GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Lecture

Relativistic kinematics, elementary reactions, cross sections

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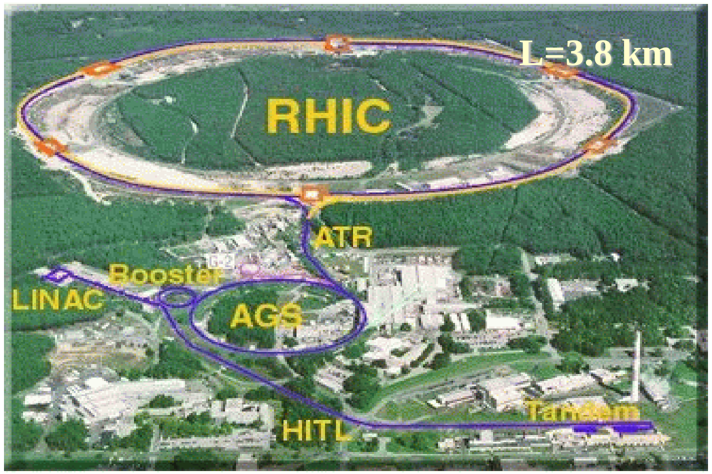
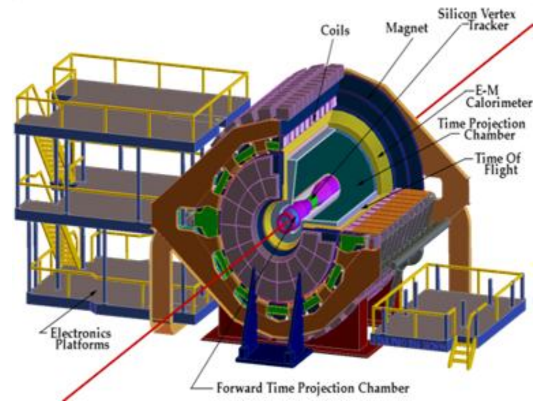
SS2025: ‚Dynamical models for relativistic heavy-ion collisions'

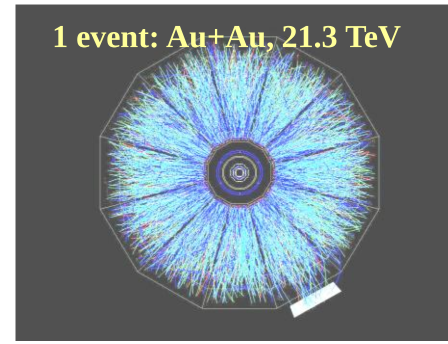
Heavy-ion accelerators

- **Relativistic-Heavy-Ion-Collider – RHIC - (Brookhaven): Au+Au at 21.3 A TeV**



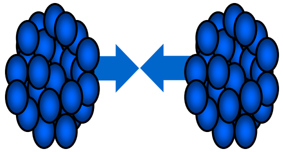
STAR detector at RHIC



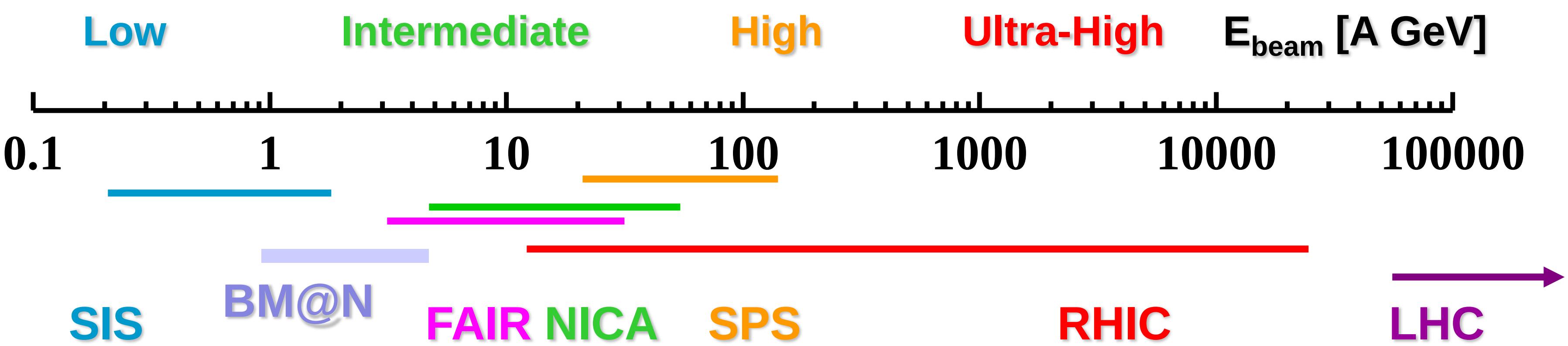


- **Large Hadron Collider - LHC - (CERN): Pb+Pb at 574 A TeV**

- **Future facilities: FAIR (GSI), NICA (Dubna)**



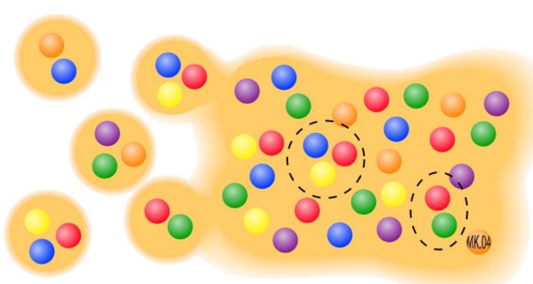
HIC experiments



Baryonic matter
||
Meson and baryon spectroscopy
In-medium effects
EoS

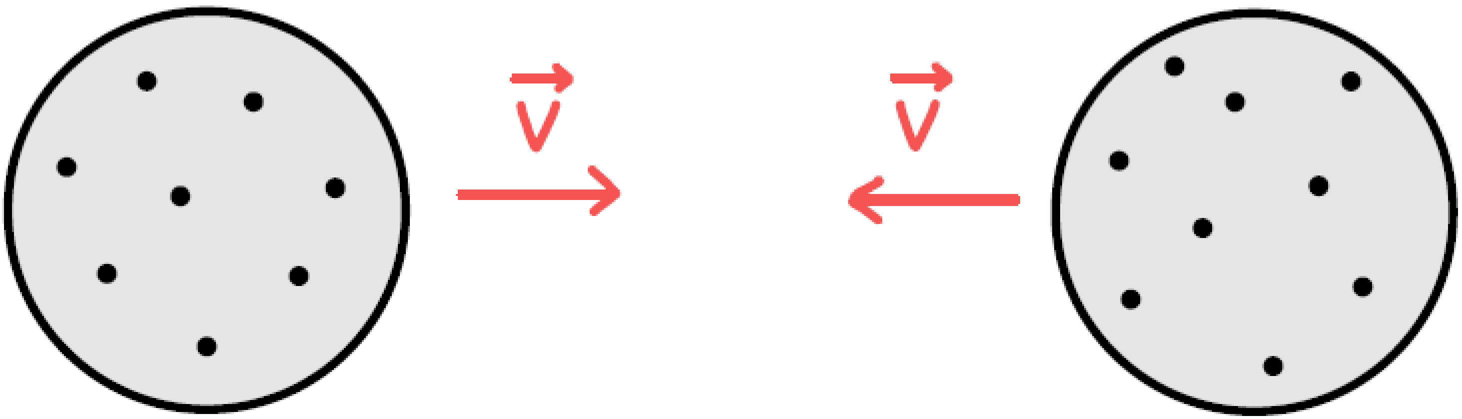
‚Mixed‘ phase:
hadrons (baryons, mesons) + quarks and gluons
||
In-medium effects
Chiral symmetry restoration
Phase transition to sQGP
Critical point in the QCD phase diagram

QGP: quarks and gluons
||
Properties of sQGP



Heavy-ion collisins

Low energy A+A collision: velocity v is small

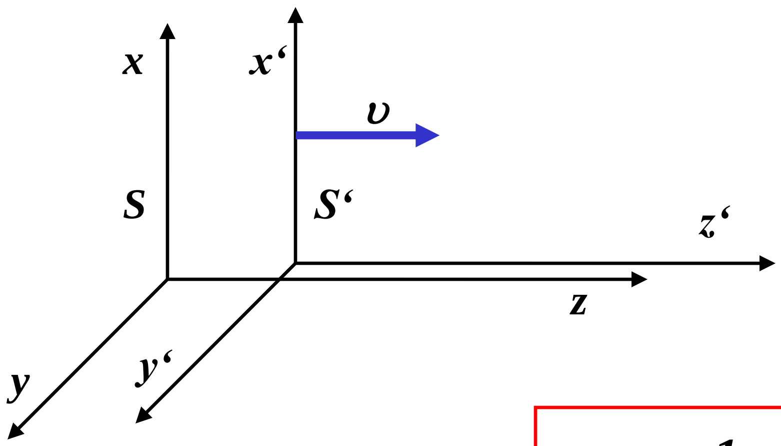


High energy A+A collision: velocity v is large

➔ **Lorentz contraction of nuclei**

Special relativity

Lorentz transformation:



❑ Consider the space-time point
- in a given frame S: (t, x, y, z)
- and in a (moving) frame S‘: (t', x', y', z')

1) S‘ moves with a constant velocity υ along z-axis

Space-time Lorentz transformation S←→S‘:

$S \Rightarrow S'$	$S' \Rightarrow S$
$x' = x$	$x = x'$
$y' = y$	$y = y'$
$z' = \gamma(z - \upsilon t)$	$z = \gamma(z' + \upsilon t')$
$t' = \gamma(t - \upsilon z)$	$t = \gamma(t' + \upsilon z')$

Lorentz factor: $\gamma = \dfrac{1}{\sqrt{1-\upsilon^2}}$

Note: units c=1

$$t' = \gamma(t - \frac{\upsilon}{c^2} z)$$

$$t = \gamma(t' + \frac{\upsilon}{c^2} z')$$

$$\upsilon_z = |\vec{\upsilon}| = \upsilon$$

❑ Consider the 4-momentum:
- in a given frame S: $p \equiv (E, p) = (E, p_x, p_y, p_z)$
- in the (moving) frame S‘: $p' \equiv (E', \vec{p}') = (E', p'_x, p'_y, p'_z)$

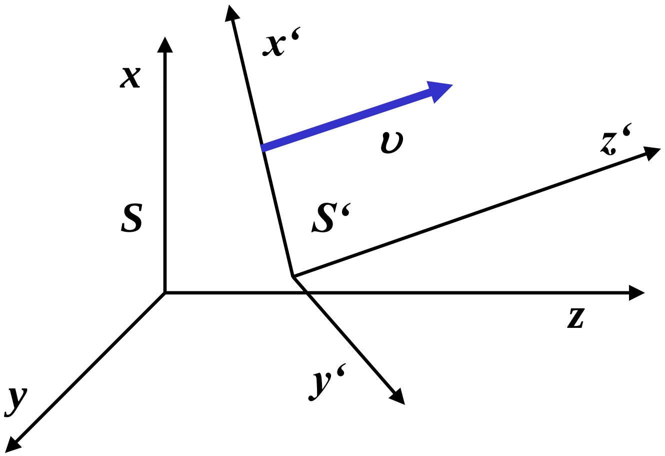
Lorentz transformation for 4-momentum S←→S‘:

$$p'_x = p_x, \quad p'_y = p_y$$

$$p'_z = \gamma(p_z - \upsilon E)$$

$$E' = \gamma(E - \upsilon p_z)$$

Special relativity



2) S' moves with a constant velocity υ in arbitary direction relative to S

$$p'_z = (\vec{p}'\vec{\upsilon})/\upsilon \qquad p_z = (\vec{p}\vec{\upsilon})/\upsilon$$

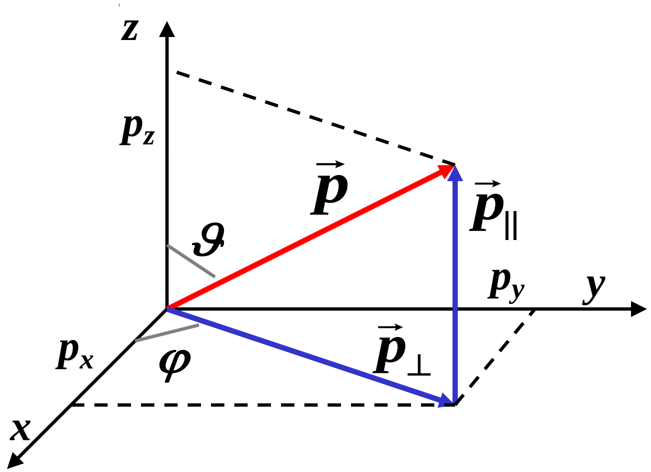
Lorentz transformation for 4-momentum S←→S':

$$\vec{p}' = \vec{p}'_{\parallel} + \vec{p}'_{\perp} \qquad \vec{p} = \vec{p}_{\parallel} + \vec{p}_{\perp}$$

$$\vec{p}'_{\perp} = (p'_x, p'_y, 0) \qquad \vec{p}_{\perp} = (p_x, p_y, 0)$$

$$\vec{p}'_{\parallel} = \frac{\vec{p}'_z\vec{\upsilon}}{\upsilon} = \frac{(\vec{p}'\vec{\upsilon})\vec{\upsilon}}{\upsilon^2} \qquad \vec{p}_{\parallel} = \frac{\vec{p}_z\vec{\upsilon}}{\upsilon} = \frac{(\vec{p}\vec{\upsilon})\vec{\upsilon}}{\upsilon^2}$$

$$\vec{p}'_{\perp} = \vec{p}_{\perp} \qquad \vec{p}' = p'_z\frac{\vec{\upsilon}}{\upsilon} + \vec{p}'_{\perp}$$



$$p_{\perp} = \sqrt{p_x^2 + p_y^2}$$

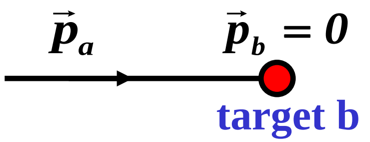
$$p_{\parallel} = p_z = p\cos\vartheta$$

$$\vec{p}' = \vec{p} + \gamma\vec{\upsilon}\left[\frac{\gamma(\vec{p}\vec{\upsilon})}{\gamma+1} - E\right]$$

$$E' = \gamma\left[E - (\vec{p}\vec{\upsilon})\right]$$

Reference frames for collision processes

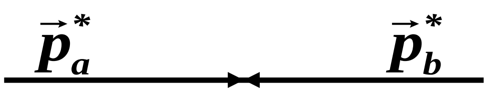
- Collision process a+b in the **laboratory frame (LS):**



$$\vec{p}_b = 0, \; E_b = m_b$$

4-momenta $\quad p_a = (E_a, \vec{p}_a), \quad p_b = (m_b, 0)$

- Collision process a+b in the **center-of-mass frame (CMS):**



$$\vec{p}_a^{\,*} + \vec{p}_b^{\,*} = 0$$

4-momenta $\quad p_a = (E_a^*, \vec{p}_a^{\,*}), \quad p_b = (E_b^*, \vec{p}_b^{\,*})$

Relative velocity of CMS and LS: $\displaystyle \vec{\upsilon} = \left. \frac{\vec{p}_a + \vec{p}_b}{E_a + E_b} \right|_{\vec{p}_b = 0} = \frac{\vec{p}_a}{E_a + m_b}$

Lorentz transformation for 4-momentum CMS←→LS:

CMS:

$$E_b^* = \frac{m_b}{\sqrt{1-\upsilon^2}} \qquad E_a^* = \frac{E_a - (\vec{p}_a \vec{\upsilon})}{\sqrt{1-\upsilon^2}}$$

$$\vec{p}_b^{\,*} = -\frac{m_b \vec{\upsilon}}{\sqrt{1-\upsilon^2}} \qquad \vec{p}_a^{\,*} = \frac{\vec{p}_a - E_a \vec{\upsilon}}{\sqrt{1-\upsilon^2}}$$

$$\vec{p}_a^{\,*} = -\vec{p}_b^{\,*}$$

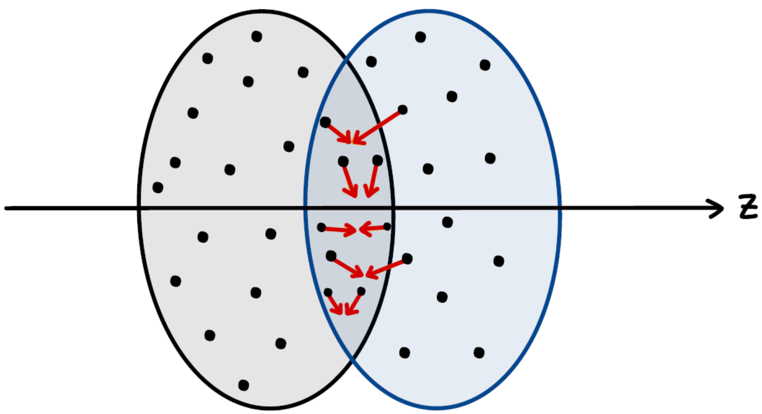
LS:

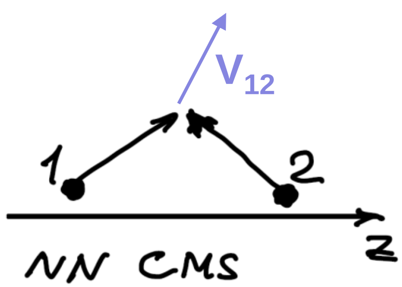
$$E_b = \frac{E_b^* + (\vec{p}_b^{\,*} \vec{\upsilon})}{\sqrt{1-\upsilon^2}} \qquad E_a = \frac{E_a^* + (\vec{p}_a^{\,*} \vec{\upsilon})}{\sqrt{1-\upsilon^2}}$$

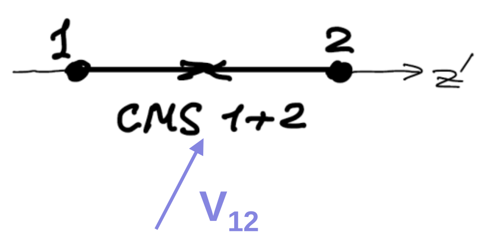
$$\vec{p}_b = \frac{\vec{p}_b^{\,*} + E_b^* \vec{\upsilon}}{\sqrt{1-\upsilon^2}} \qquad \vec{p}_a = \frac{\vec{p}_a + E_a^* \vec{\upsilon}}{\sqrt{1-\upsilon^2}}$$

Reference frames for A+A

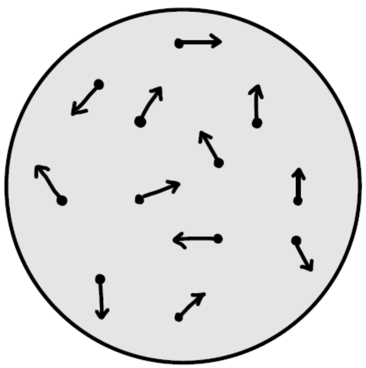
Reference frame for A+A:
N+N center-of-mass frame!



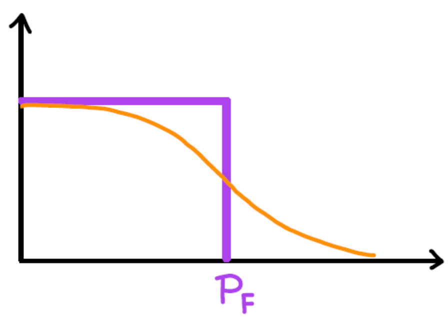




Fermi-motion of nuclons
in the initial nuclei (V=0)



Fermi-distribution of nucleons



Kinematical variables

Rapidity $$y = \frac{1}{2} \ln\left[\frac{1+\upsilon_z}{1-\upsilon_z}\right]$$

Since $\vec{\upsilon} = \frac{\vec{p}}{E}$ ⇨ $$y = \frac{1}{2} \ln\left[\frac{E+p_z}{E-p_z}\right] = \frac{1}{2} \ln\left[\frac{E+p_{||}}{E-p_{||}}\right]$$

Rapidity is additive under Lorentz transformation:

$$y = y^* + \Delta y$$

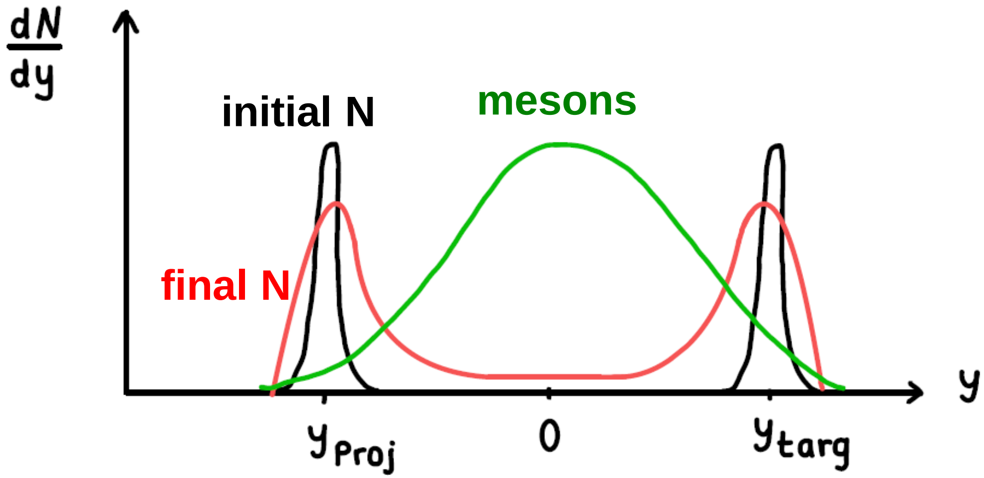
!

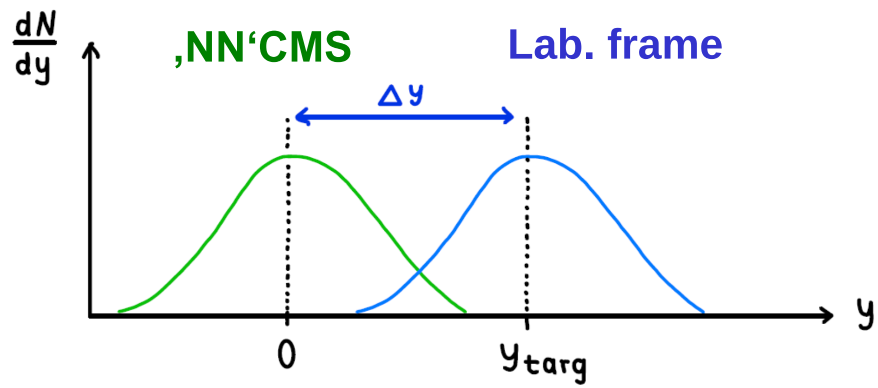
y rapidity in Lab frame = y* in cms + Δy relative rapidity of cms vs. Lab

Transverse mass: $$M_\perp = \sqrt{p_\perp^2 + m^2}$$

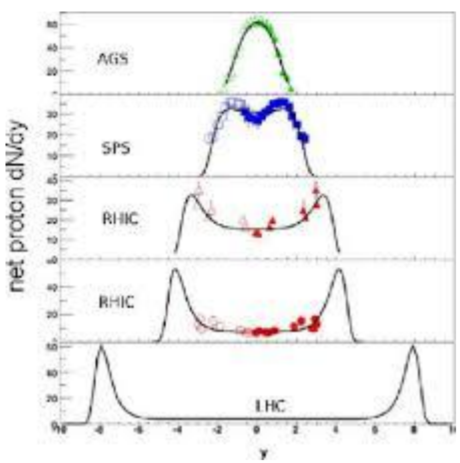
$$p_{||} = M_\perp \sinh(y)$$
$$E = M_\perp \cosh(y)$$

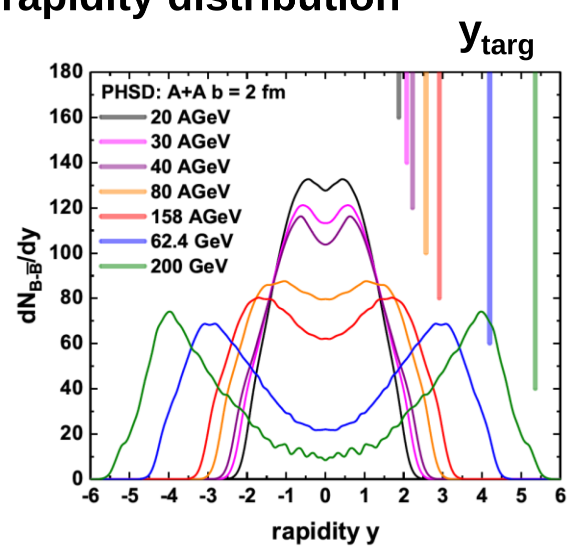
Rapidity distributions in HIC



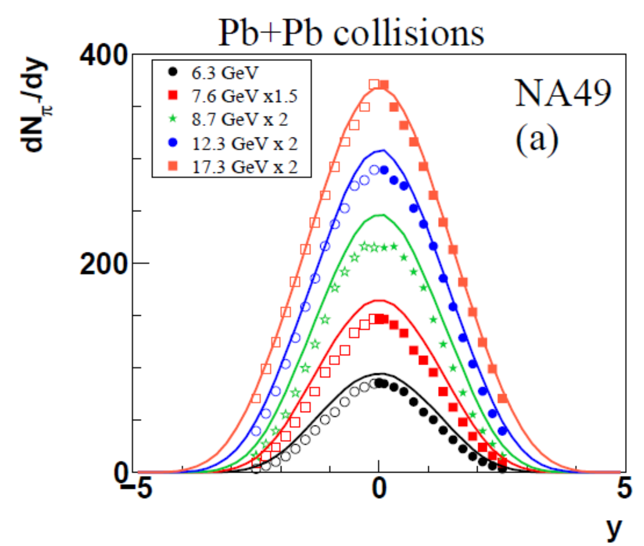


Net-proton (p-$\overline{p}$) rapidity distribution





Pion rapidity distribution



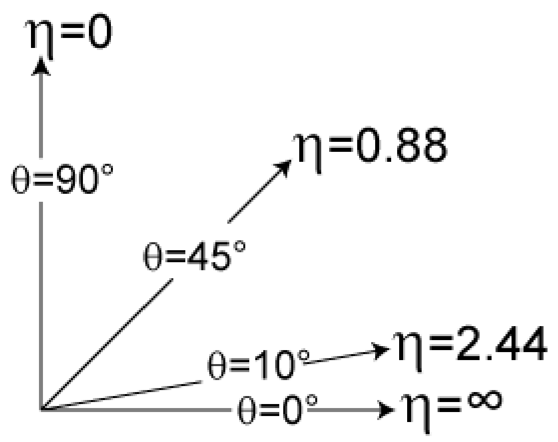
Kinematical variables

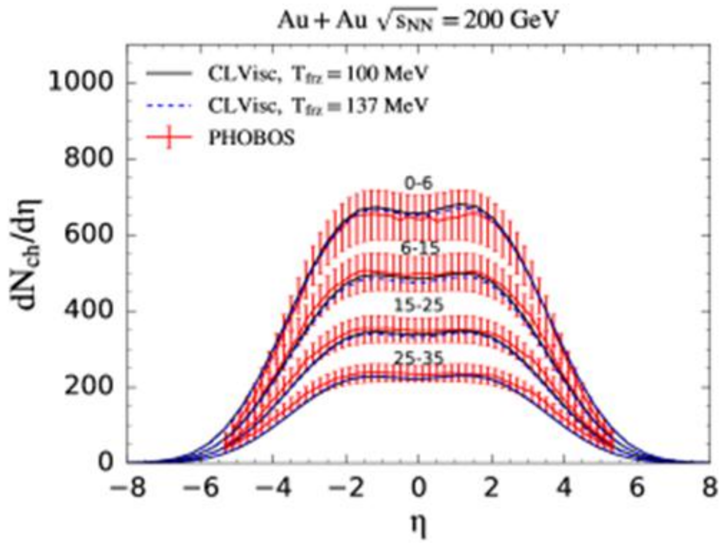
Pseudo-rapidity $$\eta \equiv -\ln\left[\tan\left(\frac{\theta}{2}\right)\right]$$

θ is the angle between the particle three-momentum and the positive direction of the beam axis

$$\eta = \frac{1}{2}\ln\left(\frac{|\mathbf{p}| + p_\mathrm{L}}{|\mathbf{p}| - p_\mathrm{L}}\right) = \operatorname{arctanh}\left(\frac{p_L}{|\mathbf{p}|}\right)$$

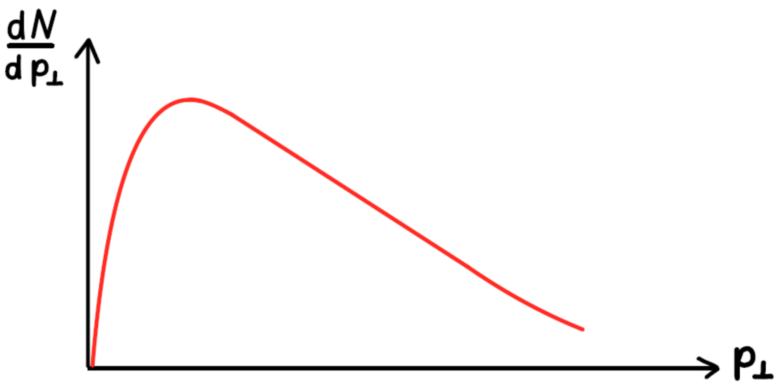
$$m \ll |\mathbf{p}| \Rightarrow E \approx |\mathbf{p}| \Rightarrow \eta \approx y$$



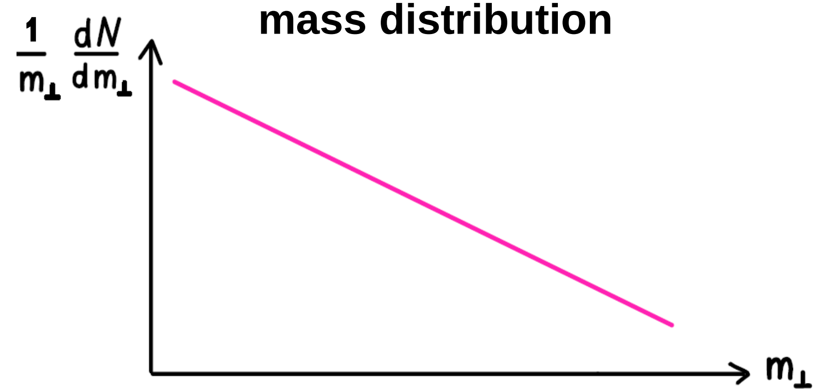


Transverse momentum distributions in A+A

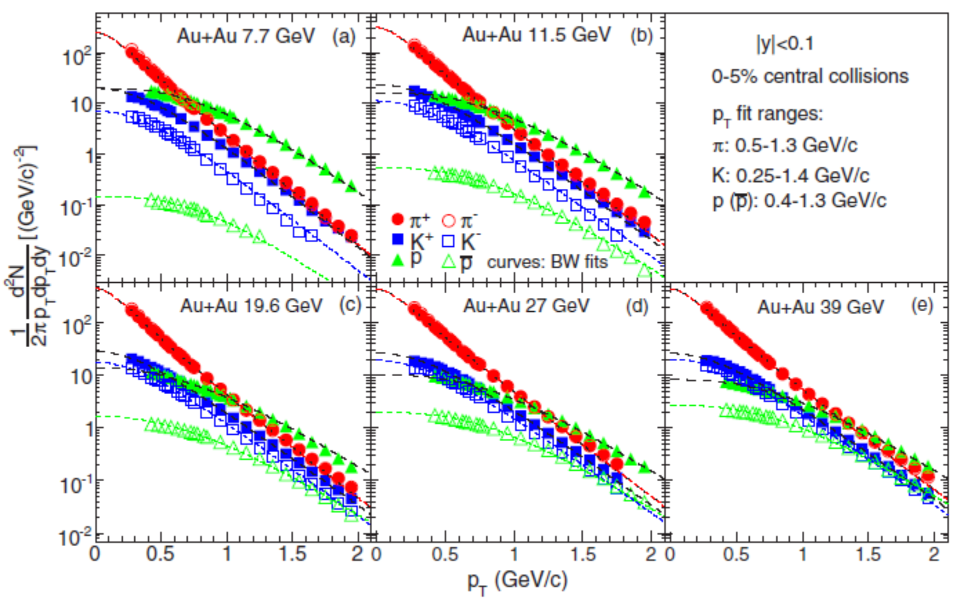
Transverse momentum distribution



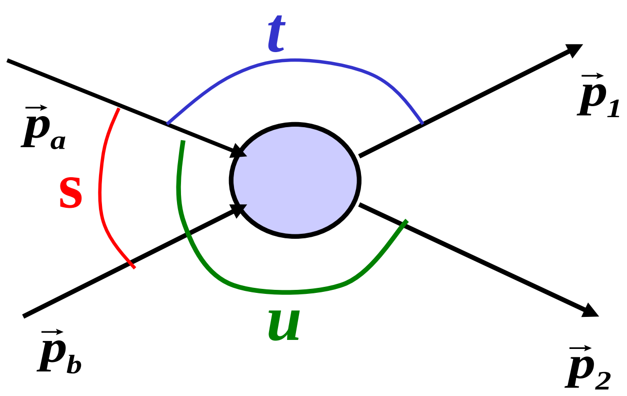
Lorentz invariant transverse mass distribution



Lorentz invariant transverse momentum distribution at midrapidity |y|<0.1



Mandelstam variables for 2→2 scattering



Definitions of the Lorentz invariants (s,t,u) for a+b→1+2:

4-vectors

$$s = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2(p_a \cdot p_b)$$

$$= (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2(p_1 \cdot p_2)$$

$$\xrightarrow{cms} = (E_a^* + E_b^*)^2 - (\underbrace{\vec{p}_a^{\,*} + \vec{p}_b^{\,*}}_{=0})^2 = (E_a^* + E_b^*)^2$$

$$\xrightarrow{Lab.\,frame} = m_a^2 + m_b^2 + 2m_b E_a$$

$$t = (p_a - p_1)^2 = (p_b - p_2)^2$$

$$\xrightarrow{cms} = m_a^2 + m_1^2 - 2E_a E_1 + 2p_a p_1 \cos\vartheta_{a1}$$

$$\xrightarrow{Lab.\,frame} = m_b^2 + m_2^2 - 2m_b E_2$$

$$u = (p_a - p_2)^2 = (p_b - p_1)^2$$

$$\xrightarrow{cms} = m_a^2 + m_2^2 - 2E_a E_2 + 2p_a p_2 \cos\vartheta_{a2}$$

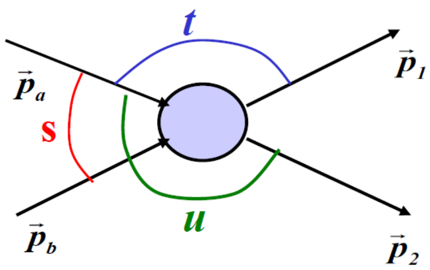
$$\xrightarrow{Lab.\,frame} = m_b^2 + m_1^2 - 2m_b E_1$$

Invariant variables for 2→2 scattering

There are two independent variables and *s,t,u* are related by:

$$s + t + u = (p_a + p_b)^2 + (p_a - p_1)^2 + (p_b - p_1)^2$$
$$= p_a^2 + p_b^2 + p_1^2 + (\underbrace{p_a + p_b - p_1}_{p_2})^2 \Rightarrow$$

$$\boxed{s + t + u = m_a^2 + m_b^2 + m_1^2 + m_2^2}$$



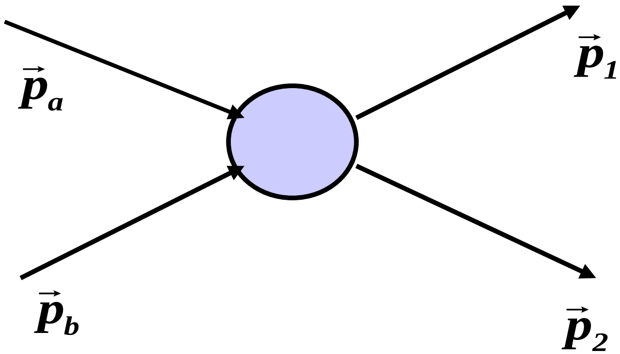
Kinematical limits:

$$s \geq max\left((m_a + m_b)^2, (m_1 + m_2)^2\right)$$ ⟸ **‚Thereshold energy‘**

$$m_a^2 + m_1^2 - 2E_aE_1 - 2p_ap_1 \leq t \leq m_a^2 + m_1^2 - 2E_aE_1 + 2p_ap_1$$
$$(\cos\vartheta_{a1} = -1) \qquad\qquad (\cos\vartheta_{a1} = 1)$$

$$m_a^2 + m_2^2 - 2E_aE_2 - 2p_ap_2 \leq u \leq m_a^2 + m_2^2 - 2E_aE_2 + 2p_ap_2$$

2→2 cross section



Differential cross section a+b→1+2:

$$d\sigma = \frac{(2\pi)^4}{F}\left|M_{if}\right|^2 d\Phi_2$$

$|M_{if}|^2$ – squared matrix element

$$F - flux: \quad F = 4\sqrt{(p_a p_b)^2 - m_a^2 m_b^2}$$

$$CMS: F = 4p_a^* \sqrt{s}$$

$$Lab.\ frame: F = 4m_b p_a$$

$$p \equiv p_\mu = (E, \vec{p})$$

$$p \equiv |\vec{p}|$$

Two-body phase space:

$$d\Phi_2 = \frac{d^3 p_1}{2E_1}\frac{d^3 p_2}{2E_2}\delta^4(p_a + p_b - p_1 - p_2)\frac{1}{\left((2\pi)^3\right)^2}$$

$$d^3 p \equiv d\vec{p} = p^2 dp d\Omega = E p dE d\Omega, \quad d\Omega = d\cos\vartheta d\varphi$$

$$E^2 = p^2 + m^2 \Rightarrow \quad d(E^2) = d(p^2) \quad \rightarrow 2EdE = 2pdp \quad \rightarrow dp = \frac{E}{p}dE$$

Two-body phase space

$$\frac{d^3p}{2E} = \int_0^\infty dEd^3p\ \delta(p_\mu p^\mu - m^2) = \int_{-\infty}^{\infty} d^4p\delta(p^2 - m^2)\theta(E) \qquad \theta(E) = \begin{cases} 1, E > 0 \\ 0, E < 0 \end{cases}$$

$$\frac{1}{2E} = \int_0^\infty dE\delta(E^2)$$

➔ Invariant under Lorentz transformation!

substitute in two-body phase space:

$$d\Phi_2 = \frac{d^3p_1}{2E_1}\int_{-\infty}^{\infty} d^4p_2\delta\left(p_2^2 - m_2^2\right)\Theta(E_2)\delta^4(p_a + p_b - p_1 - p_2)\frac{1}{(2\pi)^6}$$

$$= \frac{1}{(2\pi)^6}\frac{d^3p_1}{2E_1}\delta((p_a + p_b - p_1)^2 - m_2^2)$$

$$\frac{d^3p_1}{2E_1} = \frac{1}{2E_1}p_1E_1dE_1d\Omega = \frac{p_1}{2}dE_1d\Omega$$

(Invariant under Lorentz transformation)

! Further considerations require to choose a reference frame!

$$d^4p = dEd^3p = dE\frac{d^3p}{2E}2E = dE^2\frac{d^3p}{2E}$$

Lorentz invariants

Show that $$\frac{d^3p}{E} = p_T dp_T dy d\varphi$$

1) $$d^3p = p^2 dp\, d\cos\theta\, d\varphi = \frac{dp_T^{\,2}}{2} dp_{\parallel} d\varphi = p_T dp_T dp_{\parallel} d\varphi$$

2) Use that $$p_{\parallel} = M_{\perp} \sinh(y)$$ $$E = M_{\perp} \cosh(y)$$ $\Rightarrow$ $$dp_{\parallel} = M_{\perp} \cosh(y) dy = E dy$$

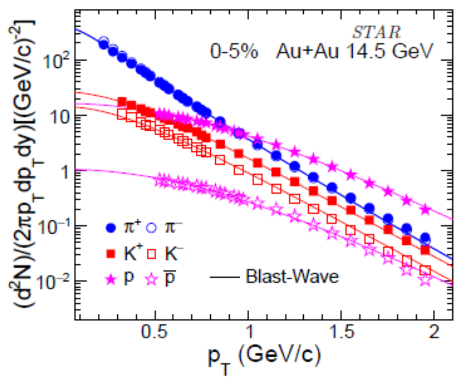
3) thus, $$d^3p = p_T dp_T E dy d\varphi$$

$$\frac{d^3p}{E} = p_T dp_T dy d\varphi$$

(Invariant under Lorentz transformation)

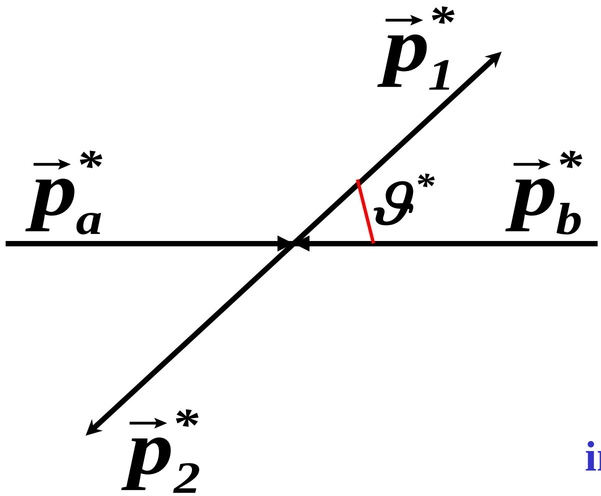
Invariant spectra: $$E\frac{d^3N}{d^3p} = \frac{d^3N}{p_T dp_T dy d\varphi} = \frac{d^3N}{m_T dm_T dy d\varphi}$$

Invariant cross section: $$E\frac{d^3\sigma}{d^3p} = \frac{d^3\sigma}{p_T dp_T dy d\varphi}$$



Two-body phase space

Let‘s choose the center-of-mass system (cms): $\vec{p}_a^{\,*} + \vec{p}_b^{\,*} = \vec{p}_1^{\,*} + \vec{p}_2^{\,*} = 0$



$$s = (p_a + p_b)^2 = (p_1 + p_2)^2$$

$$= (E_a^* + E_b^*)^2 - (\underbrace{\vec{p}_a^{\,*} + \vec{p}_b^{\,*}}_{=0 \text{ in cms}})^2 = (E_a^* + E_b^*)^2$$

Consider

$$\delta((p_a + p_b - p_1)^2 - m_2^2)$$

$$= \delta((p_a + p_b)^2 - 2p_1 \cdot (p_a + p_b) + m_1^2 - m_2^2)$$

in cms: $= \delta(s - 2E_1^*(E_a^* + E_b^*) + m_1^2 - m_2^2)$

$$= \delta(s - 2E_1^*\sqrt{s} + m_1^2 - m_2^2)$$

$$E_1^* = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}, \qquad p_1^* = \sqrt{E_1^{*2} - m_1^2} = \frac{\sqrt{\lambda(s, m_1^2, m_2^2)}}{2\sqrt{s}}$$

Kinematical function: $\lambda(x^2, y^2, z^2) = (x^2 - y^2 - z^2)^2 - 4y^2z^2$

$$= (x^2 - (y+z)^2)(x - (y-z)^2)$$

Two-body phase space

$$d\Phi_2 = \frac{1}{(2\pi)^6}\frac{d^3 p_1}{2E_1}\delta((p_a + p_b - p_1)^2 - m_2^2)$$

$$= \frac{1}{(2\pi)^6}\frac{p_1^*}{2} dE_1^* \, d\Omega^* \; \delta(s + m_1^2 - m_2^2 - 2E_1^*\sqrt{s})$$

Use that

$$\int \delta(f(x))dx = \frac{1}{|f'(x_0)|}\delta(x - x_0), \qquad f(x_0) = 0$$

$$\int \delta(ax)dx = \frac{1}{|a|}$$

Thus,

$$\int dE_1^* \; \delta(s + m_1^2 - m_2^2 - 2E_1^*\sqrt{s}) = \frac{1}{2\sqrt{s}}$$

$$d\Phi_2 = \frac{1}{(2\pi)^6}\frac{p_1^*}{2} d\Omega^* \frac{1}{2\sqrt{s}}$$

Two-body cross section in CMS

$$d\Phi_2 = \frac{1}{(2\pi)^6}\frac{1}{4\sqrt{s}}p_1^* d\Omega^*$$

Differential cross section:

$$d\sigma = \frac{(2\pi)^4}{F}\left|M_{if}\right|^2 d\Phi_2 = \frac{(2\pi)^4}{\underbrace{4p_a^*\sqrt{s}}_{F}}\left|M_{if}\right|^2 \underbrace{\frac{1}{(2\pi)^6}\frac{1}{4\sqrt{s}}p_1^* d\Omega^*}_{d\Phi_2}$$

in the cms:

$$d\sigma = \frac{1}{(2\pi)^2}\frac{p_1^*}{4^2 p_a^* s}\left|M_{if}\right|^2 d\Omega^*$$

$$p_a^* = \frac{\sqrt{\lambda(s,m_a^2,m_b^2)}}{2\sqrt{s}}$$

Cross section reads:

$$\sigma = \frac{1}{(2\pi)^2}\int \frac{p_1^*}{4^2 p_a^* s}\left|M_{if}\right|^2 d\Omega^*$$

$$p_1^* = \frac{\sqrt{\lambda(s,m_1^2,m_2^2)}}{2\sqrt{s}}$$

Two-body cross section in terms of invariants

Let's express the cross section in terms of Lorentz invariants (s,t), where

$$t = (p_a - p_1)^2 = m_a^2 + m_1^2 - 2(p_1 \cdot p_a)$$

in the cms:

$$(p_1 \cdot p_a) = E_a^* E_1^* - \vec{p}_a^{\,*} \cdot \vec{p}_1^{\,*} = E_a^* E_1^* - p_a^* \cdot p_1^* \cos\vartheta^*$$

$$t = m_a^2 + m_1^2 - 2E_a^* E_1^* + 2p_a^* \cdot p_1^* \cos\vartheta^*$$

$$\frac{dt}{d\cos\vartheta^*} = 2p_a^* \cdot p_1^* \quad \Longrightarrow \quad d\cos\vartheta^* = \frac{dt}{2p_a^* \cdot p_1^*}$$

$$d\Omega^* = d\cos\vartheta^* d\varphi^* = \frac{dt}{2p_a^* \cdot p_1^*} d\varphi^*$$

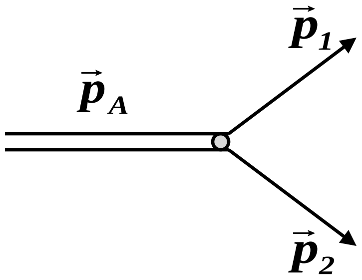
$$d\sigma = \frac{1}{(2\pi)^2} \frac{p_1^*}{4p_a^* s} \left|M_{if}\right|^2 d\Omega^* = \frac{1}{(2\pi)^2} \frac{p_1^*}{4p_a^* s} \left|M_{if}\right|^2 \frac{dt}{2p_a^* \cdot p_1^*} d\varphi^*$$

If the matrix element doesn't depend of φ: $\int_0^{2\pi} d\varphi^* = 2\pi$

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{64\pi s} \frac{1}{p_a^{*2}} \left|M_{if}\right|^2}$$

$$p_a^* = \frac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{2\sqrt{s}}$$

Decay rate A→1+2



Decay rate A→1+2

$$d\Gamma = \frac{(2\pi)^4}{F} \left| M_{if} \right|^2 d\Phi_{12}$$

$$F - flux: \quad F = 2m_A$$

$|M_{if}|^2$ – squared matrix element

$$d\Phi_{12} = \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \delta^4 \left(p_A - p_1 - p_2 \right) \frac{1}{\left((2\pi)^3 \right)^2}$$

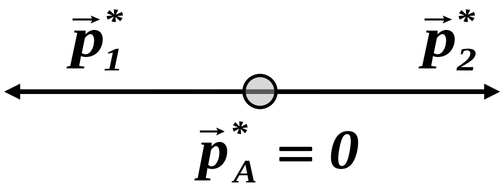
$$\frac{d^3 p}{2E} = \int_0^{\infty} dE d^3 p \ \delta(p_\mu p^\mu - m^2) = \int_{-\infty}^{\infty} d^4 p \delta(p^2 - m^2) \theta(E)$$

$$d\Phi_{12} = \frac{d^3 p_1}{2E_1} d^4 p_2 \delta(p_2^2 - m_2^2) \Theta(E_2) \delta^4 \left(p_A - p_1 - p_2 \right) \frac{1}{(2\pi)^6}$$

$$= \frac{d^3 p_1}{2E_1} \frac{1}{(2\pi)^6} \delta((p_A - p_1)^2 - m_2^2) = \frac{d^3 p_1}{2E_1} \frac{1}{(2\pi)^6} \delta(m_A^2 + m_1^2 - 2(p_A \cdot p_1) - m_2^2)$$

Futher considerations require to choose a reference frame!

Decay rate A→1+2



Rest frame of A = center-of-mass system 1+2 :

$$\vec{p}_A^{\,*} = \vec{p}_1^{\,*} + \vec{p}_2^{\,*} = 0$$

$$p_A = (E_A, \vec{p}_A) \underbrace{=}_{in\ cms\ 1+2} (m_A, 0)$$

$$(p_A \cdot p_1) = 2E_A^* E_1^* = 2m_A E_1^*$$

$$d\Phi_{12} = \frac{1}{(2\pi)^6} \frac{p_1^*}{2} dE_1^* d\Omega^* \delta(m_A^2 + m_1^2 - 2m_A E_1^* - m_2^2)$$

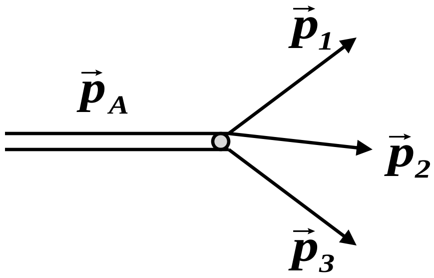
$$= \frac{1}{(2\pi)^6} \frac{p_1^*}{2} d\Omega^* \frac{1}{2m_A} = \frac{1}{(2\pi)^6} \frac{p_1^*}{2m_A} d\Omega^*$$

$$d\Gamma = \frac{(2\pi)^4}{F} |M_{if}|^2 d\Phi_{12} = \frac{(2\pi)^4}{2m_A} |M_{if}|^2 \frac{1}{(2\pi)^6}$$

Decay rate:

$$d\Gamma = \frac{1}{32\pi^2} |M_{if}|^2 \frac{p_1^*}{m_A^2} d\Omega^*$$

Dalitz decay A→1+2+3



Decay rate for Dalitz decay A→1+2+3

$$d\Gamma = \frac{(2\pi)^4}{F}\left|M_{if}\right|^2 d\Phi_{13}$$

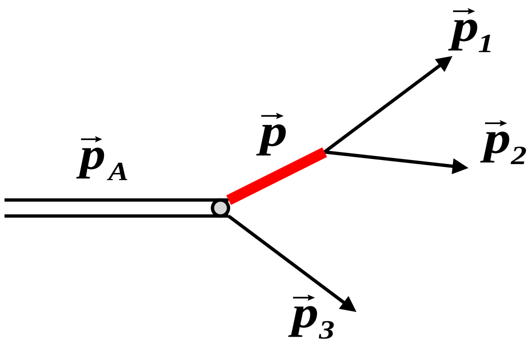
$$F - flux: \quad F = 2m_A$$

$|M_{if}|^2$ – squared **matrix element**

$$d\Phi_{13} = \frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\frac{d^3p_3}{2E_3}\delta^4(p_A - p_1 - p_2 - p_3)\frac{1}{\left((2\pi)^3\right)^3}$$

Introduce a new variable $p = p_1 + p_2$

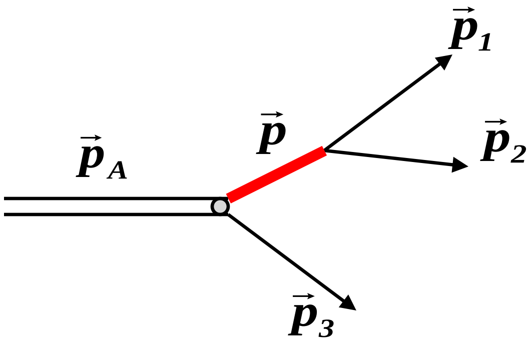
by δ-function: $\int d^4p\,\delta^4(p - (p_1 + p_2))$



Thus, A→1+2+3 is treated as **A→R_{12}+3**

$$p \equiv (E, \vec{p}),$$
$$p^2 \equiv s_{12} = E^2 - \vec{p}^2$$

Dalitz decay A→1+2+3



$$d\Phi_{13} = \frac{1}{(2\pi)^9}\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\frac{d^3p_3}{2E_3}\delta^4(p_A - p_1 - p_2 - p_3)$$
$$\times d^4p\,\delta^4(p - (p_1 + p_2))$$

$$d^4p = ds_{12}\frac{d^3p}{2E}, \qquad s_{12} \equiv (p_1 + p_2)^2 = p^2$$

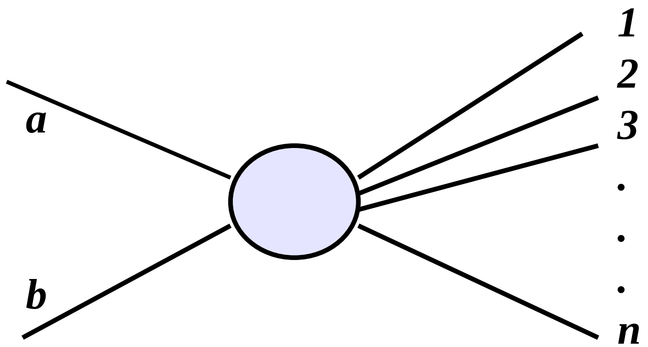
$$d\Phi_{13}(p_A, p_1, p_2, p_3) = (2\pi)^3 ds_{12}\left\{\frac{1}{(2\pi)^6}\frac{d^3p}{2E}\frac{d^3p_3}{2E_3}\delta^4(p_A - p - p_3)\right\}$$
$$\times\left\{\frac{1}{(2\pi)^6}\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\delta^4(p - p_1 - p_2)\right\}$$

$$d\Phi_{13}(p_A, p_1, p_2, p_3) = (2\pi)^3 ds_{12}\cdot d\Phi_{12}(p_A, p, p_3)\,d\Phi_{12}(p, p_1, p_2)$$

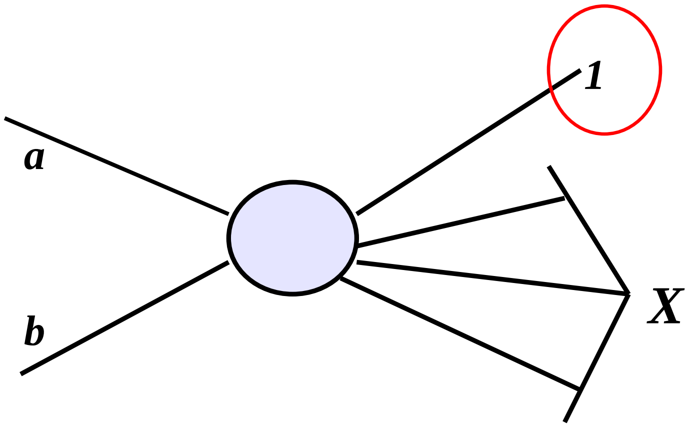
➔ **3-body phase space is replaced by the product of 2-body phase space factors by introducing an ‚intermediate state‘: A → 1+2+3 → R_{12}+3**

Inclusive reactions

Multiparticle production a+b → 1+2+3+…+n



exclusive reaction



inclusive reaction

Cross section a+b→1+2+…+n (1+X):

$$d\sigma = \frac{(2\pi)^4}{F}\left|M_{if}\right|^2 d\Phi_n$$

$$flux : F = 4\sqrt{(p_a p_b)^2 - m_a^2 m_b^2} \xrightarrow{cms} p_a^* \sqrt{s} \xrightarrow{Lab.\,frame} m_a p_a$$

n-body phase space:

$$d\Phi_n = \left[\prod_{i=1}^{n} \frac{d^3 p_i}{2E_i}\right] \delta^4\left(p_a + p_b - \sum_{i=1}^{n} p_i\right) \frac{1}{\left((2\pi)^3\right)^n}$$

References

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PDG: Kinematics:
http://pdg.lbl.gov/2019/reviews/rpp2019-rev-kinematics.pdf