Lecture

Nambu-Jona-Lasinio model

WS2018/19: 'Physics of Strongly Interacting Matter'
QCD Lagrangian:

\[ L_{QCD}(x) = \bar{\psi}(x) \left( i \gamma^\mu \left[ \partial_\mu - ig t^a A^a_\mu \right] - \hat{M}^0 \right) \psi(x) - \frac{1}{4} G^a_{\mu\nu}(x) G^{\mu\nu a}(x) \]  

(1)

Gluonic field strength tensor:

\[ G^a_{\mu\nu}(x) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu(x) A^c_\nu \]  

(2)

\[ \psi(x) \] - quark field

Flavor space \hspace{1cm} Dirac space \hspace{1cm} Color space

\[ q = u, d, s \hspace{1cm} \mu = 0, 1, 2, 3 \hspace{1cm} c = r, b, g \]

In flavor space (3 flavors):

\[ \psi(x) = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \]

Mass term:

\[ \hat{M}^0 = \begin{pmatrix} m_u^0 & 0 & 0 \\ 0 & m_d^0 & 0 \\ 0 & 0 & m_s^0 \end{pmatrix} \]  

(3)

3x3 diagonal matrix in flavor space with the bare quark masses on the diagonal.
Symmetries of QCD

1) local $SU_c(3)$ color gauge transformation (by construction)

2) global $SU_f(3)$ flavor symmetry

3) for massless quarks ONLY:
   - chiral invariance of QCD lagrangian: $SU_f(3)_R \times SU_f(3)_R$.

However, chiral symmetry is a spontaneously broken since quarks have non-zero masses.

⇒ To explore more simple effective Lagrangian with the same symmetries for the quark degrees of freedom, however, discarding the gluon dynamics completely.

QCD:
- Consider $q - \bar{q}$ interaction, where $r_c$ - range of interaction
- Consider a low energy (or small $Q^2$) ⇒ large distance $d^2 (d^2 \sim Q^2$)

If $d > r_c$ use an effective model!

![Diagram of QCD interaction]

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3
Nambu-Jona-Lasinio model: basic ideas

from q-g interaction ➔ NJL effective model

NJL model (1961):
- local quark interaction
- gluon fields are integrated out


Yoichiro Nambu 1921-2015

Giovanni Jona-Lasinio 1932

Nobel Prize in Physics (2008)
From QCD to NJL Lagrangian

\[
L_{QCD}(x) = \bar{\psi}(x) \left( i \gamma^\mu \left[ \partial_\mu - ig t^a A_\mu^a \right] - \hat{M}^0 \right) \psi(x) - \frac{1}{4} G_{\mu\nu}^a(x) G^{\mu\nu}^a(x) \tag{1}
\]

- Euler-Lagrange equations:

\[
\frac{\mathcal{L}}{\partial \varphi} - \partial_\mu \left[ \frac{\mathcal{L}}{\partial (\partial_\mu \varphi)} \right] = 0 \tag{4}
\]

for any field \( \varphi \) (the same equation for \( \varphi \)): e.g. \( \varphi = \Psi(x) \) or \( A^a_\mu(x) \).

1) Consider quark field \( \bar{\psi}(x) \)

\[
\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0, \tag{5}
\]

since the second term in Eq.(4) is equal to zero while no terms with \( \partial_\mu \bar{\psi} \).

From eqs.(1,5) follows that

\[
(i \gamma^\mu \partial_\mu - \hat{M}^0) \Psi_q(x) = -g \gamma^\mu t^a A_\mu^a(x) \bar{\psi}(x) \Psi_q(x) . \tag{6}
\]
From QCD to NJL Lagrangian

\[ L_{QCD}(x) = \bar{\psi}(x) \left( i \gamma^\mu \left[ \partial_\mu - igt^a A^a_\mu \right] - \hat{M} \right) \psi(x) - \frac{1}{4} G^a_{\mu \nu}(x) G^{\mu \nu a}(x) \]  

(1)

2) Consider field \( A^a_\nu(x) \):

Euler-Lagrange equation for gluon field:

\[ \frac{\mathcal{L}}{\partial A^a_\nu(x)} - \partial_\mu \left[ \frac{\mathcal{L}}{\partial(\partial_\mu A^a_\nu(x))} \right] = 0. \]  

(7)

- Using (1) \( \Rightarrow \) first term in eq. (7):

\[ \frac{\mathcal{L}}{\partial A^a_\nu(x)} = g \bar{\Psi} \gamma_\nu t^a \Psi + \Pi_g, \]  

(8)

where \( \Pi_g \) is the 'self-energy' of gluons:

\[ \Pi_g = \frac{\partial}{\partial A^a_\nu} \left[ -\frac{1}{4} G^a_{\mu \nu}(x) G^{\mu \nu a}(x) \right] \]  

(9)

- Using (1) \( \Rightarrow \) second term in eq. (7):

\[ \frac{\mathcal{L}}{\partial(\partial_\mu A^a_\nu(x))} = \partial_\mu A^a_\nu(x) \]  

(10)

- Substitute (8), (10) into (7):

\[ \partial_\mu \partial_\nu A^a_\nu(x) = g \bar{\Psi} \gamma_\nu t^a \Psi + \Pi_g, \]  

(11)
From QCD to NJL Lagrangian

- Use approximation: \( \Pi_g = M_g^2 \)
  
  Constituent gluon mass \( \neq 0 \) due to self-interactions of gluons.

\[
\begin{align*}
\partial_\mu \partial^\nu A^a_\nu(x) &= g\bar{\Psi}\gamma_\mu t^a \Psi + \Pi_g, \\
\left( \partial_\mu \partial^\mu + M_g^2 \right) A^a_\nu(x) &\approx -g\bar{\Psi}\gamma_\nu t^a \Psi,
\end{align*}
\]

Then from eq. (11) \( \Rightarrow \)

- Solution of eq. (12):
  \[
  A^a_\nu(x) = -\int d^4x \ G(x - x') \ g\bar{\Psi}(x')\gamma_\nu t^a \Psi(x')
  \]

  Green function:
  \[
  G(x - x') = -\int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(x-x')}}{q^2 - \Pi_g}.
  \]

- Approximation: consider **low energy physics**: i.e. small momentum or large distance

\[
\begin{align*}
q^2 &\ll M_g^2 \\
G(x - x') &= -\int \frac{d^4q}{(2\pi)^4} \ \left. \frac{e^{iq(x-x')}}{q^2 - \Pi_g} \right|_{q^2 \to 0} \quad \rightarrow \quad -\frac{1}{M_g^2} \int \frac{d^4q}{(2\pi)^4} e^{iq(x-x')} \delta(x-x')
\end{align*}
\]
From QCD to NJL Lagrangian

From eq. (15) \( \Rightarrow \)

\[ G(x - x') \Rightarrow \delta^4(x - x') \cdot M_g^{-2}. \]  

\[ (16) \]

- Substitute (16) into (13):

\[ A^a_\nu(x) = -\frac{g}{M_g^2} \bar{\Psi}(x) \gamma_\nu t^a \Psi(x). \]  

\[ (17) \]

- Substitute (17) into (6):

\[ (i \gamma^\mu \partial_\mu - \hat{M}^0) \Psi(x) - G_c^2 \gamma^\mu t^a (\bar{\Psi}(x) \gamma_\mu t^a \Psi(x)) \Psi(x) = 0 \]

\[ \sim A^a_\mu(x) \]  

\[ (18) \]

where the low energy coupling constant:

\[ G_c^2 = g^2 / M_g^2 \]  

\[ (19) \]

\[ \mathcal{L}_{eff} = \bar{\Psi}(x)(i \gamma^\mu \partial_\mu - \hat{M}^0) \Psi(x) - G_c^2 \sum_{a=1}^{8} \left( \bar{\Psi}(x) \gamma^\mu t^a \Psi(x) \right)^2. \]  

\[ (20) \]

**NJL Lagrangian**
• Consider solution of (20) in mean-field level

**Reminder:**

mean-field level = 4-point function convolutes as a product of 2-point functions:

\[
\langle \bar{\Psi}(1)\bar{\Psi}(2)\Psi(2)\Psi(1) \rangle \Rightarrow \langle \bar{\Psi}(2)\bar{\Psi}(2) \rangle \langle \bar{\Psi}(1)\bar{\Psi}(1) \rangle - \langle \bar{\Psi}(2)\bar{\Psi}(1) \rangle \langle \bar{\Psi}(1)\bar{\Psi}(2) \rangle
\]

Hartree term - Fock term (antisymmetrization)

! **Problems** with formulation of quark dynamics on mean-field level –

antisymmetrization generates a further mixing of color, flavor and Dirac indices.

Thus, it is more convenient to **introduce a Fiertz transformation**, i.e. to antisymmetrize the 4-point interaction
Fiertz transformation

\[ \bar{u}(p') O u(k) \bar{u}(k') O' u(p) = \sum_{i,l} a_i^l \bar{u}(p') \Gamma_i u(k) \bar{u}(k') \Gamma_l u(p) \]  

where the coefficients

\[ a_i^l = \frac{1}{Tr \Gamma_i \Gamma^i \cdot Tr \Gamma_l \Gamma^l \cdot Tr O \Gamma_i O' \Gamma^l} \]  

Here \( O, O' \) are the 4 × 4 matrices, \( \Gamma^i \) - Dirac matrices, \( i(l) = S, V, T, A, P \) - index \( i(l) \) corresponds to scalar, vector, tensor, axial-vector and pseudo-scalar parts:

\[ \Gamma^S = 1 \]
\[ \Gamma^V = \gamma^\mu \]
\[ \Gamma^T = \sigma^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu] \]
\[ \Gamma^A = \gamma^\mu \gamma_5 \]
\[ \Gamma^P = \gamma_5 \]

Dirac matrices \( \Gamma^i \) follow Dirac-Clifford algebra.
The Fiertz transformation generates color singlet and color octet terms:

\[ \mathcal{L}_{N\!J\!L} = \mathcal{L}_{q}^{\text{singlet}} + \mathcal{L}_{q}^{\text{octet}} \]  \hspace{1cm} (25)  

- **Color singlet term:**

\[ \mathcal{L}_{q}^{\text{singlet}} = \tilde{\Psi}_q(x) \left( i \gamma^\mu \partial_\mu - \hat{M}^0 \right) \Psi_q(x) \]

\[ = G_s^2 \sum_{i=1}^{8} \left\{ \left( \frac{\tilde{\Psi}_q(x) \gamma_i}{2} \Psi_q(x) \right)^2 + \left( \frac{\tilde{\Psi}_q(x) i \gamma_5 \gamma_i}{2} \Psi_q(x) \right)^2 \right\} \]

\[ + G_v^2 \sum_{i=1}^{8} \left\{ \left( \frac{\tilde{\Psi}_q(x) \gamma_\mu}{2} \Psi_q(x) \right)^2 + \left( \frac{\tilde{\Psi}_q(x) \gamma_5 \gamma_\mu}{2} \Psi_q(x) \right)^2 \right\} \]

Here \( G_s^2 = 2G_v^2 = \frac{8}{9} G_C^2 \)
Fiertz transformation for the NJL Lagrangian

\[ \mathcal{L}_{NJL} = \mathcal{L}_q^{\text{singlet}} + \mathcal{L}_q^{\text{octet}} \]  

(25)

**Color octet term:**

\[ \mathcal{L}_q^{\text{octet}} = - G_C^2 \sum_{a=1}^{8} \left( \bar{\Psi}_q(x) \gamma_\mu t^a \Psi_q(x) \right)^2 \]

\[ - \frac{2}{3} G_C^2 \sum_{a=1}^{8} \sum_{i=1}^{8} \left\{ \left( \bar{\Psi}_q(x) \frac{\lambda_i}{2} t^a \Psi_q(x) \right)^2 + \left( \bar{\Psi}_q(x) i \gamma_5 \frac{\lambda_i}{2} t^a \Psi_q(x) \right)^2 \right\} \]

\[ + \frac{1}{3} G_C^2 \sum_{a=1}^{8} \sum_{i=1}^{8} \left\{ \left( \bar{\Psi}_q(x) \gamma_\mu \frac{\lambda_i}{2} t^a \Psi_q(x) \right)^2 + \left( \bar{\Psi}_q(x) \gamma_\mu \gamma_5 \frac{\lambda_i}{2} t^a \Psi_q(x) \right)^2 \right\} \]

(27)
• Consider only the color singlet term (16)

- Perform the summation over the flavor \[ \sum_{i=1}^{8} \ldots \lambda_i \ldots . \]
- Let’s write (16) in matrix form:

\[ L_q = \sum_{k=u,d,s} \bar{\Psi}_k(x) (i\gamma^\mu \partial_\mu - m_k^0) \Psi_k(x) \]

\[ + \sum_{k=u,d,s} \left\{ \frac{G_S^2}{2} \left[ \left( \bar{\Psi}_k(x) \Psi_k(x) \right)^2 + \left( \bar{\Psi}_k(x)i\gamma_5 \Psi_k(x) \right)^2 \right] \right\} \]

\[ - \frac{G_V^2}{2} \left[ \left( \bar{\Psi}_k(x) \gamma_\mu \Psi_k(x) \right)^2 + \left( \bar{\Psi}_k(x)i\gamma_5 \gamma_\mu \Psi_k(x) \right)^2 \right] \right\} \]

- **Approximation**: mean-field level – keep only Hartree term:

\[ \langle \bar{\Psi}(1)\bar{\Psi}(2)\hat{O}_1\hat{O}_2\Psi(2)\Psi(1) \rangle \Rightarrow \langle \bar{\Psi}(1)\hat{O}_1\Psi(1) \rangle \langle \bar{\Psi}(2)\hat{O}_2\Psi(2) \rangle \]

\[ \text{if } \hat{O}_1 = \hat{O}_2, \langle 4 \rangle \Rightarrow \langle 2 \rangle^2. \]

**Note**: In Hartree limit for the systems of positive parity the pseudoscalar and pseudovector terms (\( \sim \gamma_5 \)) vanish, i.e. \( \langle \gamma^5 \ldots \rangle = 0, \langle \gamma^5 \gamma_\mu \ldots \rangle = 0 \)

We will consider positive parity \( P = +1 \) which corresponds to vacuum properties, nucleon, nuclear matter.
Consider energy density in Hartree mean-field level

- **Hamiltonian density:**

$$\mathcal{H}(x) = \sum_i \partial^0 \varphi_i \frac{\partial \mathcal{L}(x)}{\partial (\partial^0 \varphi_i)} - \mathcal{L}(x),$$  \hspace{1cm} (30)

where $\varphi$ any field operator, $\partial^0$ is time derivative.

For Dirac field $\varphi$:

$$\frac{\partial \mathcal{L}(x)}{\partial (\partial^0 \varphi_i)} = i\bar{\varphi}(x)\gamma^0 = i\varphi^+(x)$$  \hspace{1cm} (31)

$$\frac{\partial \mathcal{L}(x)}{\partial (\partial^0 \varphi_i)} = 0$$

- Substitute (31) into (30) for $P = +1$, i.e. for S,V only:

$$\mathcal{H}_{NJL}(x) = \sum_k \bar{\Psi}_k(x)(i\gamma^\mu \partial_\mu + m_k^0)\Psi_k(x) - \frac{G_S^2}{2} \sum_k \left( \bar{\Psi}_k(x)\Psi_k(x) \right)^2$$

$$+ \frac{G_V^2}{2} \sum_k \left( \bar{\Psi}_k(x)\gamma_\mu \Psi_k(x) \right)^2$$  \hspace{1cm} (33)
\[ \varepsilon = \langle \mathcal{H}_{NJL}(x) \rangle = \sum_k \langle \bar{\Psi}_k(x) (i \gamma^\mu \partial_\mu + m_0^k) \Psi_k(x) \rangle \]

\[ = \sum_k \langle \bar{\Psi}_k(x) (i \gamma^\mu \partial_\mu + m_0^k - G^2_S \langle \bar{\Psi}_k(x) \Psi_k(x) \rangle) \Psi_k(x) \rangle \]

\[ - \frac{G^2_S}{2} \sum_k \langle \bar{\Psi}_k(x) \Psi_k(x) \rangle^2 + \frac{G^2_V}{2} \sum_k \langle \bar{\Psi}_k(x) \gamma_\mu \Psi_k(x) \rangle^2 \]

\[ \varepsilon \quad = \quad \sum_k \langle \bar{\Psi}_k(x) (i \gamma^\mu \partial_\mu + m_0^k - G^2_S \langle \bar{\Psi}_k(x) \Psi_k(x) \rangle) \Psi_k(x) \rangle \]

\[ - \frac{G^2_S}{2} \sum_k \langle \bar{\Psi}_k(x) \Psi_k(x) \rangle^2 \quad + \quad \frac{G^2_V}{2} \sum_k \langle \bar{\Psi}_k(x) \gamma_\mu \Psi_k(x) \rangle^2 \]

\[ \text{scalar interaction density} \quad \text{vector interaction density} \]

\[ \Rightarrow \text{Gap equation for the effective mass:} \]

\[ m_k = m_0^k - G^2_S \langle \bar{\Psi}_k(x) \Psi_k(x) \rangle \]
Consider Equation of motion (EoM) for field $\bar{\Psi}_k$:

$$\frac{\partial \mathcal{L}_{NJL}}{\partial \bar{\Psi}} - \partial^{\mu} \left[ \frac{\partial \mathcal{L}_{NJL}}{\partial (\partial^{\mu} \bar{\Psi})} \right] = 0$$

(37)

Second term in (37) equal to zero since there is no terms with $\partial^{\mu} \bar{\Psi}$ in the NJL Lagrangian. Thus,

$$\frac{\partial \mathcal{L}_{NJL}}{\partial \bar{\Psi}} = 0$$

(38)

• Substitute in (38) an explicit form of Lagrangian (28) keeping only scalar and vector terms:

$$\frac{\partial \mathcal{L}_{NJL}}{\partial \bar{\Psi}} = (i\gamma^{\mu} \partial_{\mu} - m^{0}_k) \Psi_k + \frac{G^2_S}{2} 2(\bar{\Psi}_k \Psi_k) \Psi_k - \frac{G^2_V}{2} 2(\bar{\Psi}_k \gamma^{\mu} \Psi_k) \Psi_k = 0$$

(39)

Thus, we obtain an equation of motion for the field $\Psi_k$:

$$(i\gamma^{\mu} \partial_{\mu} - m^{0}_k) \Psi_k + G^2_S(\bar{\Psi}_k \Psi_k) \Psi_k - G^2_V(\bar{\Psi}_k \gamma^{\mu} \Psi_k) \Psi_k = 0$$

(40)
Now consider the mean-field level of EoM (40):

\[
\langle \bar{\Psi}_k \Psi_k \rangle \rightarrow \langle \bar{\Psi}_k \Psi_k \rangle \\
\langle \bar{\Psi}_k \gamma_\mu \Psi_k \rangle \rightarrow \langle \bar{\Psi}_k \gamma_\mu \Psi_k \rangle
\]

NJL equation-of-motion (EoM):

\[
\left[ -i \gamma^\mu \left( \partial_\mu - G_V^2 \langle \bar{\Psi}_k \gamma_\mu \Psi_k \rangle \right) + \left( m_k^0 - G_S^2 \langle \bar{\Psi}_k \Psi_k \rangle \right) \right] \Psi_k = 0
\]

Here \( p_k \) is an effective momentum and \( m_k \) is an effective mass in Hartree mean-field limit of NJL-model:

\[
p_k = \partial_\mu - G_V^2 \langle \bar{\Psi}_k \gamma_\mu \Psi_k \rangle \\
m_k = m_k^0 - G_S^2 \langle \bar{\Psi}_k \Psi_k \rangle
\]
The Eq. (44) is a gap equation for the effective mass $m_k$:

$$
    m_k = m_k^0 - G_S^2 \langle \bar{\Psi}_k \Psi_k \rangle
$$

(45)

Here all occupied states

$$
    \langle \rangle = \langle Dirac \ see \rangle + \langle\text{valence quarks} \rangle.
$$

(46)

I. Consider term $\langle \bar{\Psi}_k \Psi_k \rangle$ averaged over vacuum states only $|0\rangle$, i.e. Dirac see. Use a general definition:

$$
    \langle \bar{\Psi}_k \Psi_k \rangle_0 = -i \ \text{Tr} \{ S_F(0) \},
$$

where $S_F$ is a quark propagator in coordinate space:

$$
    S_F(x - y) = -i \ \langle T \left[ \Psi_k(x), \bar{\Psi}_k(y) \right] \rangle,
$$

(47)

Here $T$ is a time ordering operator.
A quark propagator in momentum space:

\[
S_F(p) = g \int_0^\Lambda \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{\hat{p} - m + i\epsilon}
\]

where degeneracy factor \( g = 6 \). Using that

\[
\int_0^\Lambda d^4p = \int d^4p \Theta(\Lambda^2 - p_0^2 + \vec{p}^2)
\]

where \( p_0 = (\vec{p}^2 + m_k^2)^{1/2} \) is energy, one obtains

\[
\langle \bar{\Psi}_k \Psi_k \rangle = -\frac{g}{(2\pi)^3} \int d^3p \frac{m_k}{\sqrt{\vec{p}^2 + m_k^2}} \Theta(\Lambda_S - |\vec{p}|)
\]

Thus, for the effective mass \( m_k \) in the presents of Dirac see only, the gap equation (45) in the momentum space is

\[
m_k = m_k^0 + G_S^2 \frac{g}{(2\pi)^3} \int d^3p \frac{m_k}{\sqrt{\vec{p}^2 + m_k^2}} \Theta(\Lambda_S - |\vec{p}|)
\]

\( \Lambda_S \) is a cutoff parameter to regularize the divergent integral over Dirac see.
The coupling constants $G_S$ and $\Lambda_S$ can be determined using the Gell-Mann–Oakes–Renner relation for ’micro-makro worlds’, i.e. scalar quark condensate and pion coupling constant:

$$m_\pi^2 f_\pi^2 = -(m_u^0 + m_d^0)\langle \bar{u}u \rangle,$$

(52)

using that $\langle \bar{u}u \rangle \simeq \langle \bar{d}d \rangle$.

Here $f_\pi = 93.3$ MeV is the pion decay constant, $m_\pi = 140$ MeV is the pion mass. Choosing $m_u^0 \simeq 7$ MeV, we get $\langle \bar{u}u \rangle^{1/3} \simeq -230$ MeV.

From the other hand, such value of scalar quark condensate one can get from gap equation (45)-(52) by choosing a cutoff $\Lambda_S=0.6$ GeV and coupling constant $G_S= 4.95$ GeV$^{-1}$.

$m_u = \text{Const}$ up to $p = \Lambda_S$ in NJL model.

From Dyson-Schwinger approach we know that $m_u$ drops at large $p$.

Thus, NJL model can be used only for low momentum $p$ or low energy $E$. 

\[ \text{NJL} \]
II. Gap equation in the presence of valence quarks on the top of Dirac see:

\[
m_k(\vec{r}) = m_k^0 - G_S^2 \frac{g}{(2\pi)^3} \int d^3 p \frac{m_k(\vec{r})}{\sqrt{p^2 + m_k^2(\vec{r})}} f_k(\vec{r}, \vec{p}) \quad \Delta \text{valence}
\]

\[
+ G_S^2 \frac{g}{(2\pi)^3} \int d^3 p \frac{m_k(\vec{r})}{\sqrt{p^2 + m_k^2(\vec{r})}} \Theta(\Lambda_S - |\vec{p}|) \quad \Delta \text{see}
\]

Here \( f_k(\vec{p}) \) is a phase-space distribution of a single \( k \)-quark, \( k = u, d, s \).

One can write Eq. (53) in a schematic form:

\[
m_k(\vec{r}) = m_k^0 - \Delta_{\text{valence}} + \Delta_{\text{see}}
\]

As follows from (54): the see term increases \( m_k \), whereas the valence term decreases \( m_k \).
Let’s consider energy for \( k = u \)-quark:

\[
E = \pm \sqrt{p^2 + m_u^2}
\]

For \( p = 0 \), \( E^- = -m_u, E^+ = m_u \).

In NJL model a Dirac see is taken up to \( E_{\lambda_s} \), where

\[
E_{\lambda_s} = -\sqrt{\lambda_s^2 + m_u^2}
\]

As illustrated in Figure, occupied states exist above \( E^+ \) and below \( E^- \), so the gap is

\[
\Delta E_{\text{gap}} = 2m_k
\]
I. Nuclear matter

Let’s consider a cold nuclear matter $T = 0, \mu_q \neq 0$

where $T$ is a temperature, $\mu_q$ is a quark chemical potential.

The condition $\mu_q$ is non-zero means that the number of quarks is not equal to the number of antiquarks - there are the baryons in the system.

Note that the baryon density is equal 3 times quark density: $\rho_B = 3\rho_q$

Operator of quark density $\hat{\rho}_q = q^+ q$

and quark density: $\rho_q = \langle q^+ q \rangle$

• Consider the phase-space distribution function of single $u$-quarks:

$$f_u(\vec{r}, \vec{p}) = \Theta(p_F(\vec{r}) - |\vec{p}|)$$

Here $p_F$ is a Fermi momentum depends on density:

$$p_F(\vec{r}) = \left( \frac{6}{g\pi^2} \right)^{1/3} \rho_u^{1/3}(\vec{r})$$
The gap equation relates scalar quark condensate to the quark (or baryon) density:

$$\langle q^+ q \rangle = \rho_q \sim p_F^{1/3}$$

As demonstrated in Figure the quark condensate drops to zero at $3 \div 4 \rho_0$, where $\rho_0 = 0.17 \text{ fm}^{-3}$ is the normal nuclear density.

⇒ **NJL for cold nuclear matter:** restoration of chiral symmetry at large quark density!
II. Meson gas

Let’s consider meson gas

\[ T \neq 0, \mu_q = 0 \]

- Consider the phase-space distribution function of single \( u \)-quarks for meson gas:

\[ f_u(p, T) = \frac{1}{1 + \exp(\frac{\varepsilon(p)}{T})} \]

where \( \varepsilon(p) = \sqrt{p^2 + m_u^2} \)

Note: for homogeneous system we neglect the \( r \)-dependence.

\( \rightarrow \) NJL for meson gas: a restoration of chiral symmetry at \( T \sim 160 \text{ MeV} \) (consistent with lattice QCD)
III. Nuclear matter + meson gas

\[ T \neq 0, \mu_q \neq 0 \]

- Consider the phase-space distribution function of single \( u \)-quarks for nuclear matter + meson gas:

\[
f_u(\vec{p}, T) = \frac{1}{1 + \exp \left( \frac{\varepsilon(\vec{p}) - \mu_u}{T} \right)} + \frac{1}{1 + \exp \left( \frac{\varepsilon(\vec{p}) + \mu_u}{T} \right)}
\]

- **Number of quarks:**

\[
N_q = \frac{g}{(2\pi)^3} \int_0^\infty dp \left\{ \frac{4\pi p^2}{1 + \exp \left( \frac{\varepsilon(\vec{p}) - \mu_u}{T} \right)} + \frac{4\pi p^2}{1 + \exp \left( \frac{\varepsilon(\vec{p}) + \mu_u}{T} \right)} \right\}
\]

- **Density of quarks:**

\[
\rho_q = \frac{N_q}{V}
\]

- The mass of a constituent \( u \)-quark as a function of temperature \( T \) and density \( \rho \) (in units of \( \rho_0 \))

\[ \Rightarrow \text{NJL for hot nuclear matter:}
\]

restoration of chiral symmetry at large temperature and quark density!