GOETHE
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Lecture

Models for heavy-ion collisions: (Part 3): transport models

SS2024: ‚Dynamical models for relativistic heavy-ion collisions‘

BUU (VUU) equation

Reminder: Lecture 4

Boltzmann (Vlasov)-Uehling-Uhlenbeck equation (NON-relativistic formulation!)
- free propagation of particles in the self-generated HF mean-field potential with an on-shell collision term:

$$\frac{d}{dt} f(\vec{r},\vec{p},t) \equiv \frac{\partial}{\partial t} f(\vec{r},\vec{p},t) + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}}\; f(\vec{r},\vec{p},t) - \vec{\nabla}_{\vec{r}} U(\vec{r},t)\;\vec{\nabla}_{\vec{p}} f(\vec{r},\vec{p},t) = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

Collision term for 1+2→3+4 (let's consider fermions) :

Probability including Pauli blocking of fermions

$$I_{coll} \equiv \left(\frac{\partial f}{\partial t}\right)_{coll} \Rightarrow \frac{1}{((2\pi)^3)^3}\int d^3p_2\, d^3p_3\, d^3p_4\;\cdot w(1+2\rightarrow 3+4)\cdot P$$

$$\times(2\pi)^3\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\;(2\pi)\delta\left(\frac{\vec{p}_1}{2m_1}+\frac{\vec{p}_2}{2m_2}-\frac{\vec{p}_3}{2m_3}-\frac{\vec{p}_4}{2m_4}\right)$$

Transition probability for *1+2→3+4*: $w(1+2\rightarrow 3+4)\Rightarrow \upsilon_{12}\cdot\frac{d^3\sigma}{d^3q}$

where $\upsilon_{12} = \frac{\hbar}{m}\,|\,\vec{p}_1-\vec{p}_2\,|$ **- relative velocity of the colliding nucleons**

$\frac{d^3\sigma}{d^3q}$ **- differential cross section, *q* – momentum transfer** $\vec{q}=\vec{p}_1-\vec{p}_3$

2. Density-matrix formalism: Correlation dynamics

Density-matrix formalism

- **Schrödinger equation for a system of N fermions:**

$$i\hbar \frac{\partial}{\partial t} \Psi_N(1,..,N;t) = H_N(1,..,N)\Psi_N(1,..,N;t) \qquad (1)$$

antisymmetric N-body wave function

or in Dirac notation: $$i\hbar \frac{\partial}{\partial t} |\Psi_N(t)> = H_N|\Psi_N(t)>$$

Notation:
***i* – particle index of many body system (*i=1,…,N*) :**

$$i \equiv \mathbf{r}_i, \sigma_i, \tau_i$$

***i* = coordinate, spin, isospin,...**

- **Hamiltonian operator:** $$H_N = \sum_{i=1}^{N} h^0(i) + \sum_{i<j}^{N-1} v(ij),$$

2-body interaction

One-body Hamiltonian:

$$h^0(i) = t(i) + U^0(i)$$

kinetic energy operator + (possible) external mean-field potential
(e.g. external electromagnetic field)

Hermitian Hamiltonian: $$H_N = H_N^\dagger$$

Consider (1) - hermitean conjugate:

$$-i\hbar \frac{\partial}{\partial t'} \Psi_N^*(1',..,N';t) = H_N(1',..,N'), \Psi_N^*(1',..,N';t') \qquad (2)$$

(1)*Ψ*$_N$ – Ψ$_N$*(2) :

$$i\hbar \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t'}\right) \Psi_N(1,..,N;t)\ \Psi_N^*(1',..,N';t') =$$

$$(H_N(1,..,N) - H(1',..,N';t'))\,\Psi_N(1,..,N;t)\Psi_N^*(1',..,N';t') \qquad (2.1)$$

Density-matrix formalism

- **Introduce two time density :**

$$\rho_N(1,..,N,1',..,N';t,t') = \Psi_N(1,..,N;t)\Psi_N^*(1',..,N';t') \qquad (2.2)$$

or in Dirac notation:

$$\rho_N(1,..,N,1',..,N';t,t') = \langle 1',..,N'|\rho_N(t,t')|1,..,N\rangle$$

Substitute (2.2) in (2.1):

$$i\hbar\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t'}\right)\rho_N = \left(H_N(1,..,N) - H(1',..,N';t')\right)\rho_N$$

$$i\hbar\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t'}\right)\rho_N = [H_N,\rho_N]$$

- **Restrict to *t'=t* :**

$$\rho_N(1,..,N,1',..,N';t) = \rho_N(1,..,N,1',..,N';t,t')\ \delta(t-t')$$

- **Schrödinger eq. in density-operator representation →**

von Neumann (or Liouville) eq. (in matrix representation) describes an N-particle system in- or out-off equilibrium :

$$i\hbar\frac{\partial}{\partial t}\rho_N(1,..,N;1'..N';t) = [H_N,\rho_N(1,..,N;1'..N';t)] \qquad (2.3)$$

Density-matrix formalism

❑ **Introduce a reduced density matrices $\rho_n(1\ldots n,1'\ldots n';\, t)$ by taking the trace (integrate) over particles $n+1,\ldots N$ of ρ_N:**

$$\rho_n = \frac{1}{(N-n)!}\mathrm{Tr}_{n+1,\ldots,N}\,\rho_N = \frac{1}{n+1}\mathrm{Tr}_{n+1}\{\rho_{n+1}\} \qquad (3)$$

Recurrence

Here the relative normalization between ρ_n and ρ_{n+1} is fixed and it is useful to choose the normalization

$$Tr_{1,\ldots,N}\ \rho_N = N!$$

which leads to the following normalization for the one-body density matrix:

$$Tr_{(1=1')}\rho(11';t) = \sum_i < a_i^\dagger a_i >= N$$

$$\mathrm{Tr} \to \int \frac{d^3p}{(2\pi\hbar)^3} \int d^3r$$

i.e. the particle number for the *N*-body Fermi system.

The normalization of the two-body density matrix then reads as

$$Tr_{(1,2)}\rho_2 = \sum_{i,j} < a_i^\dagger a_j^\dagger a_j a_i >= -\sum_{i,j} < a_i^\dagger a_j^\dagger a_i a_j >= \sum_{i,j}\{< a_i^\dagger a_i a_j^\dagger a_j > - < a_i^\dagger a_j > \delta_{ij}\}$$

$$= (N-1)\sum_j < a_j^\dagger a_j >= N(N-1)$$

The traces of the density matrices ρ_n (for n < N) : $$Tr_{(1,\ldots,n)}\rho_n = \frac{N!}{(N-n)!}$$

Density matrix formalism: BBGKY-Hierarchy

Taking corresponding traces (i.e. $Tr_{(n+1,\ldots N)}$) of the von-Neumann equation we obtain the BBGKY-Hierarchy (Bogolyubov, Born, Green, Kirkwood and Yvon)

$$i\frac{\partial}{\partial t}\,\rho_n = [\sum_{i=1}^{n} h^0(i), \rho_n] + [\sum_{1=i\langle j}^{n-1} v(ij), \rho_n] + \sum_{i=1}^{n} \mathrm{Tr}_{n+1}[v(i,n+1), \rho_{n+1}] \qquad (4)$$

for $1 \le n \le N$ with $\rho_{N+1} = 0$.

- **This set of equations is equivalent to the von-Neumann equation**
- **The approximations or truncations of this set will reduce the information about the system**

❑ **The explicit equations for *n=1, n=2* read:**

$$i\frac{\partial}{\partial t}\,\rho_1 = [h^0(1), \rho_1] + \mathrm{Tr}_2[v(12), \rho_2], \qquad (5)$$

$$i\frac{\partial}{\partial t}\,\rho_2 = [\sum_{i=1}^{2} h^0(i), \rho_2] + [v(12), \rho_2] + \mathrm{Tr}_3[v(13) + v(23), \rho_3] \qquad (6)$$

Eqs. (5,6) are not closed since eq. (6) for ρ_2 requires information from ρ_3. Its equation reads:

$$i\frac{\partial}{\partial t}\,\rho_3 = [\sum_{i=1}^{3} h^0(i), \rho_3] + [v(12) + v(13) + v(23), \rho_3] + Tr_4[v(14) + v(24) + v(34), \rho_4] \qquad (7)$$

Correlation dynamics

Introduce the cluster expansion ➔ Correlation dynamics:

- 1-body density matrix: $\rho_1(11') = \rho(11'),$

1 – initial state of particle „1“
1' – final state of the same particle „1“

- 2-body density matrix (consider fermions):

(8) $$\rho_2(12,1'2') = \rho(11')\rho(22') - \rho(12')\rho(21') + c_2(12,1'2') = \rho_{20}(12,1'2') + c_2(12,1'2')$$

2PI= 2-particle-irreducible approach

(9) $$\rho_2(12,1'2') = \mathcal{A}_{12}\rho(11')\rho(22') + c_2(12,1'2')$$

2-body antisymmetrization operator:

$$\mathcal{A}_{ij} = 1 - P_{ij}$$

Permutation operator

1PI = 1-particle-irreducible approach (TDHF approximation) + 2-body correlations

By neglecting c_2 in (9) we get the limit of independent particles (Time-Dependent Hartree-Fock). This implies that all effects from collisions or correlations are incorporated in c_2 and higher orders in c_2 etc.

- 3-body density matrix:

$$\begin{aligned}\rho_3(123,1'2'3') &= \rho(11')\rho(22')\rho(33') - \rho(12')\rho(21')\rho(33')\\ &-\rho(13')\rho(22')\rho(31') - \rho(11')\rho(32')\rho(23') + \rho(13')\rho(21')\rho(32') + \rho(12')\rho(31')\rho(23')\\ &+\rho(11')c_2(23,2'3') - \rho(12')c_2(23,1'3') - \rho(13')c_2(23,2'1') + \rho(22')c_2(13,1'3')\\ &-\rho(21')c_2(13,2'3') - \rho(23')c_2(13,1'2') + \rho(33')c_2(12,1'2') - \rho(31')c_2(12,3'2')\\ &-\rho(32')c_2(12,1'3') + c_3(123,1'2'3').\end{aligned} \qquad (10)$$

3-body correlations

Correlation dynamics

The goal: from BBGKY-hierarchy obtain closed equation **for 1-body density matrix** within 2PI **discarding explicit three-body correlations c_3**

- **for that we reformulate eq. (5) for ρ_1 using the cluster expansion (correlation dynamics):**

(5) $$i\hbar\frac{\partial}{\partial t}\rho_1 = [h^0(1), \rho_1] + Tr_2[v(12), \rho_2]$$

substitutute eq. (8) for ρ_2 in eq. (5)

(8) $$\rho_2(12, 1'2') = \rho(11')\rho(22') - \rho(12')\rho(21') + c_2(12, 1'2') = \rho_{20}(12, 1'2') + c_2(12, 1'2')$$

➔ **we obtain EoM for the one-body density matrix:**

$$i\frac{\partial}{\partial t}\rho(11';t) = [h^0(1) - h^0(1')]\rho(11';t)$$
$$+\mathrm{Tr}_{(2=2')}[v(12)\mathcal{A}_{12} - v(1'2')\mathcal{A}_{1'2'}]\rho(11';t)\rho(22';t) + \mathrm{Tr}_{(2=2')}[v(12) - v(1'2')]c_2(12, 1'2';t)$$

2-body correlations

*** How to obtain the 2-body correlation matrix c_2 ?**

Correlation dynamics

- **In order to obtain the 2-body correlation matrix c_2, we start with eq. (6) for ρ_2**

$$i\hbar \frac{\partial}{\partial t} \rho_2 = [\sum_{i=1}^{2} h^0(i), \rho_2] + [v(12), \rho_2] + Tr_3[v(13) + v(23), \rho_3] \qquad (6)$$

$$\rho_3(123, 1'2'3') = \rho(11')\rho(22')\rho(33') - \rho(12')\rho(21')\rho(33') \qquad (10)$$
$$+\rho(13')\rho(22')\rho(31') - \rho(11')\rho(32')\rho(23') + \rho(13')\rho(21')\rho(32') + \rho(12')\rho(31')\rho(23')$$
$$+\rho(11')c_2(23, 2'3') - \rho(12')c_2(23, 1'3') - \rho(13')c_2(23, 2'1') + \rho(22')c_2(13, 1'3')$$
$$-\rho(21')c_2(13, 2'3') - \rho(23')c_2(13, 1'2') + \rho(33')c_2(12, 1'2') - \rho(31')c_2(12, 3'2')$$
$$-\rho(32')c_2(12, 1'3') + c_3(123, 1'2'3')$$

substitute eq. (10) for ρ_3
and discarding explicit 3-body correlations c_3

➔ **we obtain EoM for the two-body correlation matrix c_2 :**

$$i\frac{\partial}{\partial t} c_2(12, 1'2'; t) = [h^0(1) + h^0(2) - h^0(1') - h^0(2')]c_2(12, 1'2'; t) \qquad (12)$$
$$+\mathrm{Tr}_{(3=3')}[v(13)\mathcal{A}_{13} + v(23)\mathcal{A}_{23} - v(1'3')\mathcal{A}_{1'3'} - v(2'3')\mathcal{A}_{2'3'}]\rho(33'; t)c_2(12, 1'2'; t)$$
$$+[v(12) - v(1'2')]\rho_{20}(12, 1'2')$$
$$-\mathrm{Tr}_{(3=3')}\{v(13)\rho(23'; t)\rho_{20}(13, 1'2'; t) - v(1'3')\rho(32'; t)\rho_{20}(12, 1'3'; t)$$
$$+v(23)\rho(13'; t)\rho_{20}(32, 1'2'; t) - v(2'3')\rho(31'; t)\rho_{20}(12, 3'2'; t)\}$$

$$\rho(11')\rho(22') - \rho(12')\rho(21') = \rho_{20}(12, 1'2')$$

$$+[v(12) - v(1'2')]c_2(12, 1'2'; t)$$
$$-\mathrm{Tr}_{(3=3')}\{v(13)\rho(23'; t)c_2(13, 1'2'; t) - v(1'3')\rho(32'; t)c_2(12, 1'3'; t)$$
$$+v(23)\rho(13'; t)c_2(32, 1'2'; t) - v(2'3')\rho(31'; t)c_2(12, 3'2'; t)\}$$

$$+\mathrm{Tr}_{(3=3')}\{[v(13)\mathcal{A}_{13}\mathcal{A}_{1'2'} - v(1'3')\mathcal{A}_{1'3'}\mathcal{A}_{12}]\ \rho(11'; t)c_2(32, 3'2'; t)$$
$$+[v(23)\mathcal{A}_{23}\mathcal{A}_{1'2'} - v(2'3')\mathcal{A}_{2'3'}\mathcal{A}_{12}]\ \rho(22'; t)c_2(13, 1'3'; t)\}.$$

Correlation dynamics

To reduce the complexity we introduce useful notations:

- **a one-body Hamiltonian by**

$$h(i) = t(i) + U^s(i) = t(i) + \mathrm{Tr}_{(n=n')} v(in)\mathcal{A}_{in}\rho(nn';t), \quad (13)$$

$$h(i') = t(i') + U^s(i') = t(i') + \mathrm{Tr}_{(n=n')} v(i'n')\mathcal{A}_{i'n'}\rho(nn';t)$$

kinetic term + interaction with the self-generated time-dependent mean field

- **Pauli-blocking operator is uniquely defined by** (14)

$$Q^{=}_{ij} = 1 - \mathrm{Tr}_{(n=n')}(P_{in} + P_{jn})\rho(nn';t); \qquad Q^{=}_{i'j'} = 1 - \mathrm{Tr}_{(n=n')}(P_{i'n'} + P_{j'n'})\rho(nn';t),$$

- **Effective 2-body interaction <u>in the medium</u>:**

$$V^{=}(ij) = Q^{=}_{ij}\underline{v(ij)}; \qquad V^{=}(i'j') = Q^{=}_{i'j'}v(i'j'), \quad (15)$$

Resummed interaction ➔ G-matrix approach

Correlation dynamics

❑* EoM for the **one-body density matrix**:

$$i\frac{\partial}{\partial t}\rho(11';t) = [h(1) - h(1')]\rho(11';t) + \mathrm{Tr}_{(2=2')}[v(12) - v(1'2')]c_2(12,1'2';t) \qquad (16)$$

TDHF — 2-body correlations

EoM (16) describes the propagation of a particle in the self-generated mean field U^s with additional 2-body correlations that are further specified in the EoM (17) for c_2 :

❑* EoM for the **2-body correlation matrix**:

$$\begin{aligned} i\frac{\partial}{\partial t}c_2(12,1'2';t) = &\Big[\sum_{i=1}^{2} h(i) - \sum_{i'=1'}^{2'} h(i')\Big]c_2(12,1'2';t) \\ &+[V^{=}(12) - V^{=}(1'2')]\rho_{20}(12,1'2';t) \\ &+[V^{=}(12) - V^{=}(1'2')]c_2(12,1'2';t) \\ &+\mathrm{Tr}_{(3=3')}\{[v(13)\mathcal{A}_{13}\mathcal{A}_{1'2'} - v(1'3')\mathcal{A}_{1'3'}\mathcal{A}_{12}]\ \rho(11';t)c_2(32,3'2';t) \\ &+[v(23)\mathcal{A}_{23}\mathcal{A}_{1'2'} - v(2'3')\mathcal{A}_{2'3'}\mathcal{A}_{12}]\ \rho(22';t)c_2(13,1'3';t)\}. \end{aligned} \qquad (17)$$

- Propagation of two particles 1 and 2 in the mean field U^s
- Born term: bare 2-body scattering
- resummed in-medium interaction with intermediate Pauli blocking (G-matrix theory)
- 2-Particle-2-hole interactions (important for groundstate correlations) and damping of low energy modes

Note: Time evolution of c_2 depends on the distribution of a **third particle**, which is integrated out in the trace! The third particle is interacting as well!

*: EoM is obtained after the ‚cluster' expansion and neglecting the explicit 3-body correlations c_3

Vlasov equation

BBGKY-Hierarchie - 1PI ➔
eq.(11) with $c_2(1,2,1',2')=0$

$$i\hbar\frac{\partial}{\partial t}\rho(11';t) = [h^0(1) - h^0(1')]\rho(11';t)$$

TDHF

$$\frac{\partial}{\partial t}\rho(\vec{r},\vec{r}',t)+\frac{i}{\hbar}\left[\frac{\hbar^2}{2m}\vec{\nabla}_r^2+U(\vec{r},t)-\frac{\hbar^2}{2m}\vec{\nabla}_{r'}^2-U(\vec{r}',t)\right]\rho(\vec{r},\vec{r}',t)=0$$

- **perform Wigner transformation of one-body density distribution function $\rho(r,r',t)$** ➔

$$f(\vec{r},\vec{p},t)=\int d^3s\ exp\left(-\frac{i}{\hbar}\vec{p}\vec{s}\right)\rho\left(\vec{r}+\frac{\vec{s}}{2},\vec{r}-\frac{\vec{s}}{2},t\right) \qquad (18)$$

f(r,p,t) **is the single particle phase-space distribution function**

After the 1st order gradient expansion ➔ Vlasov equation of motion
- free propagation of particles in the self-generated HF mean-field potential *U(r,t):*

$$\frac{\partial}{\partial t}f(\vec{r},\vec{p},t)+\frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}}\ f(\vec{r},\vec{p},t)-\vec{\nabla}_{\vec{r}}U(\vec{r},t)\ \vec{\nabla}_{\vec{p}}f(\vec{r},\vec{p},t)=0 \qquad (19)$$

$$U(\vec{r},t)=\frac{1}{(2\pi\hbar)^3}\sum_{\beta_{occ}}\int d^3r'd^3pV(\vec{r}-\vec{r}',t)f(\vec{r}',\vec{p},t)$$

Uehling-Uhlenbeck equation: collision term

$$i\frac{\partial}{\partial t}\rho(11';t) = [h(1) - h(1')]\rho(11';t) + \mathrm{Tr}_{(2=2')}[v(12) - v(1'2')]c_2(12,1'2';t) \qquad (21)$$

TDHF – Vlasov equation

2-body correlations

Collision term: $I(11',t) = \mathrm{Tr}_{(2=2')}[v(12) - v(1'2')]c_2(12,1'2';t) \qquad (22)$

- perform Wigner transformation
- Formally solve the EoM for c_2 (with some approximations in momentum space):

$$\begin{aligned} i\frac{\partial}{\partial t}c_2(12,1'2';t) &= [\sum_{i=1}^{2} h(i) - \sum_{i'=1'}^{2'} h(i')]c_2(12,1'2';t) \\ &+[V^{=}(12) - V^{=}(1'2')]\rho_{20}(12,1'2';t) \\ &+[V^{=}(12) - V^{=}(1'2')]c_2(12,1'2';t) \\ &+\mathrm{Tr}_{(3=3')}\{[v(13)\mathcal{A}_{13}\mathcal{A}_{1'2'} - v(1'3')\mathcal{A}_{1'3'}\mathcal{A}_{12}]\ \rho(11';t)c_2(32,3'2';t) \\ &+[v(23)\mathcal{A}_{23}\mathcal{A}_{1'2'} - v(2'3')\mathcal{A}_{2'3'}\mathcal{A}_{12}]\ \rho(22';t)c_2(13,1'3';t)\}. \end{aligned} \qquad (23)$$

- and insert obtained c_2 in the expression (22) for *I(11′,t)* : → BUU EoM

Boltzmann (Vlasov)-Uehling-Uhlenbeck (B(V)UU) equation : Collision term

$$\frac{d}{dt} f(\vec{r},\vec{p},t) \equiv \frac{\partial}{\partial t} f(\vec{r},\vec{p},t) + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}}\, f(\vec{r},\vec{p},t) - \vec{\nabla}_{\vec{r}} U(\vec{r},t)\, \vec{\nabla}_{\vec{p}} f(\vec{r},\vec{p},t) = \left(\frac{\partial f}{\partial t}\right)_{coll} \qquad (24)$$

Collision term for 1+2→3+4 (let‘s consider fermions) :

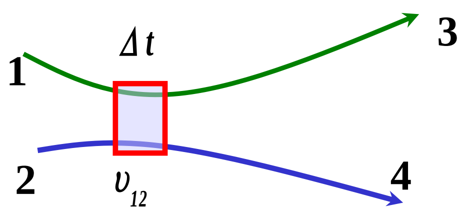
$$I_{coll} = \frac{4}{(2\pi)^3}\int d^3p_2\, d^3p_3 \int d\Omega\ |\upsilon_{12}|\,\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\cdot\frac{d\sigma}{d\Omega}(1+2\rightarrow 3+4)\cdot P \qquad (25)$$

Probability including Pauli blocking of fermions:

$$P = f_3 f_4 (1-f_1)(1-f_2) - f_1 f_2 (1-f_3)(1-f_4) \qquad (26)$$

Gain term (first term): **3+4→1+2**

Loss term (second term): **1+2→3+4**



For particle 1 and 2:
Collision term = Gain term – Loss term

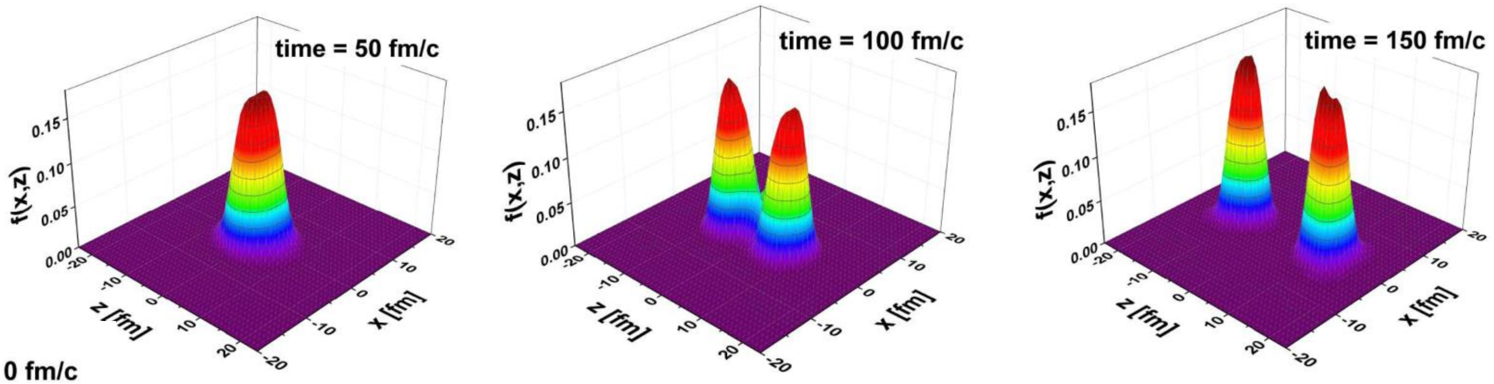
$$I_{coll} = G - L$$

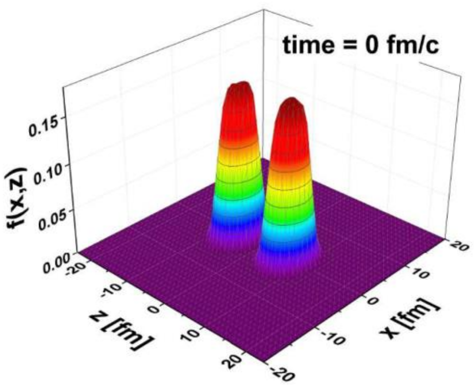
The BUU equations (24) describes the propagation in the self-generated mean-field U(r,t) as well as mutual two-body interactions respecting the Pauli-principle

Density distribution: Vlasov vs. Boltzmann equation

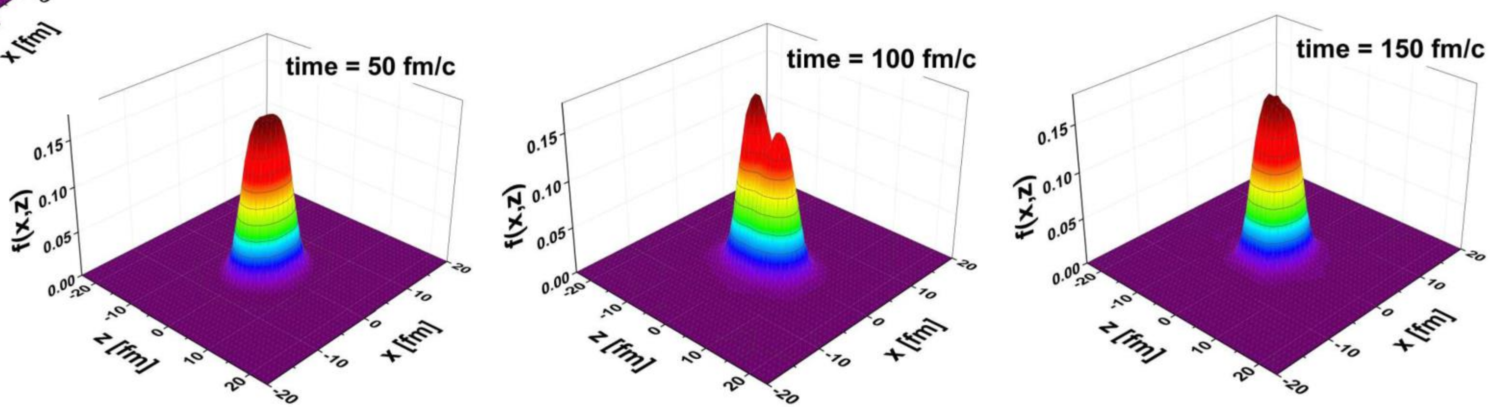
Ca+Ca, 40 A MeV

Vlasov →





Boltzmann →



Covariant transport equation

From non-relativistic to a relativistic formulation of on-shell transport equations:

Non-relativistic Schrödinger equation → relativistic Dirac equation

Non-relativistic dispersion relation:	**Relativistic dispersion relation:**

Non-relativistic dispersion relation:

$$E = \frac{\vec{p}^{\,2}}{2m} + U(\vec{r})$$

U(r) – density dependent potential (with attractive and repulsive parts)

! Not Lorentz invariant, i.e. dependent on the frame

Relativistic dispersion relation:

$$E^{*2} = m^{*2} + \vec{p}^{\,*2}$$

$$m^* = m + U_S$$

$$\vec{p}^{\,*} = \vec{p} + \vec{U}_V$$

$$E^* = E - U_0$$

U_S – scalar potential (attractive)

$U_\mu = (U_0, \vec{U}_V)$ vector 4-potential (repulsive)

$\mu = 0,1,2,3$

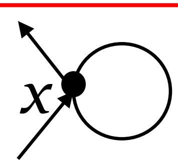
! Lorentz invariant, i.e. independent on the frame

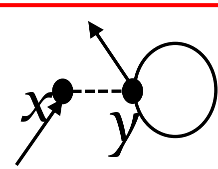
➔ **Consider the Dirac equation with local and non-local mean fields:**

$$(i\gamma^\mu \partial_\mu - m)\psi(x) - U^{MF}(x)\psi(x) - \int d^4y\, U^{MD}(x,y)\psi(x) = 0 \qquad (12)$$

here

$x \equiv (t, \vec{r}) \quad y \equiv (t, \vec{r}\,')$





Covariant transport equation

Perform a Wigner transformation of eq. (12) ➔ in phase-space:

$$U_S^{MD}(x,p) = C_S \frac{4}{(2\pi)^3} \int d^4p' \, D_S^{MD}(p-p') \, m^*(x,p') \, f(x,p')$$

$$U_{V\mu}^{MD}(x,p) = C_V \frac{4}{(2\pi)^3} \int d^4p' \, D_V^{MD}(p-p') \, \Pi_\mu(x,p') \, f(x,p')$$

where $f(x,p)$ is the single particle phase-space distribution function

$$m^*(x,p) = m + U_S^{MF}(x) + U_S^{MD}(x,p)$$ **- effective mass**

$$\Pi_\mu(x,p) = p_\mu - U_{V\mu}^{MF}(x) - U_{V\mu}^{MD}(x,p)$$ **- effective momentum**

where

$$D_{S,V}^{MD}(p-p') \equiv \int d^4z \, D_{S,V}^{MD}(z) e^{iz(p-p')}$$

Fourier transform

‚Ansatz' for *D*-function:

$$D_{S(V)}^{MD}(p-p') = \frac{\Lambda_{S(V)}^2}{\Lambda_{S(V)}^2 - (p-p')^2}$$

Covariant transport equation

Covariant relativistic on-shell BUU equation :

from many-body theory by connected Green functions in phase-space + mean-field limit for the propagation part (VUU)

$$\left\{ \left(\Pi_\mu - \Pi_\nu (\partial_\mu^p U_V^\nu) - m^* (\partial_\mu^p U_S^\nu)\right) \partial_x^\mu + \left(\Pi_\nu (\partial_\mu^x U_V^\nu) + m^* (\partial_\mu^x U_S^\nu)\right) \partial_p^\mu \right\} f(x,p) = I_{coll}$$

$$I_{coll} \equiv \sum_{2,3,4} \int d2\ d3\ d4\ \ [G^+G]_{1+2\to 3+4}\ \delta^4(\Pi + \Pi_2 - \Pi_3 - \Pi_4)$$

$$d2 \equiv \frac{d^3 p_2}{E_2}$$

$$\times \{ f(x,p_3)\ f(x,p_4)\ (1 - f(x,p))\ (1 - f(x,p_2))$$

Gain term $3+4 \rightarrow 1+2$

$$- f(x,p)\ f(x,p_2)\ (1 - f(x,p_3))\ (1 - f(x,p_4))\ \}$$

Loss term $1+2 \rightarrow 3+4$

where $\partial_\mu^x \equiv (\partial_t, \vec{\nabla}_r)$

$m^*(x,p) = m + U_s(x,p)$ - effective mass

$\Pi_\mu(x,p) = p_\mu - U_\mu(x,p)$ - effective momentum

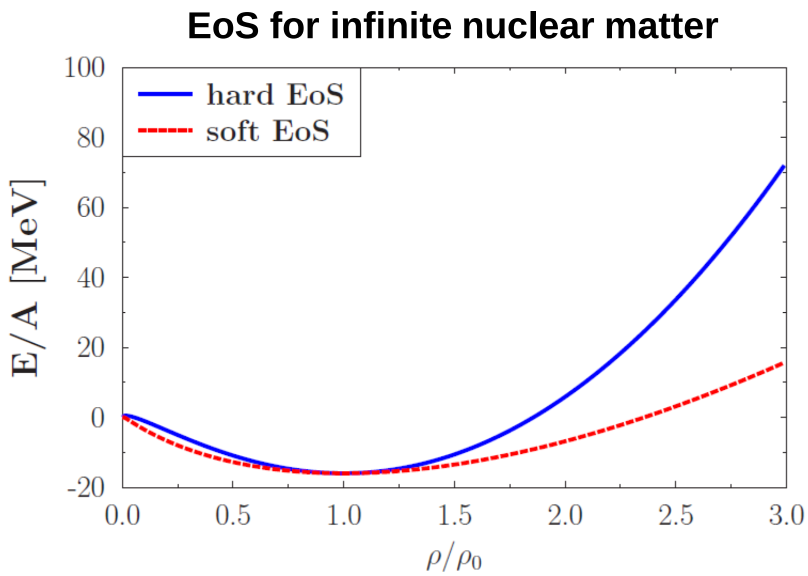
$U_s(x,p)$, $U_\mu(x,p)$ are scalar and vector part of particle self-energies

$\delta(\Pi_\mu \Pi^\mu - m^{*2})$ – mass-shell constraint

W. Ehehalt, W. Cassing, Nucl. Phys. A 602 (1996) 449

Nuclear equation of state (EoS)

EoS: $$\frac{E_B}{A} = \frac{\varepsilon}{\rho_N} - M_0$$



- **Energy density:** $$\mathcal{E} = \langle : T^{00} : \rangle,$$

$$P = \langle : T^{ii} : \rangle = \frac{1}{3}\sum_{i=1}^{3}\langle : T^{ii} : \rangle$$

- **Nucleon density:**

$$\rho_N(x) = 2\langle : \bar{\Psi}\gamma^0\Psi : \rangle = d\int \frac{d^3\Pi}{(2\pi)^3}\left(f_p^*(x,\Pi) - f_a^*(x,\Pi)\right)$$

$f^*(x,\Pi)$ – Fermi distribution with effective masses and energies

Momentum dependence of the Schrödinger-equivalent potential:

$$U_{\rm SEP} = U_S(\rho_0,P) + U_0(\rho_0,P) + \frac{1}{2M_N}\left(U_S(\rho_0,P)^2 - U_0(\rho_0,P)^2\right) + U_0(\rho_0,P)\,\frac{\sqrt{P^2+M_N^2}-M_N}{M_N}$$

Compression modulus:

$$K = -V\frac{dP}{dV} = 9\rho^2\frac{\partial^2[E/A(\rho)]}{(\partial\rho)^2}\bigg|_{\rho=\rho_0}$$

Soft EoS: K=200 MeV
Hard EoS: K=380 MeV

Useful literature

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W. Cassing and E.L. Bratkovskaya, ‘Hadronic and electromagnetic probes of hot and dense nuclear matter’, Phys. Reports 308 (1999) 65-233.
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W. Cassing, `Transport Theories for Strongly-Interacting Systems’, Springer Nature: Lecture Notes in Physics 989, 2021;
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