n$ cancel

$$\approx\vec s\cdot\vec\nabla_{\vec r}U(\vec r,t)$$

Classical limit: keep only the first term $n=1$ (good approximation for hadronic potentials)

# Vlasov equation-of-motion

From (2) and (3),(4) obtain

$$\frac{\partial}{\partial t} f(\vec{r},\vec{p},t) + \frac{i}{2m}\frac{\hbar^2}{\hbar}\int d^3s \; exp\left(-\frac{i}{\hbar}\vec{p}\vec{s}\right) \; 2\vec{\nabla}_{\vec{r}}\cdot\vec{\nabla}_{\vec{s}} \; \rho\left(\vec{r}+\frac{\vec{s}}{2},\vec{r}-\frac{\vec{s}}{2},t\right) \qquad (5)$$

$$+\frac{i}{\hbar}\int d^3s \; exp\left(-\frac{i}{\hbar}\vec{p}\vec{s}\right) \; \vec{s}\cdot\vec{\nabla}_{\vec{r}} \, U(\vec{r},t) \; \rho\left(\vec{r}+\frac{\vec{s}}{2},\vec{r}-\frac{\vec{s}}{2},t\right) = 0$$

(see Task 2)

(*): $f(\vec{r},\vec{p},t) = \int d^3s \; exp\left(-\frac{i}{\hbar}\vec{p}\vec{s}\right) \; \rho\left(\vec{r}+\frac{\vec{s}}{2},\vec{r}-\frac{\vec{s}}{2},t\right)$

using (*)

**Vlasov equation**

**- free propagation of particles in the self-generated HF mean-field potential:**

$$\frac{\partial}{\partial t} f(\vec{r},\vec{p},t) + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}} \; f(\vec{r},\vec{p},t) - \vec{\nabla}_{\vec{r}} U(\vec{r},t) \; \vec{\nabla}_{\vec{p}} \, f(\vec{r},\vec{p},t) = 0 \qquad (6)$$

**Eq.(6) is entirely classical (lowest order in *s* expansion).**
**Here *U* is a self-consistent potential associated with *f* phase-space distribution:**

$$U(\vec{r},t) = \frac{1}{(2\pi\hbar)^3}\int d^3r' d^3p \, V(\vec{r}-\vec{r}',t) \, f(\vec{r}',\vec{p},t) \qquad (7)$$

# Vlasov EoM

**Vlasov equation of motion**
**- free propagation of particles in the self-generated HF mean-field potential:**

$$\frac{\partial}{\partial t} f(\vec{r},\vec{p},t) + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}}\, f(\vec{r},\vec{p},t) - \vec{\nabla}_{\vec{r}} U(\vec{r},t)\, \vec{\nabla}_{\vec{p}} f(\vec{r},\vec{p},t) = 0$$

**Vlasov EoM is equivalent to:**

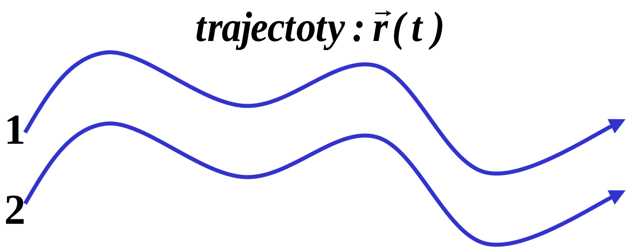
$$\frac{d}{dt} f(\vec{r},\vec{p},t) = 0 = \left[\frac{\partial}{\partial t} + \dot{\vec{r}}\vec{\nabla}_{\vec{r}} + \dot{\vec{p}}\vec{\nabla}_{\vec{p}}\right] f(\vec{r},\vec{p},t) = 0$$

**with**

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{\vec{p}}{m}$$

$$\dot{\vec{p}} = \frac{d\vec{p}}{dt} = -\vec{\nabla}_{\vec{r}} U(\vec{r},t)$$

**➔ Classical equations of motion**



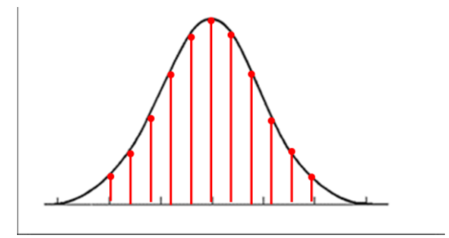

**Note: the quantum physics plays a role in the initial conditions for *f*:**
**the initial *f* in case of fermions must respect the Pauli principle**

# Numerical solution of Vlasov EoM

**Testparticle method or method of parallel ensembles :**

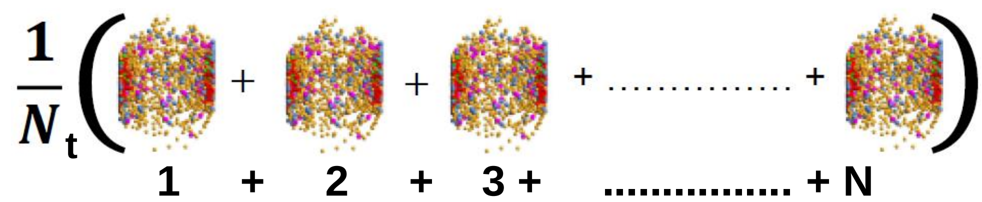
**the distribution functions of the system of N particles can be described as a sum of point-like particles ($\delta$ –functions)**

**In the limit of large number of parallel ensembles** $N_t \to \infty$



$$f(\vec{r},\vec{p},t)=\frac{1}{N_t}\sum_{i=1}^{N\cdot N_t}\delta(\vec{r}-\vec{r}_i(t))\,\delta(\vec{p}-\vec{p}_i(t))$$

**is a solution of Vlasov EoM**



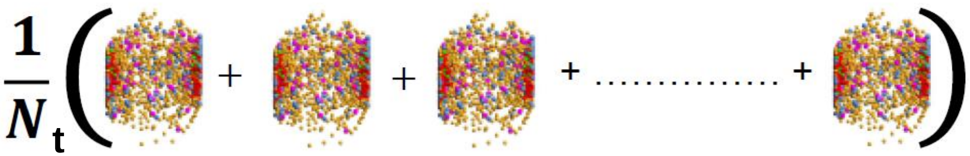

➔ **Propagation of test-particles in time following ‘classical’ EoM:**

$$\dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} = \frac{\vec{p}_i}{m_i}$$

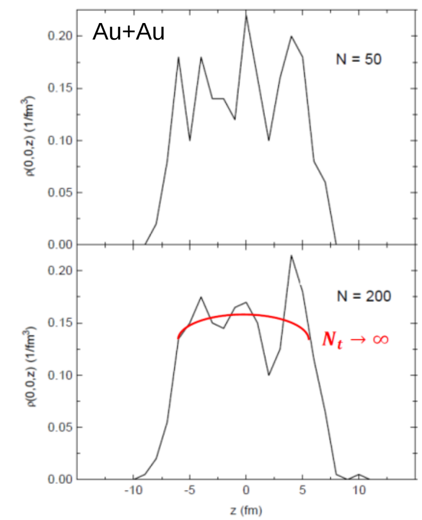
$$\dot{\vec{p}}_i = \frac{d\vec{p}_i}{dt} = -\vec{\nabla}_{\vec{r}_i} U(\vec{r}_i, t)$$

# Mean-field potential

- **Testparticle method** provides a smooth density distribution for calculation of **mean-field potential** for particle propagation
  **(No exchange of particles between the parallel ensembles)**



**1 + 2 + 3 + ................ + N**




**Mean-field potential:**

$$U(\vec{r},t) = \frac{1}{(2\pi\hbar)^3}\int d^3r' d^3p V(\vec{r}-\vec{r}',t) f(\vec{r}',\vec{p},t)$$

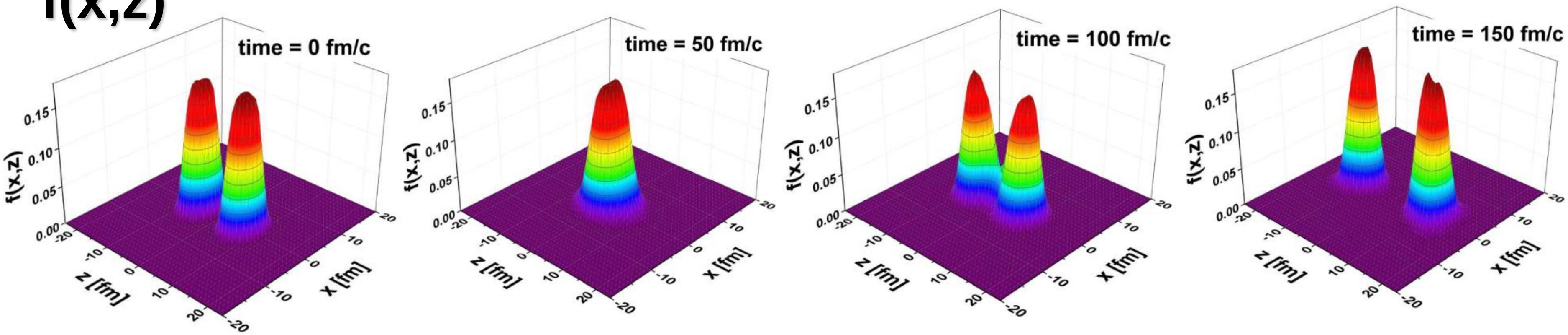
- **Effective two-body interaction** with
  **a finite-range Yukawa, Skyrme-type and Coulomb interactions:**

$$v(\mathbf{x}-\mathbf{x}_2) = -A_0\frac{\exp(-\mu|\mathbf{x}-\mathbf{x}_2|)}{\mu|\mathbf{x}-\mathbf{x}_2|} + B_0\delta^3(\mathbf{x}-\mathbf{x}_2)\rho(\mathbf{x}-\mathbf{x}_2)^2 + \frac{e^2}{4\pi}\frac{\delta_{pp}}{|\mathbf{x}-\mathbf{x}_2|}$$

# Density distribution: Vlasov equation

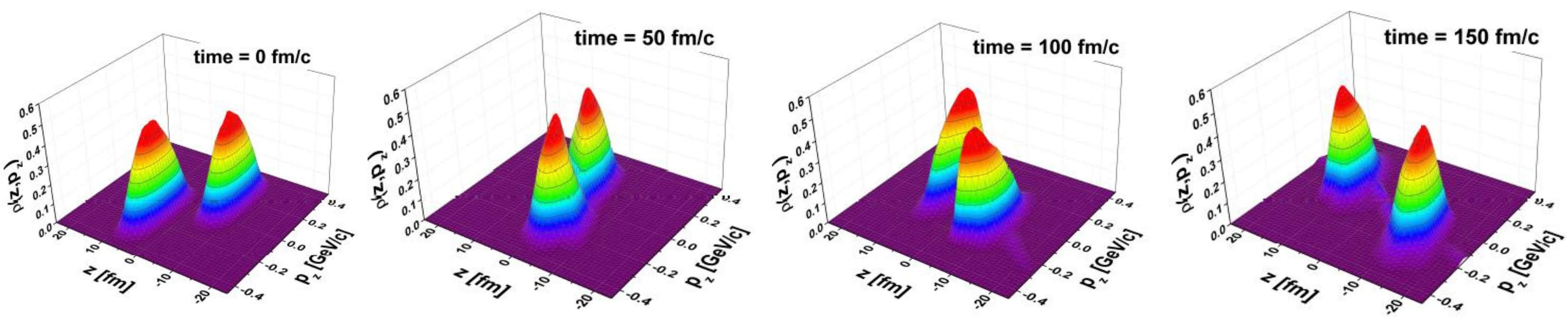
## Ca+Ca, 40 A MeV

## f(x,z)




time →

## f(z,$p_z$)



# Useful literature

**L. P. Kadanoff, G. Baym, ‚*Quantum Statistical Mechanics*‘, Benjamin, 1962**

**M. Bonitz, ‚*Quantum kinetic theory*‘, B.G. Teubner Stuttgart, 1998**

**W. Cassing, `Transport Theories for Strongly-Interacting Systems’,
Springer Nature: Lecture Notes in Physics 989, 2021;
DOI: 10.1007/978-3-030-80295-0**

# *Task 1

**1) Show that**

$$\vec{\nabla}_x^2 - \vec{\nabla}_{x'}^2 = \vec{\nabla}^2_{\mathbf{r}+\mathbf{s}/2} - \vec{\nabla}^2_{\mathbf{r}-\mathbf{s}/2} = 2\vec{\nabla}_{\mathbf{s}} \cdot \vec{\nabla}_{\mathbf{r}},$$

where $\mathbf{x} = \mathbf{r} + \mathbf{s}/2, \quad \mathbf{x}' = \mathbf{r} - \mathbf{s}/2.$

❖ **Solution:**

1) We have (i=1,2,3)

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial r_i}\frac{\partial r_i}{\partial x_i} + \frac{\partial}{\partial s_i}\frac{\partial s_i}{\partial x_i} = \frac{1}{2}\frac{\partial}{\partial r_i} + \frac{\partial}{\partial s_i},$$

$$\frac{\partial}{\partial x'_i} = \frac{\partial}{\partial r_i}\frac{\partial r_i}{\partial x'_i} + \frac{\partial}{\partial s_i}\frac{\partial s_i}{\partial x'_i} = \frac{1}{2}\frac{\partial}{\partial r_i} - \frac{\partial}{\partial s_i}.$$

This leads to

$$\vec{\nabla}_x^2 - \vec{\nabla}_{x'}^2 = \vec{\nabla}^2_{\mathbf{r}+\mathbf{s}/2} - \vec{\nabla}^2_{\mathbf{r}-\mathbf{s}/2} = \sum_{i=1}^{3}\left( \left(\frac{1}{2}\frac{\partial}{\partial r_i} + \frac{\partial}{\partial s_i}\right)^2 - \left(\frac{1}{2}\frac{\partial}{\partial r_i} - \frac{\partial}{\partial s_i}\right)^2 \right)$$

$$= \sum_{i=1}^{3}\left( \frac{1}{4}\frac{\partial^2}{\partial r_i^2} + \frac{\partial}{\partial r_i}\frac{\partial}{\partial s_i} + \frac{\partial^2}{\partial s_i^2} - \frac{1}{4}\frac{\partial^2}{\partial r_i^2} + \frac{\partial}{\partial r_i}\frac{\partial}{\partial s_i} - \frac{\partial^2}{\partial s_i^2} \right) = \underline{2\vec{\nabla}_{\mathbf{r}} \cdot \vec{\nabla}_{\mathbf{s}}}.$$

# *Task 2

2) What is the Wigner transform of $\vec{\nabla}_{\mathbf{s}} \cdot \vec{\nabla}_{\mathbf{r}}\, \rho(\mathbf{r}+\mathbf{s}/2, \mathbf{r}-\mathbf{s}/2)$ when assuming that $\rho(\mathbf{r},\mathbf{s})$ vanishes at $s_i \to \pm\infty$ for $i = x, y, z$?

❖ **Solution:**

The Wigner transform of $\vec{\nabla}_{\mathbf{s}} \cdot \vec{\nabla}_{\mathbf{r}} \rho(\mathbf{r}+\mathbf{s}/2, \mathbf{r}-\mathbf{s}/2)$ is given by

$$\int d^3s\ \exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s})\vec{\nabla}_{\mathbf{s}} \cdot \vec{\nabla}_{\mathbf{r}}\ \rho(\mathbf{r}+\mathbf{s}/2, \mathbf{r}-\mathbf{s}/2)$$
$$= \vec{\nabla}_{\mathbf{r}} \cdot \int d^3s\ \exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s})\vec{\nabla}_{\mathbf{s}} \rho(\mathbf{r}+\mathbf{s}/2, \mathbf{r}-\mathbf{s}/2). \qquad (4)$$

Use that

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx}.$$

Then

$$\vec{\nabla}_{\mathbf{s}} \left(\exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s})\ \rho(\mathbf{r}+\mathbf{s}/2, \mathbf{r}-\mathbf{s}/2)\right) =$$
$$= \exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s})\vec{\nabla}_{\mathbf{s}}\ \rho(\mathbf{r}+\mathbf{s}/2, \mathbf{r}-\mathbf{s}/2) + \ \rho(\mathbf{r}+\mathbf{s}/2, \mathbf{r}-\mathbf{s}/2)\vec{\nabla}_{\mathbf{s}} \exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s}).$$

# *Task 2

Since

$$\vec{\nabla}_{\mathbf{s}} \exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s}) = -\frac{i}{\hbar}\mathbf{p}\exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s}),$$

we obtain that

$$\begin{aligned}&\exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s})\vec{\nabla}_{\mathbf{s}}\ \rho(\mathbf{r}+\mathbf{s}/2,\mathbf{r}-\mathbf{s}/2) \qquad (5)\\ &= \vec{\nabla}_{\mathbf{s}}\left(\exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s})\ \rho(\mathbf{r}+\mathbf{s}/2,\mathbf{r}-\mathbf{s}/2)\right) + \frac{i}{\hbar}\mathbf{p}\exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s})\ \rho(\mathbf{r}+\mathbf{s}/2,\mathbf{r}-\mathbf{s}/2).\end{aligned}$$

Substitute (5) to eq.(4):

$$\begin{aligned}(4):\quad &= \vec{\nabla}_{\mathbf{r}}\cdot\int d^3s\vec{\nabla}_{\mathbf{s}}\left(\exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s})\ \rho(\mathbf{r}+\mathbf{s}/2,\mathbf{r}-\mathbf{s}/2)\right) \qquad (6)\\ &-\vec{\nabla}_{\mathbf{r}}\cdot\int d^3s\ (\vec{\nabla}_{\mathbf{s}}\exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s}))\ \rho(\mathbf{r}+\mathbf{s}/2,\mathbf{r}-\mathbf{s}/2)\end{aligned}$$

# *Task 2

The first term in (6) is vanishing in the limits of partial integration since $\rho(\mathbf{r},\mathbf{s}) \to \mathbf{0}$ when $s_i \to \pm\infty$ for all components $i = 1, 2, 3$:

$$\int d^3s \vec{\nabla}_{\mathbf{s}} \left( \exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s})\ \rho(\mathbf{r}+\mathbf{s}/2, \mathbf{r}-\mathbf{s}/2) \right) \to 0$$

$$\text{since} \int\limits_{-\infty}^{\infty} ds_i \frac{\partial}{s_i} \left( \exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s})\ \rho(\mathbf{r}+\mathbf{s}/2, \mathbf{r}-\mathbf{s}/2) \right)$$

$$\to \exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s})\ \rho(\mathbf{r}+\mathbf{s}/2, \mathbf{r}-\mathbf{s}/2)|_{s_i\to-\infty}^{s_i\to\infty}$$

Thus, $\int d^3s\ \exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s})\vec{\nabla}_{\mathbf{s}}\cdot\vec{\nabla}_{\mathbf{r}}\ \rho(\mathbf{r}+\mathbf{s}/2, \mathbf{r}-\mathbf{s}/2)$ ⇒

$$(4):\quad = -\vec{\nabla}_{\mathbf{r}}\cdot\int d^3s\ (-\frac{i}{\hbar}\mathbf{p}\ \exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s})\ \rho(\mathbf{r}+\mathbf{s}/2, \mathbf{r}-\mathbf{s}/2)$$

$$= \frac{i}{\hbar}\mathbf{p}\cdot\vec{\nabla}_{\mathbf{r}}\int d^3s\ \exp(-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s})\ \rho(\mathbf{r}+\mathbf{s}/2, \mathbf{r}-\mathbf{s}/2)$$

$$= \frac{i}{\hbar}\mathbf{p}\cdot\vec{\nabla}_{\mathbf{r}} f(\mathbf{r},\mathbf{p}). \qquad (7)$$