I.N. Mishustin

# I. Lecture notes on Hydrodynamic Model

To derive hydro equations start from kinetic Bolzmann equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f + \underset{\text{ext. force}}{\vec{F}}\,\frac{\partial f}{\partial \vec{p}} = I_{coll}(\underset{\text{1-body distrib. function}}{f}) \quad (1)$$

Here the collision integral is defined as

$$I_{coll}(f) = \int w'(f' f_1' - f f_1)\, d\Gamma_1\, d\Gamma'\, d\Gamma_1' ,$$

where $w'(\Gamma, \Gamma_1; \Gamma', \Gamma_1')$ is the invariant amplitude of 2-body scattering $(p + p_1 \to p' + p_1')$ satisfying the detailed balance condition

$$\int w(\Gamma', \Gamma_1'; \Gamma, \Gamma_1)\, d\Gamma' d\Gamma_1' = \int w'(\Gamma, \Gamma_1; \Gamma', \Gamma_1')\, d\Gamma' d\Gamma_1' \quad (2)$$

In the case of $2 \leftrightarrow 2$ process with structureless particles ($\Gamma \to \vec{p}$, $d\Gamma = d^3p$, $E = \sqrt{m^2 + p^2}$)

$$w\, d^3p'\, d^3p_1' = v_{rel}\, d\sigma ,$$

$$I_{coll}(f) = \int v_{rel}(f' f_1' - f f_1)\, d\sigma\, d^3p_1 , \quad (3)$$

where $v_{rel} = |\vec{v} - \vec{v}_1|$ is the relative velocity of colliding particles and differential cross section is

(4) $\frac{d\sigma_\ell}{d\Omega} = |F_\ell(\varphi, \theta)|^2$, $F_\ell$ is scattering amplitude with ang. mom. $\ell$, $d\Omega = \sin\theta\, d\theta\, d\varphi$

It is easy to prove that (Landau & Lifshitz, Physical kinetics)

$$\int I_{coll}(f)\,d\Gamma = 0, \quad \int I_{coll}(f)\,E\,d\Gamma = 0, \quad \int I_{coll}(f)\,\vec{p}\,d\Gamma = 0 \qquad (5a,b,c)$$

particle number cons. Energy cons. momentum cons.

Now hydrodynamic equations can be obtained by integrating eq. (1) with 1, E and $\vec{p}$ factors:

$$\frac{\partial N}{\partial t} + \nabla(N\vec{v}) = 0 \quad , \text{ baryon current cons.} \qquad (6a)$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla(\mathcal{E}\vec{v}) = -\nabla(P\vec{v}) \quad \text{energy cons.} \qquad (6b)$$

$$\frac{\partial \vec{M}}{\partial t} + \nabla\cdot(\vec{M}\cdot\vec{v}) = -\nabla P, \quad \text{momentum cons.} \qquad (6c)$$

Here quantities $N$, $\mathcal{E}$ and $\vec{M}$ are defined as

$$N = \gamma n \quad , \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1}, \quad N = \int f\,d\Gamma \qquad (7a)$$

$$\vec{M} = \gamma^2(\varepsilon + P)\vec{v} \quad , \quad \vec{v} \text{ is collective velocity} \qquad (7b)$$

$$\mathcal{E} = \gamma^2(\varepsilon + P v^2) \quad \text{of the fluid} \qquad (7c)$$

$n$, $\varepsilon$ are baryon density and energy density in the rest frame of the fluid, $P$ is pressure

Equations (6a,b,c) can be derived directly from the general equation expressing energy-momentum cons.

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0 \qquad \text{(derivation see on p. 2a)} \qquad (8)$$

where

$$(9) \qquad T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu - P g^{\mu\nu}, \quad u^\mu = \gamma(1, v_x, v_y, v_z)$$

4-velocity

$$g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

Relativistic hydro equations in 3-vector form

Generally hydro equations can be written as

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0 \quad , \quad \mu, \nu = 0, 1, 2, 3; \quad i, j = 1, 2, 3$$

$$T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu - p g^{\mu\nu} \quad , \quad u^\mu = \gamma(1, \vec{v})$$

energy-momentum tensor 4-velocity, $\gamma = \frac{1}{\sqrt{1-v^2}}$

$$T^{00} = (\varepsilon + p)\gamma^2 - p = \gamma^2(\varepsilon + p v^2) \equiv \mathcal{E}$$

$$T^{ij} = (\varepsilon + p)\gamma^2 v_i v_j + p\delta_{ij} \equiv M_i v_j + p\delta_{ij} = v_i M_j + p\delta_{ij}$$

$$T^{0i} = (\varepsilon + p)\gamma^2 v_i \equiv M_i$$

$$\mu = 0: \quad \frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0i}}{\partial x^i} = \frac{\partial \mathcal{E}}{\partial t} + \frac{\partial M_i}{\partial x^i} \qquad (*)$$

$$\mu = j: \quad \frac{\partial T^{j0}}{\partial t} + \frac{\partial T^{ji}}{\partial x^i} = \frac{\partial M_j}{\partial t} + \frac{\partial (M_i v_j)}{\partial x^i} + \delta_{ij}\frac{\partial p}{\partial x^i} \qquad (**)$$

Introduce 3-vectors: $\vec{v} = (v_1, v_2, v_3)$, $\vec{M} = (M_1, M_2, M_3)$

$$\vec{M} = (\varepsilon + p)\gamma^2 \vec{v} = [(\varepsilon + p)\gamma^2 - p + p]\vec{v} = \mathcal{E}\vec{v} + p\vec{v}$$

$$(*) \to \frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \vec{M} = \frac{\partial \mathcal{E}}{\partial t} + \nabla(\mathcal{E}\vec{v}) + \nabla(p\vec{v}) = 0$$

$$(**) \to \frac{\partial \vec{M}}{\partial t} + \nabla(\vec{M} \cdot \vec{v}) + \nabla p = 0$$

since $M_i v_j = v_i M_j$

$$= \vec{v}(\nabla \vec{M}) + (\vec{M} \cdot \nabla)\vec{v}$$

The necessary condition for applicability of hydro model is

$$\lambda = \frac{1}{n\sigma_{tot}} \ll L, \qquad (10)$$

mean free path characteristic size of a system

For hadrons $\sigma_{tot} \simeq d^2$, where $d$ is the size (diameter) of a hadron ($\sigma_{NN} \simeq 40\,mb = 4\,fm^2$, $\sigma_{\pi\pi} \simeq 15\,mb \simeq 1.5\,fm^2$)
Introducing a mean distance between the particles $\bar{r} = n^{-1/3}$, we can write

$$\lambda = \frac{\bar{r}^3}{d^2} = \bar{r}\left(\frac{\bar{r}}{d}\right)^2 \qquad (11)$$

In nuclear collisions at intermediate energies ($\frac{E_{cm}}{A} \sim 1\,GeV$), $\lambda \sim \bar{r} \sim d$ and eq. (10) is satisfied, i.e.

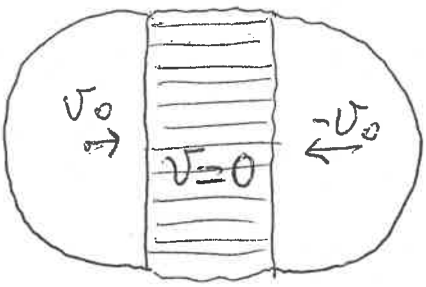
$n_0 = 0.15\,fm^{-3}$, $\lambda_{NN} \simeq \frac{1}{0.15\,fm^{-3}\cdot 4\,fm^2} \simeq 1.6\,fm \ll R \simeq 6\,fm$

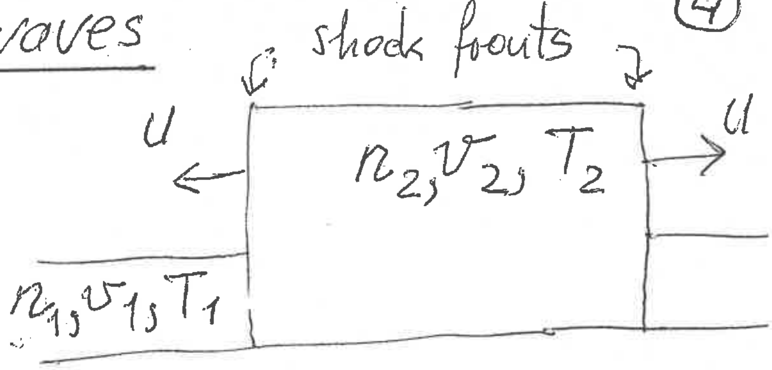
However at very high energies ($\frac{E_{cm}}{A} = \frac{\sqrt{s}}{2} \gtrsim 10\,GeV$) the differential NN cross section is forward-backward peaked and baryon stopping becomes unefficient. The momentum transfer is better characterized by the transport cross section

$$\sigma_{tr} = \frac{1}{\sigma_{NN}} \int_0^{\pi/2} (1-\cos\theta_{cm}) \frac{d\sigma}{d\Omega} d\Omega = \frac{\langle \Delta p_{\parallel} \rangle}{p_{\parallel}} \qquad (12)$$

At high energies $\sigma_{tr}$ becomes small and correspondingly, the mean free path $\lambda_{tr} = (n\sigma_{tr})^{-1}$, large. One fluid model is not applicable under such conditions and one must consider multi-fluid models.
A 3-fluid model contains 1) projectile fluid, 2) target fluid and 3) fireball fluid.
Recent development in this direction see in nucl-th/0503088

# Plane shock waves




In central collisions of heavy nuclei a compressed zone is produced when the collision is supersonic. The characteristics of compressed matter can be found by applying the conservation laws across the shock front. Most simple way to write them is in the rest frame of the front. Then, assuming the stationary situation ($\frac{\partial}{\partial t}=0$), after integrating eqs. (6a,b,c) over a small interval $\Delta z$, one can obtain the following equations connecting quantities from left and right sides of the front:

$$\left.\begin{aligned} N_1 v_1 &= N_2 v_2 \equiv j \\ \mathcal{E}_1 v_1 + P_1 v_1 &= \mathcal{E}_2 v_2 + P_2 v_2 \\ M_1 v_1 + P_1 &= M_2 v_2 + P_2 \end{aligned}\right\} \underset{(7a,b,c)}{\overset{\text{use}}{\Longrightarrow}} \quad \begin{aligned} n_1\gamma_1 v_1 &= n_2\gamma_2 v_2 \equiv j && (13a) \\ (\varepsilon_1+P_1)\gamma_1^2 v_1 &= (\varepsilon_2+P_2)\gamma_2^2 v_2 && (13b) \\ (\varepsilon_1+P_1)\gamma_1^2 v_1^2 + P_1 &= (\varepsilon_2+P_2)\gamma_2^2 v_2^2 + P_2 && (13c) \end{aligned}$$

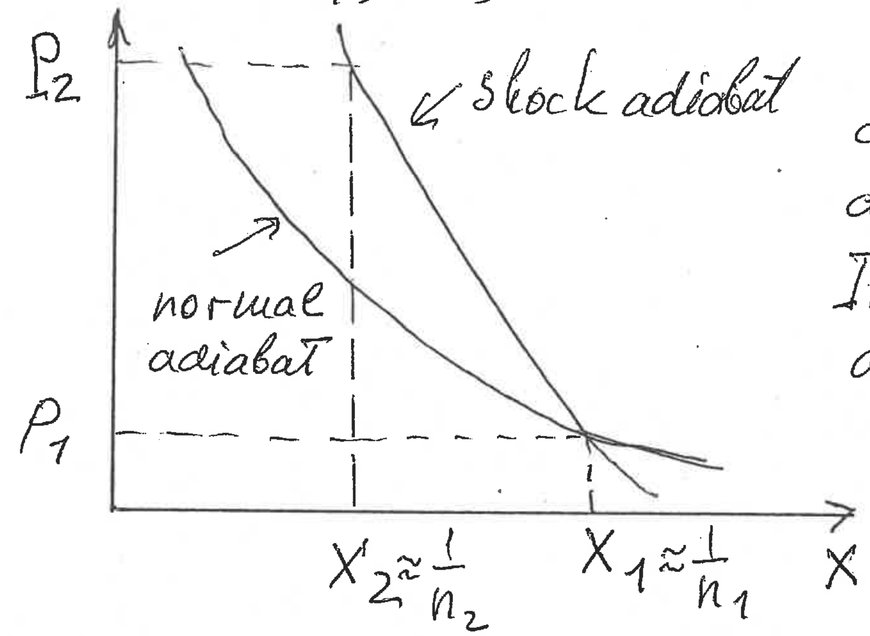
Now we introduce notation $X = \frac{\varepsilon+P}{n^2}$, and eliminate velocities from these equations:

$$(13c) \Rightarrow j^2 = -\frac{P_2-P_1}{X_2-X_1}, \quad (13a) \Rightarrow \frac{1}{(\gamma v)^2} = \frac{1}{\gamma^2-1} = \frac{n^2}{j^2}.$$

Finally, from (13b) we get so called Rankine–Hugoniot–Taub adiabat

$$(X_2 n_2)^2 - (X_1 n_1)^2 - (P_2-P_1)(X_1+X_2) = 0 \qquad (14)$$

For a given equation of state $P(\mathcal{E})$ and initial state at $n_1, P_1$, the RHT adiabat gives $n_2, P_2$




In a shock wave the compression is always accompagned with heating. In contrast to normal adiabat, corresponding to $S = const$, in shock adiabat the entropy grows. Accordingly, the pressure is larger than in normal adiabat.

The shock front velocity $u$ in the frame where shocked matter is at rest can be calculated as relative velocity

$$v_{12} = \frac{v_1 - v_2}{1 - v_1 \cdot v_2} = \frac{(P_2 - P_1)(\mathcal{E}_2 - \mathcal{E}_1)}{(P_1 + \mathcal{E}_2)(P_2 + \mathcal{E}_1)} \qquad (15)$$

Accordingly, the c.m. Lorentz-factor is

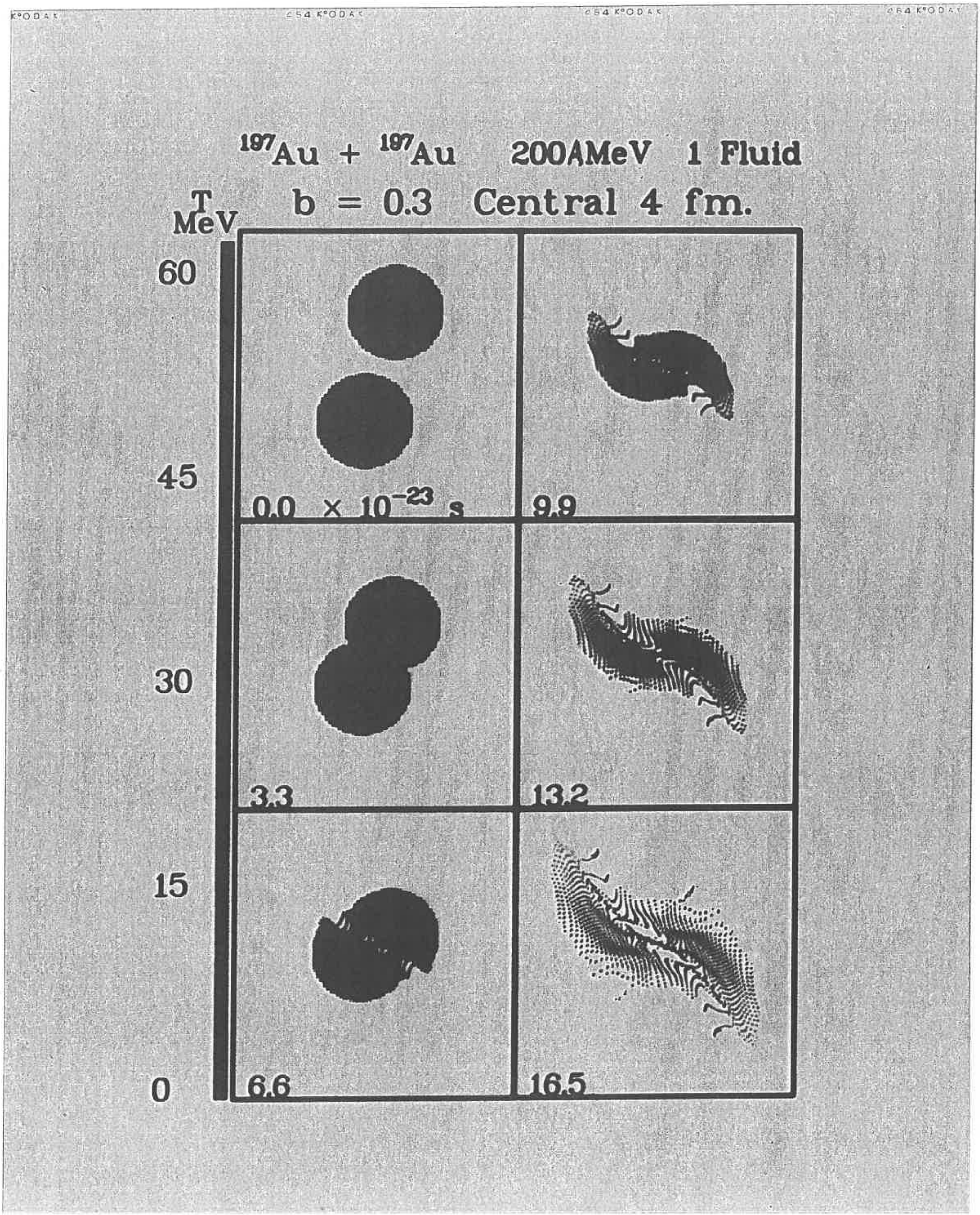
$$\gamma_{CM}^2 = \left(\frac{1}{\sqrt{1 - \frac{v_{CM}^2}{c^2}}}\right)^2 = \frac{(P_1 + \mathcal{E}_2)(P_2 + \mathcal{E}_1)}{(P_1 + \mathcal{E}_1)(P_2 + \mathcal{E}_2)} \qquad (16)$$

This $\gamma_{CM}$ gives the beam energy $E_{Lab}^{kin}/A = 2m_N(\gamma_{CM}^2 - 1)$ which is needed to produce shocked matter at $(n_2, P_2)$. For the shocked matter described by equation of state of ideal gas $P = \frac{\mathcal{E}}{3}$ the compression ratio at high beam energies is given by

$$\frac{n_2}{n_0} = 4\gamma_{CM} - 3/\gamma_{CM}, \quad \frac{\mathcal{E}_2}{\mathcal{E}_0} = 4\gamma_{CM}^2 - 3 \qquad (17)$$

It is assumed here that $P_1 = 0$, $\mathcal{E}_1 = m_N n_0$ at $n = n_0 = 0.15 fm$

⑥ Early hydro calculations from 70's
1-fluid model by Nix, Strottman



1-fluid hydro by J. Brachmann et al.

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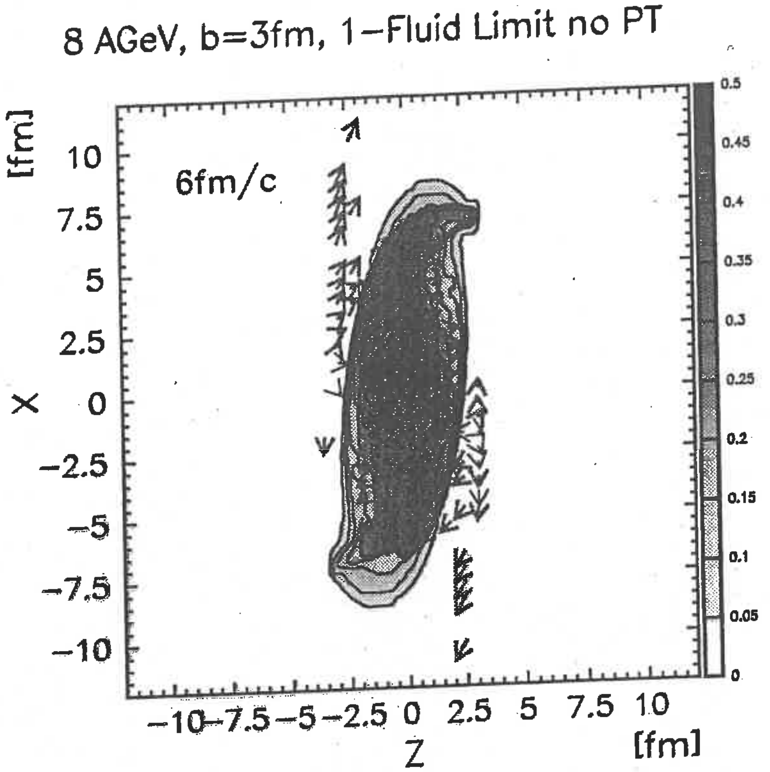
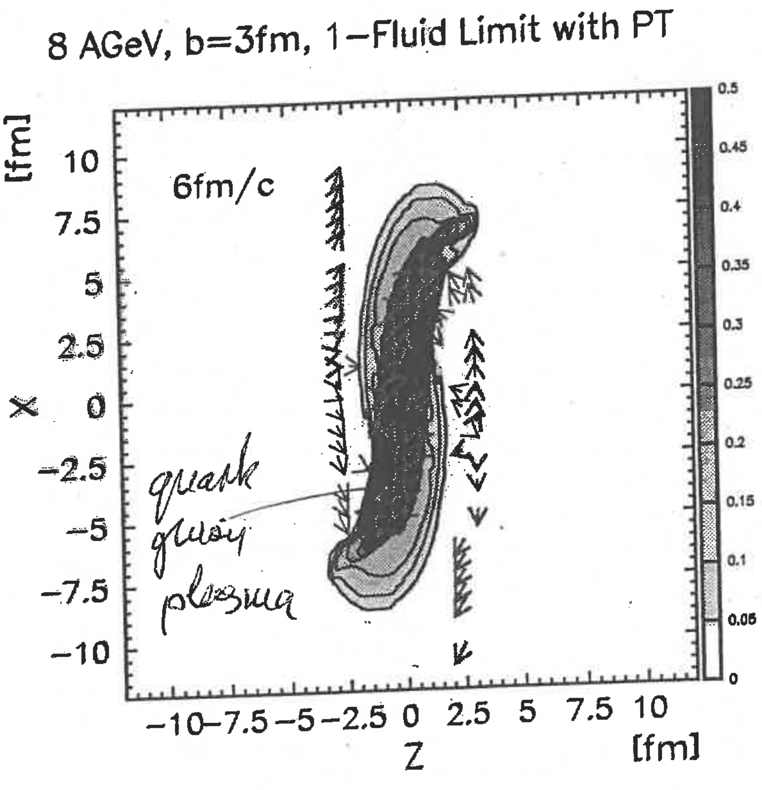


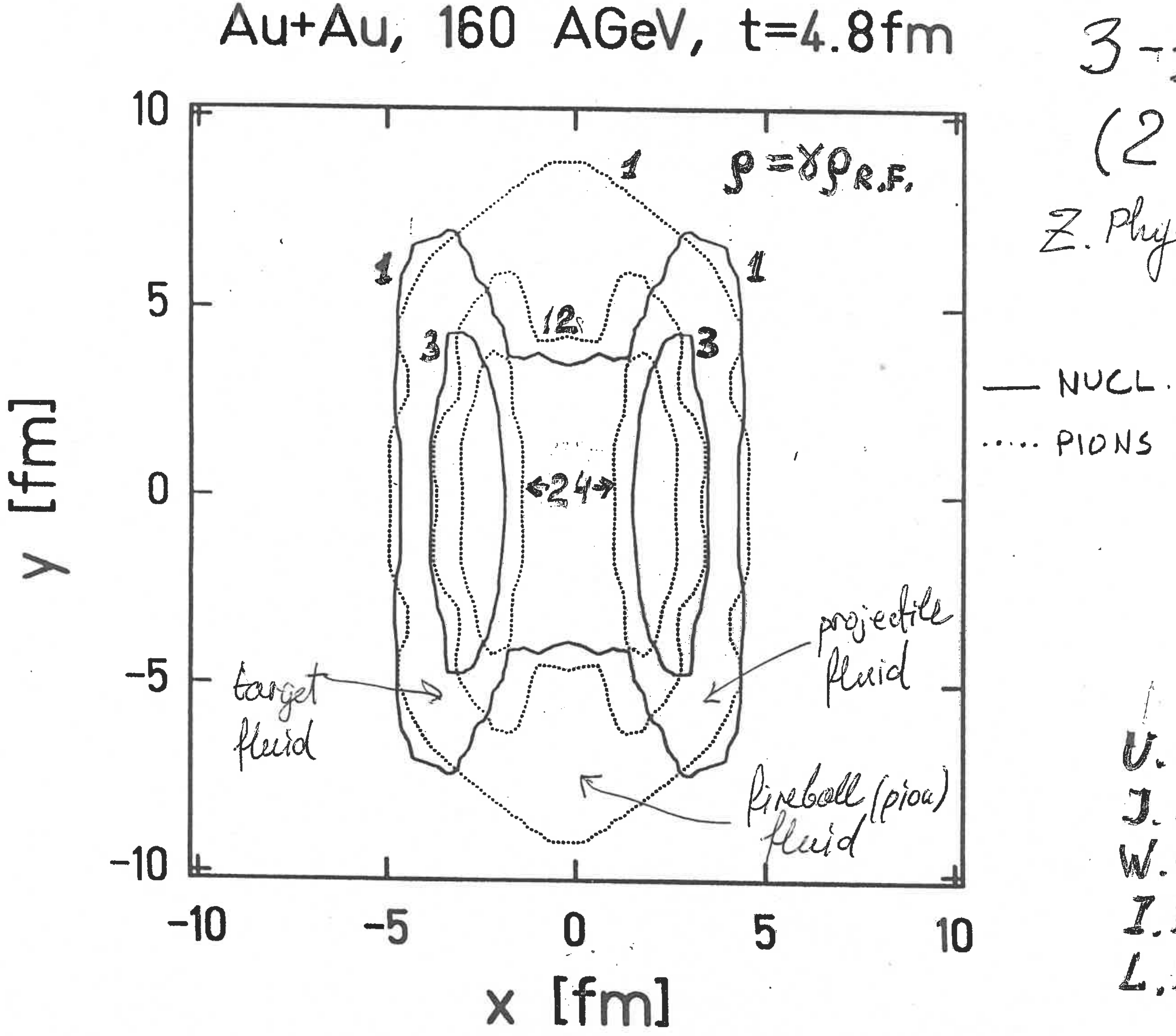

no phase transition




with phase transition

Abbildung 2.13: Baryonendichte in der Reaktionsebene am Ende der Kompressionsphase ($t_{CM} = 6$ fm/c). Antifluß - dünne Pfeile. Normaler Fluß - dicke Pfeile. Dichten in fm$^{-3}$. Oben: Rein hadronische Zustandsgleichung. Unten: Mit Phasenübergang in das QGP.

3-fluid model
(2B+1π)
Z. Phys. A346 (1993) 209




U. Katscher
J. Maruhn
W. Greiner
I. Mishustin
L. Satarov

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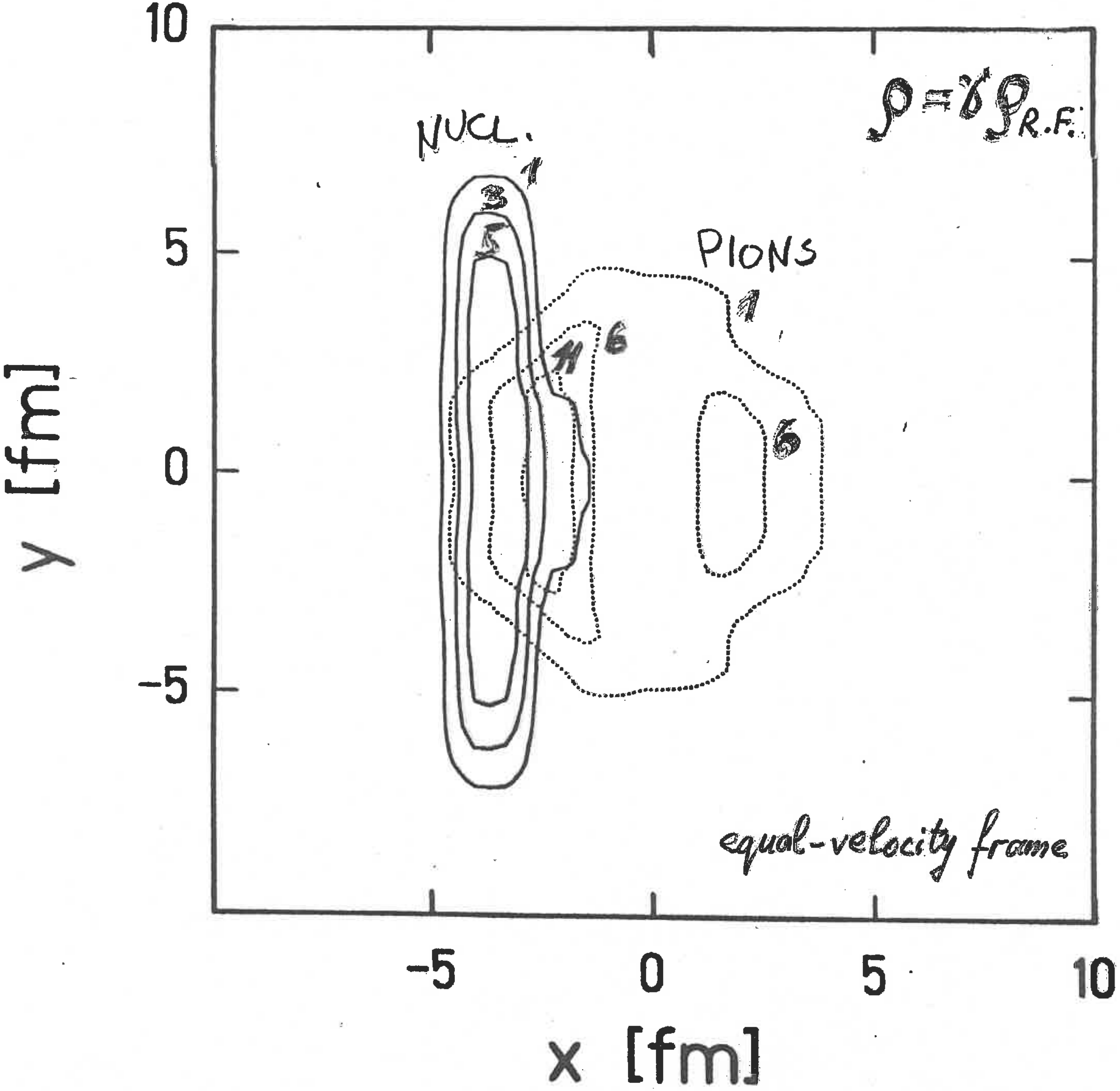
O+Au, 200 AGeV, t=4.2fm

3-fluid model
(2 B + 1 π)




U. Katscher
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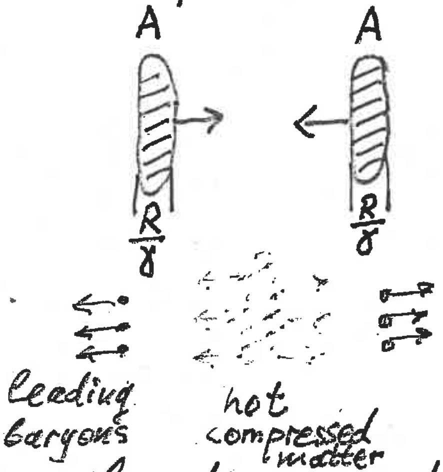
I. Mishustin
L. Satarov

⑩

# Landau picture

Izvestiya AN SSSR
17 (1953) 51

1. Complete stopping of matter in Lorentz contracted volume




Energy density (at $t=0$)

$$\mathcal{E} = \frac{2 A m_N \gamma}{(V_0/\gamma)} = 2\gamma^2 \mathcal{E}_{n.m.} (K(\gamma))$$

$$\mathcal{E}_{n.m.} = m_N \cdot \rho_0 = 0.15 \frac{GeV}{fm^3}$$

leading particle effect is taken into account by introducing the „inelasticity" coefficient

$$K(E) = \frac{(E_{cm})_{deposited}}{(E_{cm})_{initial}} \approx \frac{1}{2} \text{ at high energy}$$

2. 1-dimentional expansion $\partial_\nu T_{\mu\nu} = 0$, $\partial_\mu J_\mu^{(B)} = 0$

Exact solution - Khalatnikov I.M. (ZhETF 27 (1954) 529)

Approximate solution - Landau L.D. (1953)

$$v_{\parallel} = \frac{x_{\parallel}}{t} \qquad T \sim (t^2 - x_{\parallel}^2)^{-\frac{c^2}{2}} \quad (P = c^2 \mathcal{E})$$

EOS

Very similar to Bjorken model!

3. Break-up into secondary particles

freeze-out criterion: $T_f \approx m_\pi$

Number of particles produced: $N \sim S$ (entropy)

# Observable characteristics:

1. Transverse momentum distributions

$$\frac{dN}{dp_\perp^2} \sim \exp\left(-\frac{m_\perp}{T_f}\right), \quad m_\perp = \sqrt{m^2+p_\perp^2}$$

prediction: $\langle p_\perp \rangle \approx T_f = \mathrm{const}$

2. Rapidity distributions – Gaussian-like shape

$$\frac{dN}{dy} \sim \frac{s^\alpha}{\sqrt{L(s)}} \exp\left[-L+\sqrt{L^2-y^2}\right] \approx \frac{s^\alpha}{\sqrt{L(s)}} e^{-\frac{y^2}{2L(s)}}$$

$$\alpha = \frac{1-c^2}{2(1+c^2)}, \quad L(s) = \frac{4c^2}{3(1-c^4)} \ln\frac{s}{4m_N^2}$$

(E. Shuryak, Phys. Rep. 115 (1984) 251)

SPS: $\sqrt{s} = 20\,\mathrm{GeV}$, $L(s) = 2.5$

RHIC: $\sqrt{s} = 200\,\mathrm{GeV}$, $L(s) = 4.7$

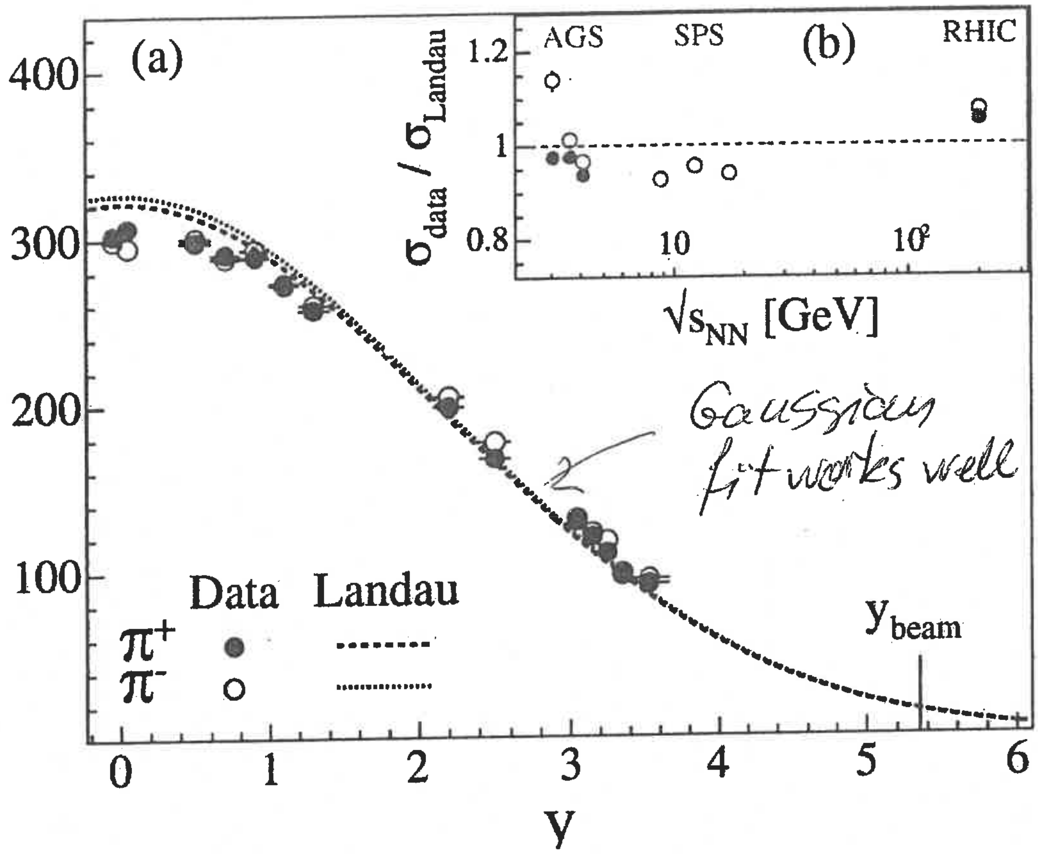
LHC: $\sqrt{s} = 5500\,\mathrm{GeV}$, $L(s) = 8.0$

for $c^2 = \frac{1}{3}$, $\alpha = \frac{1}{4}$, $L(s) = \frac{1}{2}\ln\frac{s}{4m_N^2}$

3. Secondary particle multiplicity ($N \sim S$)

$$N = \mathrm{const} \cdot A^{1/3} E_{lab}^{1/4} \quad (hA \text{ collisions})$$

(12) BRAHMS data from RHIC
Au+Au, $\sqrt{s}$ = 200 GeV




Gaussian fit works well

FIG. 4: Comparison $dN/dy(\pi)$ and Landau's prediction at $\sqrt{s_{NN}} = 200\,\mathrm{GeV}$ (a) and ratio $\sigma_{N(\pi)}/\sigma_{Landau}$ as a function of $\sqrt{s_{NN}}$ (b). Errors are statistical.

menta. Landau's full stopping assumption therefore does not hold at RHIC. The insert in Fig. 4 shows the ratio $\sigma_{data}/\sigma_{Landau}$ as a function of $\sqrt{s_{NN}}$. At all energies, the widths $\sigma_{data}$ come from Gaussian fits to the pion distributions. While the difference between theory and measurements is of the order of 10% at most from AGS to RHIC energies, it is worth noting that the overall systematic may not be trivial. The ratio at RHIC is $\sim 15\%$ higher than at SPS.

On the basis of Landau's hydrodynamic, Bjorken [2] proposed a scenario in which yields of produced particles would be boost–invariant within a region around mid–rapidity. In that approach, reactions are described

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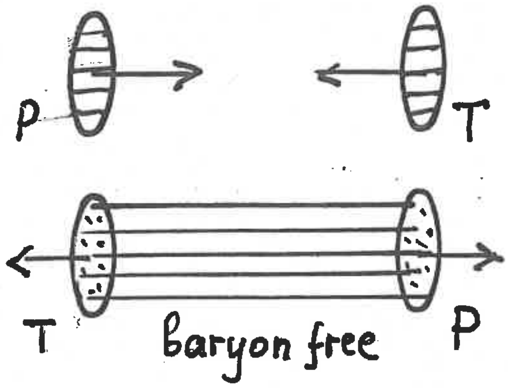
# Bjorken picture

Phys. Rev. D 27 (8: 140

Partial transparency of nuclei, finite formation time of secondary hadrons: $\tau_f = \tau_0 \gamma$, $\tau_0 \approx 1\ fm/c$




Scaling solution:

$$\varepsilon = \varepsilon(\tau),\ p = p(\tau),\ T = T(\tau)$$

$$\tau = \sqrt{t^2 - z^2},\ \text{proper time}$$

$$z_{form} = \tau_0 \sinh y$$

Fluid-dynamical equations: $\frac{\partial}{\partial x^\mu} T^{\mu\nu} = 0$

$$T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu - p g^{\mu\nu},\quad u^\mu = \frac{1}{\tau}(t, 0, 0, z) \equiv \frac{\tilde{x}^\mu}{\tau}$$

$$\frac{\partial \tau}{\partial x^\mu} = \frac{\partial \sqrt{t^2 - z^2}}{\partial x^\mu} = \frac{\tilde{x}^\mu}{\tau} = u^\mu,\quad \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial x^\mu} = u^\mu \frac{\partial}{\partial \tau}$$

$$\frac{\partial u^\mu}{\partial x^\nu} = \frac{1}{\tau}\frac{\partial \tilde{x}^\mu}{\partial x^\nu} - \frac{1}{\tau^3}\tilde{x}^\mu \tilde{x}_\nu = \frac{1}{\tau}(\tilde{\delta}^\mu_\nu - u^\mu u_\nu)$$

$$0 = \underbrace{\frac{\partial(\varepsilon + p)}{\partial x^\mu}}_{\partial(\varepsilon + p)/\partial \tau} u^\mu u^\nu + (\varepsilon + p)\left[\underbrace{\frac{\partial u^\mu}{\partial x^\mu}}_{= 1/\tau} u^\nu + \underbrace{u^\mu \frac{\partial u^\nu}{\partial x^\mu}}_{= 0}\right] - g^{\mu\nu}\frac{\partial p}{\partial x^\mu} = 0$$

Combining all together: $\boxed{\frac{\partial \varepsilon}{\partial \tau} + \frac{\varepsilon + p}{\tau} = 0}$

For ideal gas EOS $p = c_s^2 \varepsilon \Rightarrow \varepsilon(\tau) = \varepsilon(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{1 + c_s^2}$

Initial condition $\varepsilon(\tau_0) = \frac{1}{S}\frac{dE}{dz} = \frac{1}{S}\underbrace{\frac{dE}{dN}}_{m_\perp \cosh y}\frac{dN}{dy}\underbrace{\frac{dy}{dz}}_{(\tau_0 \cosh y)^{-1}}$

$$\varepsilon(\tau_0) = \frac{m_\perp}{\tau_0 \pi\, ...}\ \frac{dN}{dy}$$

# Fluid-dynamical evolution of plasma



For boost invariant initial conditions, $\varepsilon = \varepsilon(\tau)$, $p = p(\tau)$
valid at $\eta \simeq 0$ only

$$\frac{d\varepsilon}{d\tau} + \frac{\varepsilon + p}{\tau} = 0$$ Bjorken, 1983

Energy density ($\varepsilon \sim T^4$)

$$\varepsilon(\tau) = \varepsilon(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{1+c_s^2}$$

Entropy density ($s \sim T^3$)

$$s(\tau) = s(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{\frac{3}{4}(1+c_s^2)}$$

These formulae are valid until freeze-out time $\tau_f$

$\tau_0 < \tau < \tau_f$

Simulations show that

$$\frac{\tau_0}{\tau_f} \sim \frac{1}{8}$$

Transverse energy at $\eta = 0$

$$\frac{dE_T}{d\eta} = \pi R^2 \varepsilon(\tau)\tau = \underbrace{\pi R^2 \varepsilon(\tau_0)\tau_0}_{\text{initial value}} \underbrace{\left(\frac{\tau_0}{\tau_f}\right)^{1/3}}_{\text{reduction factor} \sim \frac{1}{2}} \quad \text{for } c_s^2 = \frac{1}{3}$$

Entropy at $\eta = 0$

$$\frac{dS}{d\eta} = \pi R^2 s(\tau)\tau = \pi R^2 s(\tau_0)\tau_0 \text{, independent of } \tau$$

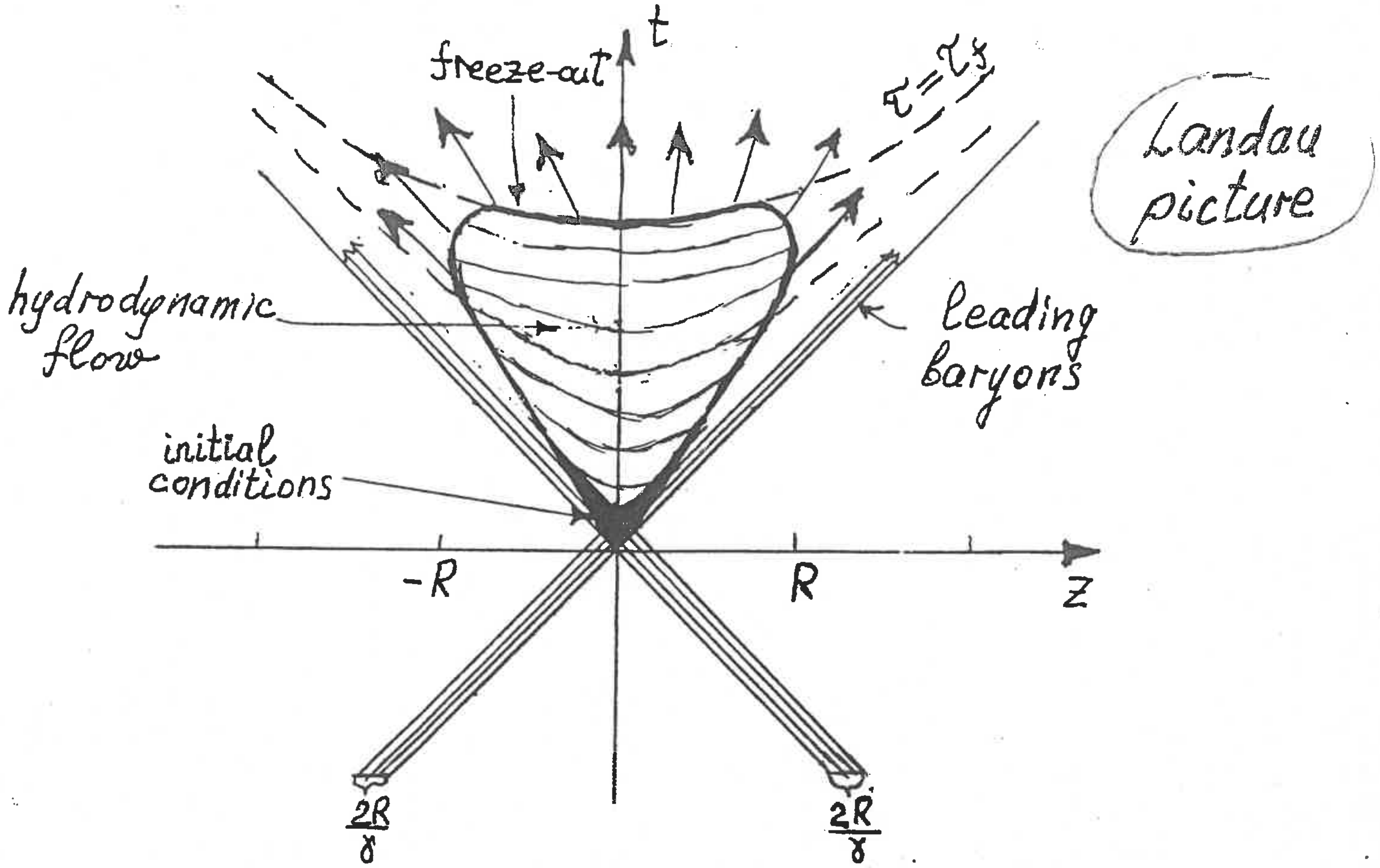
Observed hadron multiplicity $\sim S$, $N_\pi = \frac{S}{3.6}$

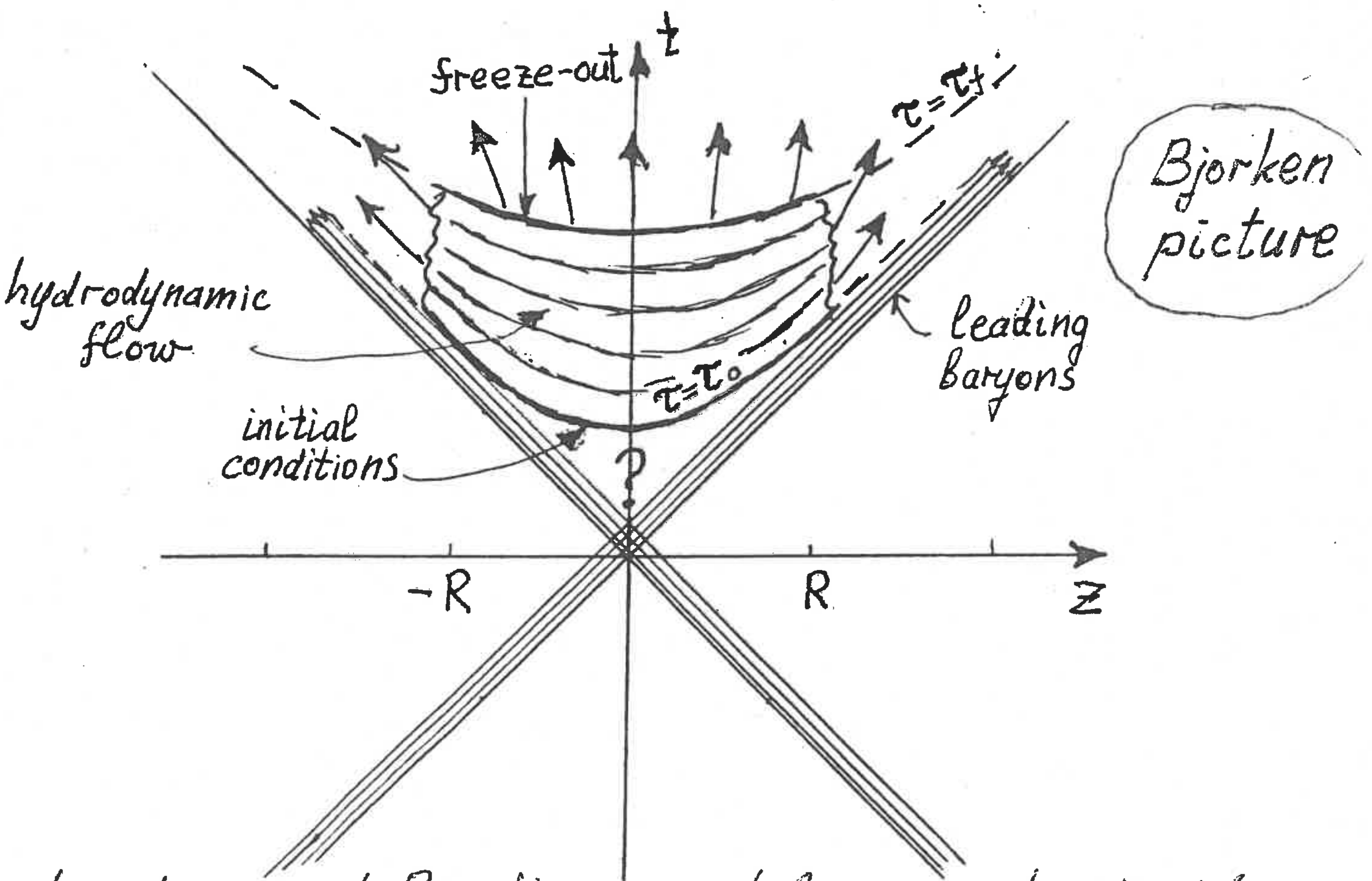
To explain RHIC data $\left(\left.\frac{dE_T}{d\eta}\right|_{\eta=0} = 600\,\text{GeV},\ \left.\frac{dN_{ch}}{dy}\right|_{y=0} = 600\right)$

one needs the initial energy of about

$\varepsilon(\tau_0) \simeq (8 \div 12)\ \frac{\text{GeV}}{\text{fm}^3}$, at $\tau_0 \simeq 1$ fm/c

# Space-time evolution of collision process (15)




Landau and Bjorken model predict similar behaviour at later time at midrapidity ($y \simeq 0$)

# Freeze-out conditions



In macroscopic models a sharp freeze-out is assumed at some density or temperature

$t<t_f$ : equilibrium $\rightarrow$ $t>t_f$ : free streaming

Often used freeze-out criteria:

$$\boxed{\bar{\rho}_B = (\tfrac{1}{3} \div \tfrac{1}{2})\rho_0} \quad \text{or} \quad \boxed{\bar{T} = m_\pi = 140\,\text{MeV}}$$

baryon-rich matter      baryon-free matter

Only very few calculations use more realistic differential criteria

$$\boxed{\lambda_i = \frac{1}{\sum_j \rho_j \sigma_{ij}} \approx L} \quad \text{or} \quad \boxed{\nu_i = \sum_j \rho_j \langle v_{ij}\, \sigma_{ij} \rangle \approx \frac{\dot{\rho}_B}{\rho_B} = \frac{1}{\tau_{ex}}}$$

(I. Mishustin, L. Satarov, 1983; U. Heinz, 1990; U. Ornik, 1994)

Microscopic models open the possibility to clear up this point by studying the space-time distribution of Last Collision Points (LCP)

Sharp freeze-out would correspond to $\delta(t-t_f)$

1) LCP distributions are rather broad even for Au+Au

2) there is a clear separation of elastic and inelastic LCP

(H. Babic, P. Gerber et al., 1992)

Freeze-out isotherms (F. Cooper, G. Frye, E. Schonberg PRD11(1975)192

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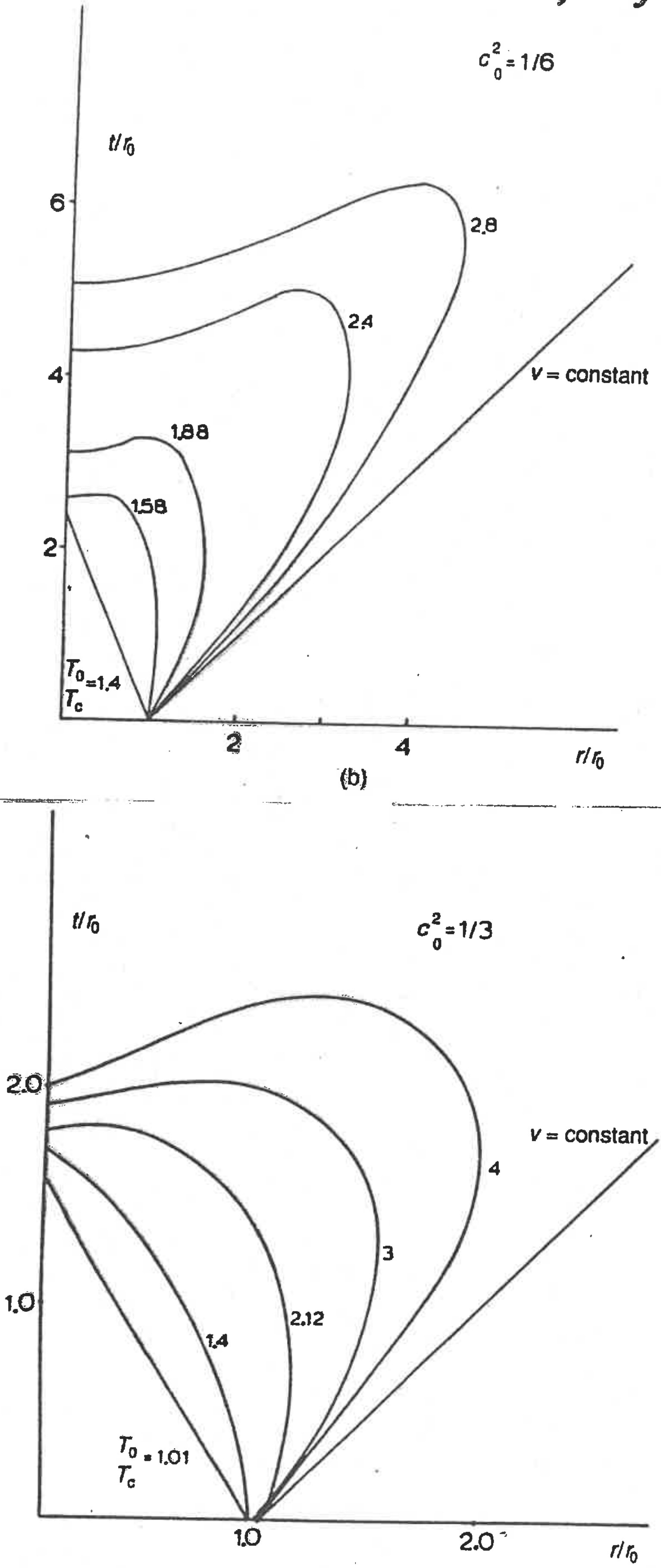


**Figure 6.12** The isotherms of the flow at the break-up when the temperature reaches $T_{bu}$(= $T_c$ in the figure). The ratio of the initial temperature, $T_0$, to the final temperature, $T_{bu}$, is the parameter of the contours. The solution depends on the sound speed, $c_0$, and on the initial thickness of the disk, $r_0$. As the initial temperature increases the solution at break-up approaches the Bjorken model solution, particularly for a soft EOS like $c_0^2 = 1/6$. Reproduced with permission from [45]

# III. Non-equilibrium phase transitions in relativistic heavy-ion collisions

## Introduction

- Strong collective expansion is observed in central heavy-ion collisions
  $v_\perp \sim 0.5c$ , $v_\parallel = \tanh \eta_{max} \approx c$
- Not sufficient time for equilibrating matter, especially at a 1-st order phase transition (due to a long barrier penetration time)

Non-equilibrium phenomena must be important

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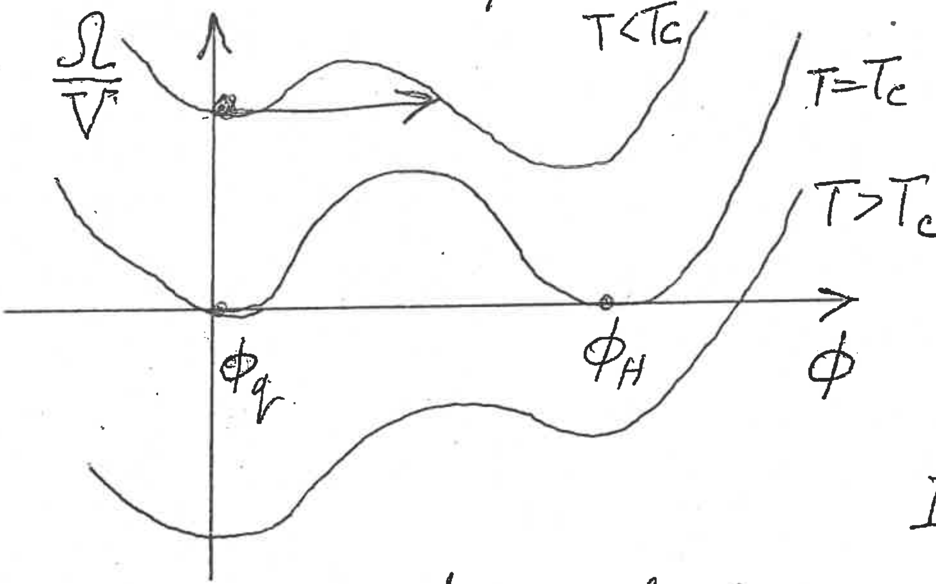
## Dynamics of a 1-st order phase transition



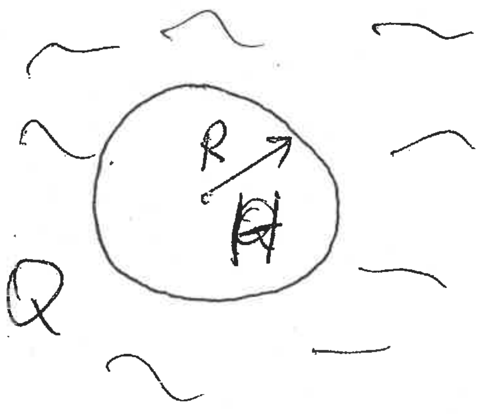
It is convenient to describe a 1-st order p.t. in terms of thermodynamic potential $\Omega = -pV$. Most generally it can be written in the form

$$\frac{\Omega}{V} = \frac{\Omega_0}{V} + A\phi + B\phi^2 + C\phi^3 + D\phi^4 + \frac{1}{2}(\nabla\phi)^2,$$

where $\phi$ is an order parameter field and coefficients $A, B, C, D$ as well as $\Omega_0$ are functions of $T, \mu$. For a 1-st order phase transition $\Omega(\phi)$ behaves as




Each minimum corresponds to a separate phase. Lower minimum corresponds to a more stable phase. In other words, more stable phase has a higher pressure. At critical temperature or/and chemical potential two minima have equal hight ($P_q = P_H$). A 1-st order phase transition proceeds through the nucleation when a bubble (nucleus) of a new phase is created in an old phase




$$\Delta\Omega = -(P_H - P_Q)\frac{4\pi}{3}R^3 + 6\,4\pi R^2$$

The bubble can grow only when

$$\frac{\partial\Delta\Omega}{\partial R} \le 0 \Rightarrow R > R_c = \frac{2\sigma}{P_H - P_Q}$$

Since two phases are separated by a potential barrier, the creation of a bubble requires the barrier penetration or thermal activation process. Therefore, it needs some time, i.e. the matter evolution should be very slow to follow the equilibrium between two phases. If the matter expands very fast, like in relativistic heavy-ion collisions, the phase equilibrium may not be reached. In this case the matter will continue expansion in a metastable phase below the critical point (super-cooling). Let us study this process on the basis of Bjorken's hydrodynamic model.

We take equation of state in the form ($\mu=0$):

1) $T<T_c$, massless pion gas, $P_H=\frac{\varepsilon_H}{3}$, $\varepsilon_H=\nu_\pi a T^4$, $s_H=\frac{4}{3}\nu_\pi a T^3$
where $\nu_\pi=3$ is the spin-isospin degeneracy factor, $a=\frac{\pi^2}{30}$

2) $T>T_c$, quark-gluon plasma described by the MIT bag model

$$P_Q=\frac{\varepsilon_{kin}}{3}-B,\ \varepsilon_Q=\varepsilon_{kin}+B,\ \varepsilon_{kin}=\nu_q a T^4,\ s_Q=\frac{4}{3}\nu_q a T^3,$$

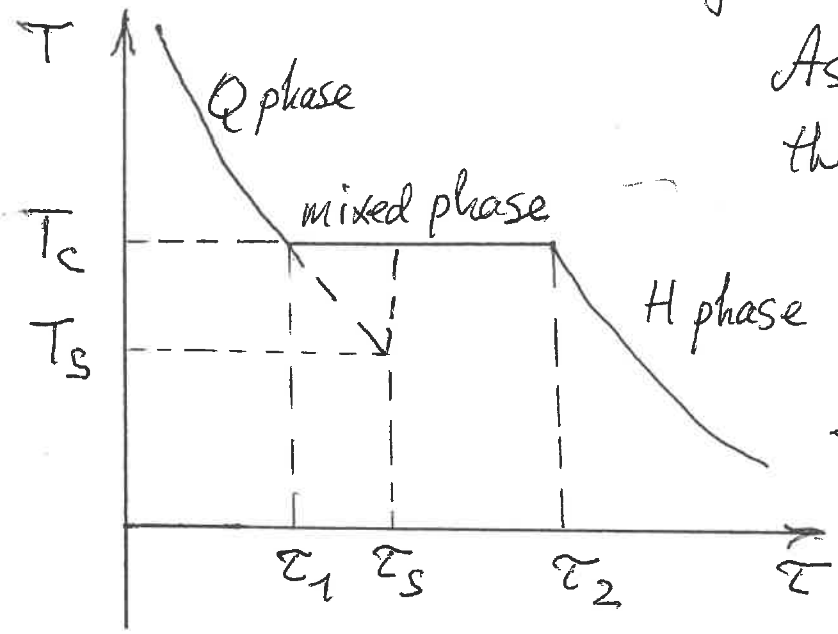
where
$$\nu_q = 2(\underset{\substack{\uparrow\\ \text{number}\\ \text{of colors}}}{N_c^2}-1)+\frac{7}{8}\,\underset{\substack{\uparrow\\ \text{spin}}}{2}\times\underset{\substack{\uparrow\\ \text{color}}}{3}\times\underset{\substack{\uparrow\\ q\bar{q}}}{2}\times N_f=\begin{cases}37 & N_f=2\\ 47.5 & N_f=3\end{cases}$$

This model predicts a 1-st order deconfinement transition at critical temperature

$$P_Q=P_H \Rightarrow T_c=\left(\frac{3B}{(\nu_Q-\nu_H)a}\right)^{1/4}\simeq 0.72\,B^{1/4}$$

To have $T_c\simeq 170$ MeV (as Lattice calculations give) one needs $B^{1/4}=236$ MeV.

We describe expansion by the Bjorken's hydro model




As was shown earlier the Bjorken model gives

$$\frac{d\varepsilon}{d\tau} + \frac{\varepsilon + P}{\tau} = 0 \quad (*)$$

In addition one should write the entropy conserv

$$\partial(s u^{\mu}) = 0$$

$$s(\tau) = s(\tau_0)\frac{\tau_0}{\tau} \Leftarrow \frac{d(s\tau)}{d\tau} = 0$$

In the case of perfect thermodynamic equilibrium the system expands in Q phase until $\tau = \tau_1$, then follows mixed phase ($\tau_1 < \tau < \tau_2$) and finally expands in H phase. In each of pure phases the temperature changes according eq.(*):

$$P = \frac{\varepsilon}{3}, \quad \frac{d\varepsilon}{d\tau} + \frac{4}{3}\frac{\varepsilon}{\tau} = 0 \Rightarrow \varepsilon(\tau) = \varepsilon(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{4/3} \Rightarrow T(\tau) = T(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{1/3}$$

In mixed phase the temperature is constant, $T = T_c$, but changes the volume fraction of the Q phase $f(\tau)$:

$$s(\tau) = s(\tau_0)\frac{\tau_0}{\tau} = s_Q(T_c) f + s_H(T_c)(1 - f)$$

This gives $f(\tau) = \frac{1}{\xi - 1}\left(\xi \frac{\tau_1}{\tau} - 1\right)$, where $\xi = \frac{\nu_Q}{\nu_\pi}$

The energy density in the mixed phase changes as

$$\varepsilon(f, T = T_c) = \varepsilon_Q(T_c) f(\tau) + \varepsilon_H(T_c)(1 - f(\tau))$$

Times $\tau_1$ and $\tau_2$ are defined by the conditions:

$f(\tau_1) = 1$, $f(\tau_2) = 0$, that gives $\frac{\tau_2}{\tau_1} = \xi$

In the case when the phase transition does not follow the equilibrium, the matter continues its expansion in Q phase to temperatures $T_S < T_C$. Then the barrier between two phases becomes smaller (it can even vanish at the spinodal point), and nucleation process starts. Assuming that after that the system rapidly transforms into a mixed phase, we can find the volume fraction of Q phase at $\mathcal{E}=\mathcal{E}_s$ and temperature $T_c$ by using the conservation of energy:

$$\mathcal{E}_Q(T_S) = \mathcal{E}(f_S, T_c) = \mathcal{E}_Q(T_c) f_S + \mathcal{E}_H(T_c)(1-f_S)$$

$$\Rightarrow f_S = \frac{3}{4}\,\frac{\xi\left(\frac{T_S}{T_c}\right)^4 - 1}{\xi - 1} + \frac{1}{4}$$

It is easy to see that the entropy is generated in this sudden change of the matter state. Indeed the entropy gain can be found from

$$\frac{S(f_S, T_c)}{S_Q(T_S)} = \frac{S_Q(T_c) f_S + S_H(T_c)(1-f_S)}{S_Q(T_S)} = \frac{1}{4}\left(\frac{T_c}{T_S}\right)^3 + \frac{3}{4}\left(\frac{T_S}{T_c}\right)$$

The calculations show that the entropy generation is very small at small supercooling ($\sim 2\%$ at $\frac{T_S}{T_c} = 0.9$), but then grows significantly with degree of supercooling ($30\%$ at $\frac{T_S}{T_c} = 0.7$).

This example shows how the entropy is generated in an nonequilibrium process in accordance with H-theorem derived by Bolzmann in 1872.

# Fluctuations of order parameter $\eta$



Equilibrium probability distribution: $P(\eta) = P_0 \exp\left[-\frac{\Delta\Omega}{T}\right]$

$$\Delta\Omega(T,\mu;\eta) = \frac{1}{2}\overbrace{a\cdot(T-T_c)}^{m^2}\eta^2 + \frac{1}{2}b(\nabla\eta)^2 + \frac{\lambda}{4}\eta^4 + \ldots$$

$$\frac{\partial\Delta\Omega}{\partial\eta} = 0 \Rightarrow \begin{cases} \bar{\eta} = 0 \,, & T > T_c \quad \text{symmetric phase} \\ \bar{\eta} = \sqrt{\dfrac{a(T_c - T)}{\lambda}} \,, & T < T_c \quad \text{broken phase} \end{cases}$$

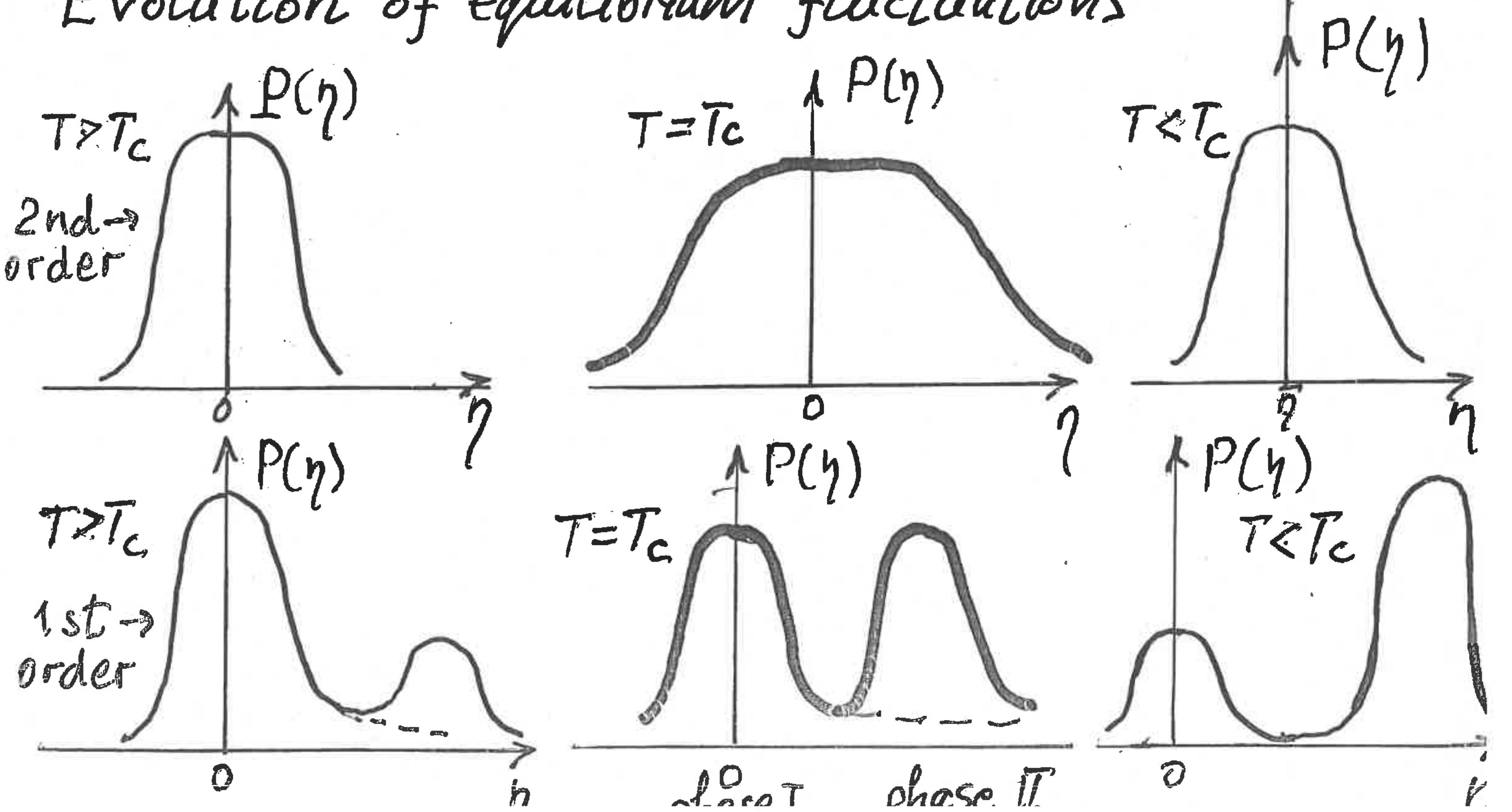
Expand $\Delta\Omega$ in powers of $\delta\eta = \eta - \bar{\eta} = \sum_k \delta\eta_k e^{i\vec{k}\vec{r}}$

Find $\quad \langle|\delta\eta_k|^2\rangle = \dfrac{T}{a(T-T_c) + bk^2} \to \infty$ at $T \to T_c$, $k = 0$

Correlation function $\quad g(\vec{r}) = \langle\delta\eta(\vec{r}_1)\,\delta\eta(\vec{r}_2)\rangle$, $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$g(r) = \frac{T}{4\pi b r}\exp\left(-\frac{r}{r_c}\right), \quad r_c(T) = \sqrt{\frac{b}{a(T-T_c)}} \to \infty \text{ at } T \to T_c$$

Evolution of equilibrium fluctuations



# Critical slowing down of relaxation near $T_c$



Time evolution of the order parameter field is described by the Langevin equation

$$\frac{d\eta}{dt} = -\gamma \frac{\partial \Omega}{\partial \eta} + \xi \longleftarrow \text{Gaussian noise}$$

$$= -\gamma \left[ a(T-T_c)\eta - b\Delta\eta + \lambda\eta^3 + \dots \right]$$

After linearization and Fourier transformation

$$\frac{d\,\delta\eta_k}{dt} = -\frac{\delta\eta_k}{\tau_k}, \quad \frac{1}{\tau_k} = \overbrace{\gamma a |T-T_c|}^{1/\tau_0} + \gamma b k^2$$

Relaxation time at $k=0$ $\tau_0 \to \infty$ at $T \to T_c$

Using scaling arguments one can write

$$m(T) = \frac{1}{r_c(T)} \sim |T-T_c|^{\nu}$$

$$\tau(k,T) = |T-T_c|^{-y} f(kr_c) \to k^{-z} \text{ at } T \to T_c,\ k \neq 0$$

$$\sim |T-T_c|^{-y} \sim [m(T)]^{-z} \quad \text{at } T \to T_c,\ k=0$$

$$\nu y = z = 2 + \alpha/\nu = 2 \div 3$$

Relaxation equation for correlation length $m^{-1}(T)$

$$\frac{d}{dt} m(t) = -\frac{1}{\tau(m)} \left[ m(t) - \underbrace{\frac{1}{r_c(T)}}_{\text{in equilibrium}} \right]$$

Berdnikov
Rajagopal
PRD 61 (2000)

! Due to critical slowing down, $\tau \to \infty$, at $T_c$ the system cannot relax to equilibrium at any time-dependent evolution

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B. Berdnikov, K. Rajagopal, Phys. Rev. D 61, 105017 (2000)
« Slowing out of equilibrium near the QCD critical point ».

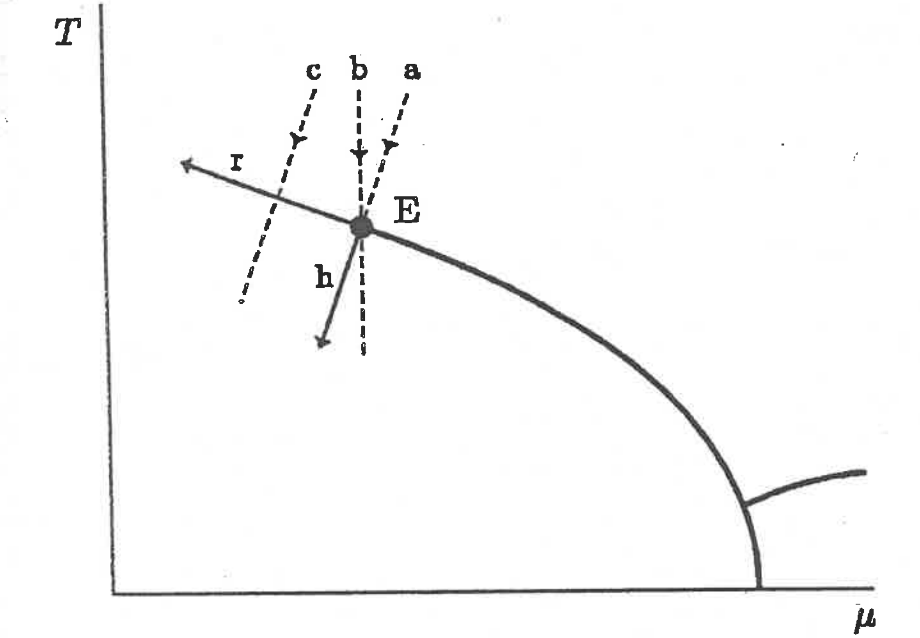



FIG. 1. Sketch of the QCD phase diagram as a function of temperature $T$ and baryon chemical potential $\mu$. Chiral symmetry is broken at low $T$ and $\mu$. As $T$ is increased, chiral symmetry is approximately restored via a smooth crossover to the left of $E$ or a first-order phase transition to the right of $E$. The symmetry is only approximately restored because the light quarks are not massless. At the critical point $E$ at which the line of first-order phase transitions ends, the transition is second order and is in the Ising universality class. (At large $\mu$ and small $T$, there are color superconducting phases which we do not discuss in this paper.) The Ising model $r$ axis and $h$ axis and the trajectories a, b, and c will be discussed in Sec. II.

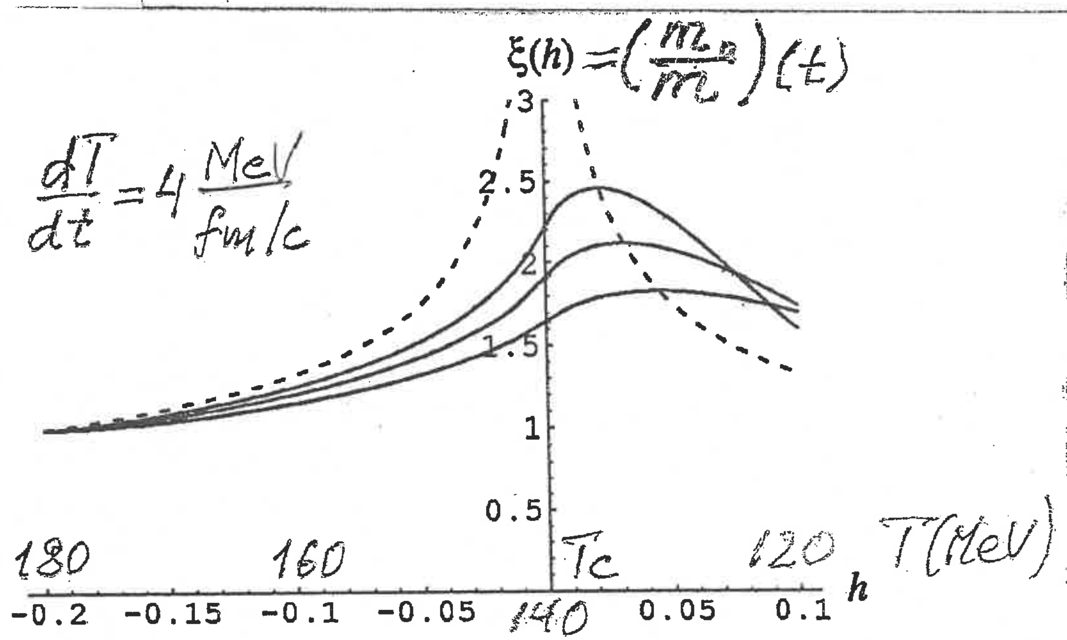



FIG. 2. Behavior of the correlation length for cooling through the critical point along the $h$ axis of Fig. 1. The equilibrium correlation length is shown as a dashed line. The true correlation length is shown for (bottom to top) $a=25,50,100$. Our units, described in the text, are such that $h=-0.2,-0.1,0,0.1$ corresponds to $T=180,160,140,120$ MeV, and $\xi$ is measured in units of $\xi_0=1.4$ fm.

1. The correlation length increases by about factor 2.5 ~~at most~~!
2. It is a hard, if impossible, task to see a 2nd order p.t. in dynamics!

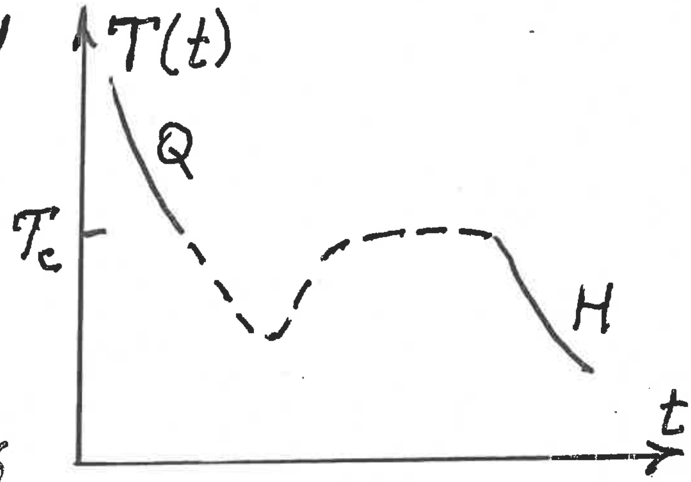
# When does the nucleation start? 

## Dynamics of a 1-st order phase transition

In standard picture (ignoring flow) — Below c.p.

supercooling is needed in order to fuel bubbles

$$R_c = \frac{2\sigma}{p_H - p_Q}$$

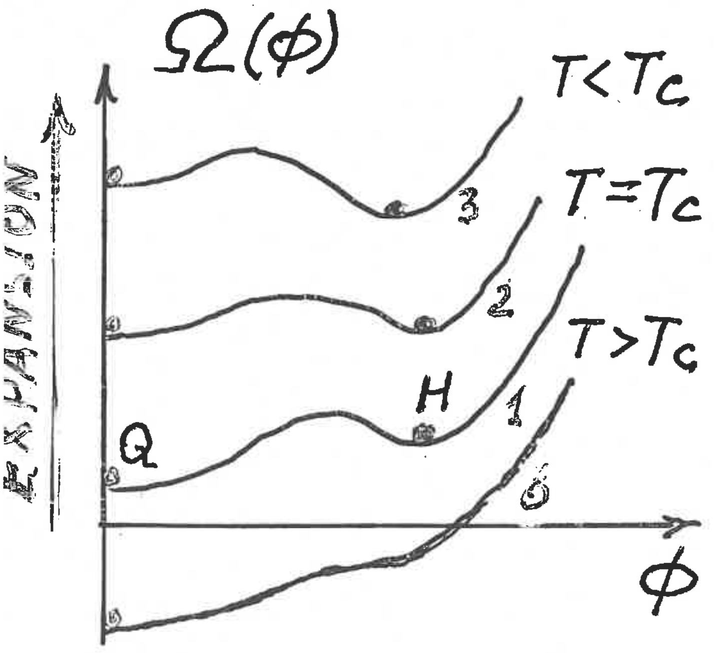


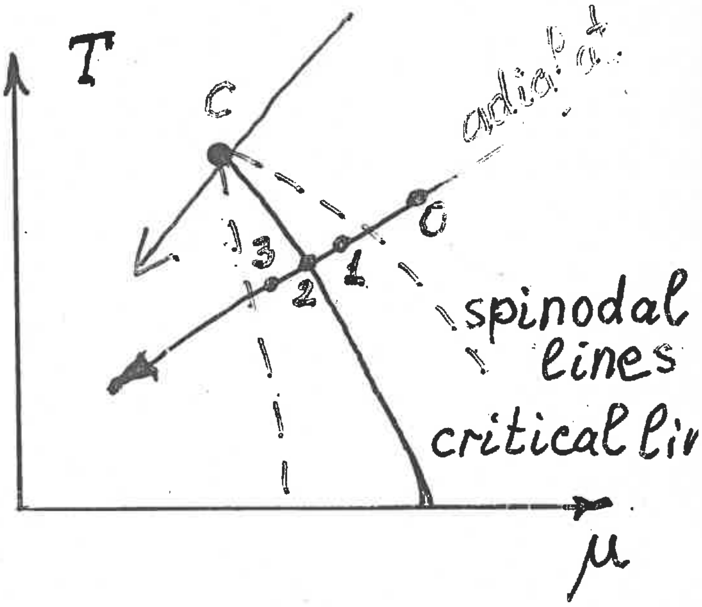

Csernai & Kapusta, PRD46(92)46

The cooling is fast $\frac{dT}{d\tau} = -\frac{T}{n\tau} \approx (5 \div 10)\ \frac{MeV}{fm/c}$ around $T_c$

In new picture (including flow) — above c.p.

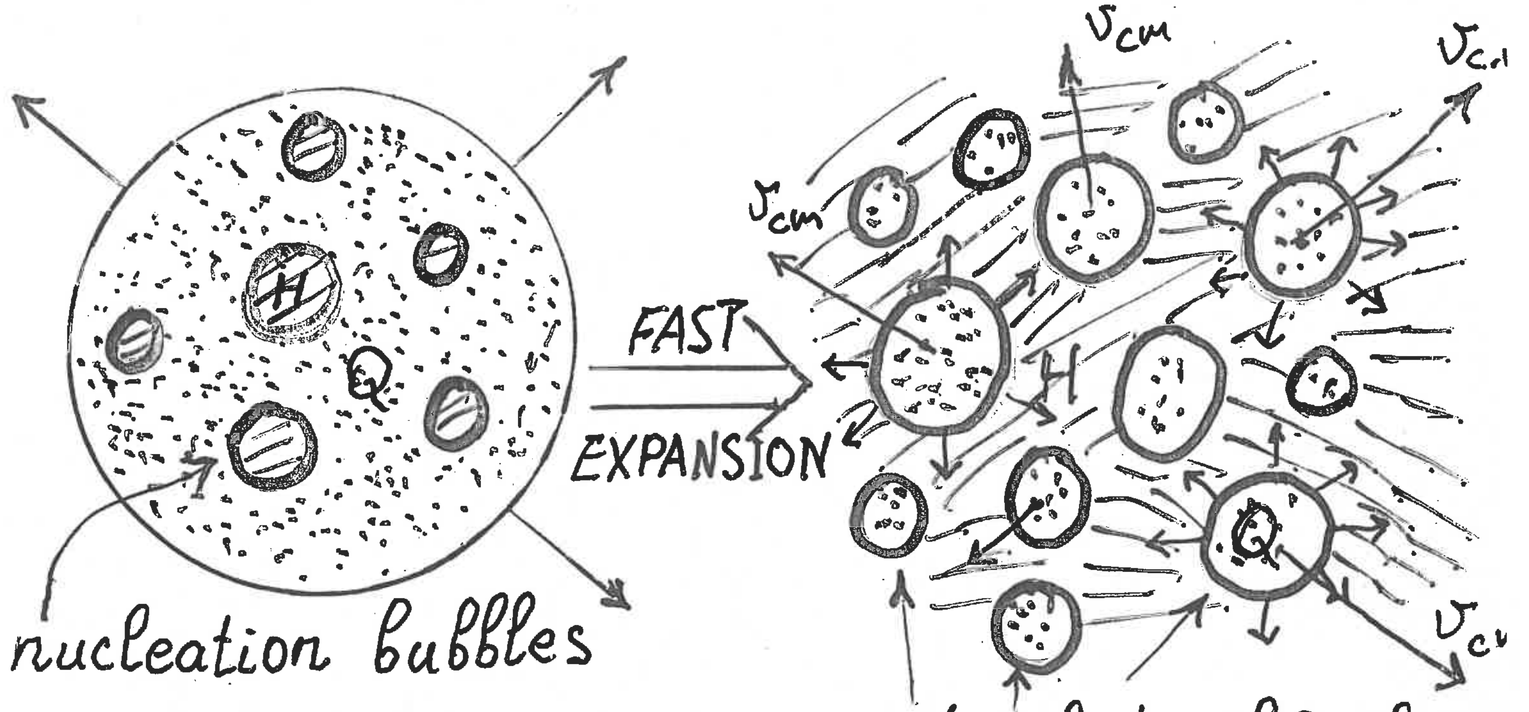
as early as a metastable H state appears in $\Omega$




Bubble growth is driven by the flow alone!

# Dynamical fragmentation of a metastable phase




$p_H < p_Q$
undersaturated hadron gas

$p_H > p_Q$

Characteristic droplet size from energy balance
(Grady, J. Appl. Phys. 53 (1981) 322)

$$\Delta\Omega = -(P_Q - P_H)V + \frac{3}{10}\,\mathcal{E}_Q V H^2 R^2 + 4\pi R^2 \sigma$$

Minimize $\frac{\Delta E}{V} \Longrightarrow \boxed{R^* = \left(\frac{5\sigma}{\mathcal{E}_Q H^2}\right)^{1/3}}$

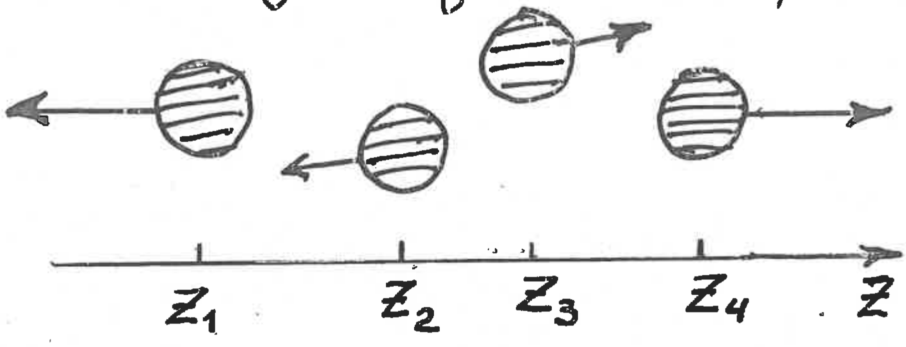
Droplet mass: $\boxed{M \approx \mathcal{E}_Q V = \frac{20\pi}{3}\frac{\sigma}{H^2}}$

fast expansion $H^{-1} \sim 6\,fm/c$ $M \sim 10\,GeV$

slow expansion $H^{-1} \sim 20\,fm/c$ $M \sim 100\,GeV$ !

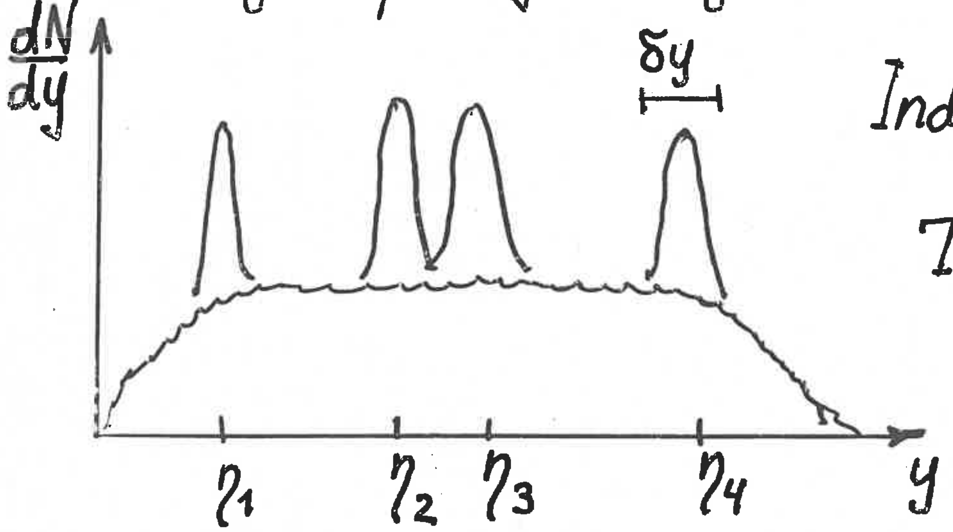
# How to see droplets of QGP?

Strong longitudinal expansion – separation in rapidity space




$\eta(z) \sim z$

Strong rapidity density fluctuations of hadrons
event-by-event



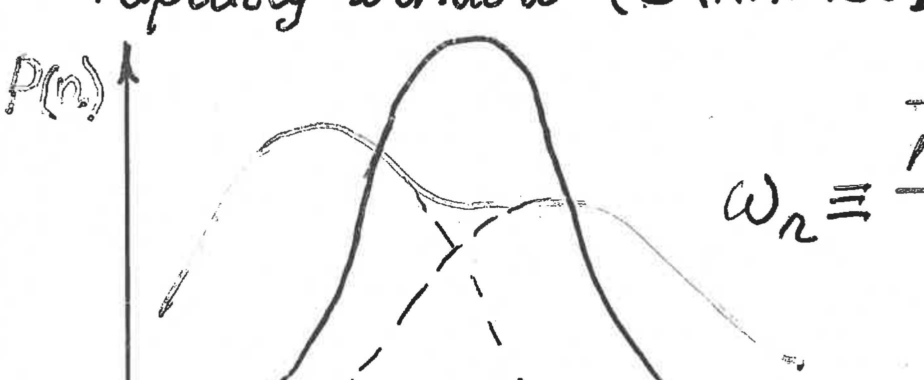

Individual emitting source:

$T \sim 100 \div 120$ MeV

$\delta y = 1 \div 2$

$\delta N \sim 100$

Measure multiplicity distributions in a given rapidity window (BRAHMS: $\Delta y = 1$, ALLICE: $\Delta y = 2$)

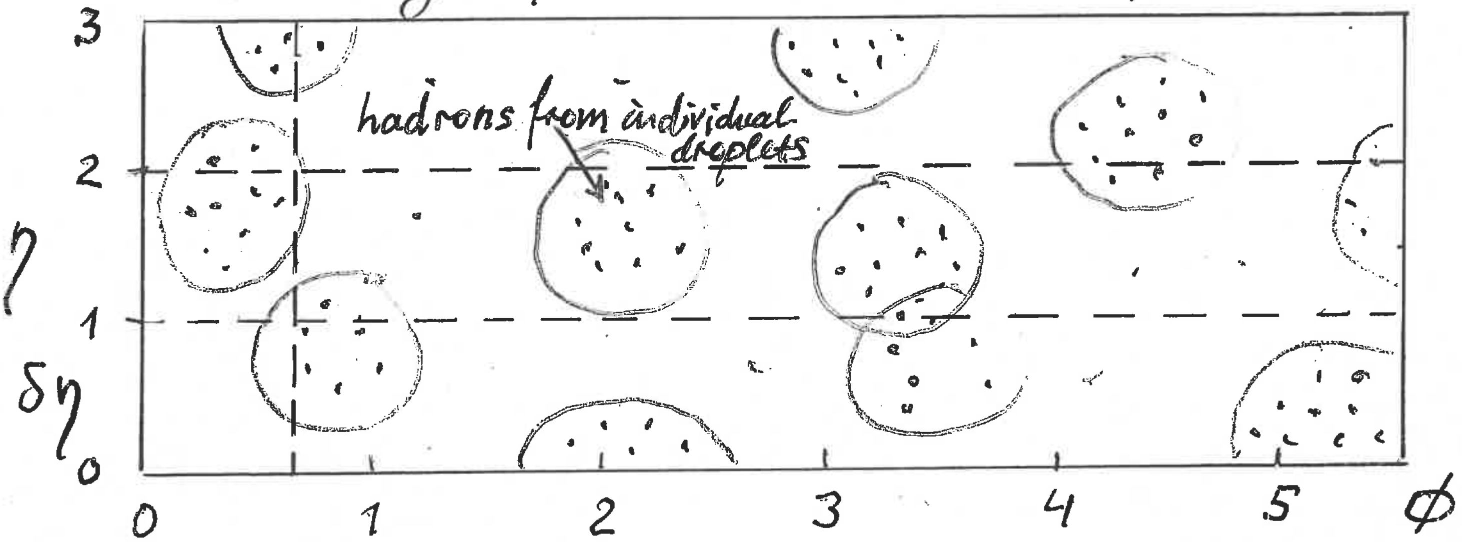



$$\omega_n \equiv \frac{\overline{n^2} - \bar{n}^2}{\bar{n}} \gg 1$$

Look for large non-statistical fluctuations!

# A strategy of how to see QGP droplets

1. Estimate the droplet extension in momentum space: $\delta\eta \approx 2\sqrt{\frac{T}{m_\pi}} \approx 1 \div 1.5$, $\delta\phi \approx \frac{m_\pi v_T}{\langle p_t \rangle} \approx 0.3 \div 0.6$
2. Plot $\eta - \phi$ distributions event by event all charged particles with $0 < p_t < 2\,\text{GeV}$




Similar to LEGO plots by J. Bjor[illegible]

3. Make cluster analysis of points using a reasonable algorithm like MST (minimum spanning tree), widely used in nuclear fragmentation studies