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Lecture
Symmetries of QCD

Elena Bratkovskaya, WS2023/24:
‚Physics of Strongly Interacting Matter‘

Symmetries and conservation laws

- The interactions are defined by symmetry principles
- Symmetries imply conservation laws, in particular conserved currents

From the invariance with respect to

1) continuous transformations, i.e. shifts in space,

➔ momentum conservation

2) time shift transformations

➔ energy conservation

3) rotations in space

➔ angular momentum conservation

Global Symmetries

Consider the free fermion Lagrangian

$$L = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi \qquad (1)$$

- I) This Lagrangian has the global U(1) invariance:

i.e. an invariance under global phase transformations

$$\psi(x) \to e^{i\alpha}\psi(x) \qquad (2)$$

α – arbitrary real constant

Phases transformations $U(\alpha) \equiv e^{i\alpha}$ with α=const and real ➔ correspond to the unitary abelian group U(1) (3)

Abelian means that the commutative multiplication low holds:

$$U(\alpha_1)\,U(\alpha_2) = U(\alpha_2)\,U(\alpha_1) \qquad (4)$$

since the complex numbers (3) commute, i.e. $[U(\alpha_1), U(\alpha_2)] = 0$

Global Symmetries

Consider now an infinitely small transformation of the U(1) group:

$$\begin{aligned}
&\psi(x) \rightarrow e^{i\alpha}\psi(x) \rightarrow (1+i\alpha+\ldots)\ \psi(x)\\
&\partial_\mu\psi(x) \rightarrow e^{i\alpha}\partial_\mu\psi(x) \rightarrow (1+i\alpha+\ldots)\ \partial_\mu\psi(x)\\
&\overline{\psi}(x) \rightarrow e^{-i\alpha}\overline{\psi}(x) \rightarrow (1-i\alpha+\ldots)\ \overline{\psi}(x)
\end{aligned} \tag{5}$$

Keep only the first term in the Taylor expansion:

$$\psi(x) \rightarrow (1+i\alpha)\ \psi(x) \tag{6}$$

The invariance of the Lagrangian under the transformation (6) means that

$$\boxed{\delta\ L = 0} \tag{7}$$

⇨

$$\begin{aligned}
&\delta\ L = \delta(i\,\overline{\psi}\,\gamma^\mu\partial_\mu\psi\ -\ m\,\overline{\psi}\psi)\\
&\Rightarrow \delta\ L(\psi, \partial_\mu\psi, \overline{\psi}, \partial_\mu\overline{\psi})
\end{aligned} \tag{8}$$

Note: $\delta\ L(f) = \dfrac{\partial L}{\partial f}\delta f$

Symmetries and Conservation Laws

$\delta\ L = \delta(i\,\overline{\psi}\,\gamma^{\mu}\partial_{\mu}\psi\ -\ m\,\overline{\psi}\psi)$

$\Rightarrow \delta\ L(\psi, \partial_{\mu}\psi, \overline{\psi}, \partial_{\mu}\overline{\psi})$

$$\delta\ L = \frac{\partial L}{\partial\psi}\delta\psi + \frac{\partial L}{\partial(\partial_{\mu}\psi)}\delta(\partial_{\mu}\psi) + \delta\overline{\psi}\frac{\partial L}{\partial\overline{\psi}} + \delta(\partial_{\mu}\overline{\psi})\frac{\partial L}{\partial(\partial_{\mu}\overline{\psi})} \tag{9}$$

Substitute (6) in (9) :

$$\delta\ L = \frac{\partial L}{\partial\psi}(i\alpha\,\psi) + \underline{\frac{\partial L}{\partial(\partial_{\mu}\psi)}(i\alpha\partial_{\mu}\psi)} + [conjugated\ \overline{\psi}] \tag{10}$$

use that

$$\partial_{\mu}\left(\frac{\partial L}{\partial(\partial_{\mu}\psi)}\psi\right) = \underline{\frac{\partial L}{\partial(\partial_{\mu}\psi)}\partial_{\mu}\psi} + \partial_{\mu}\left(\frac{\partial L}{\partial(\partial_{\mu}\psi)}\right)\psi \tag{11}$$

Substitute (11) in (10) :

$$\delta\ L = i\alpha\,\underline{\left[\frac{\partial L}{\partial\psi} - \partial_{\mu}\left(\frac{\partial L}{\partial(\partial_{\mu}\psi)}\right)\right]\psi}\ +\ i\alpha\ \partial_{\mu}\left(\frac{\partial L}{\partial(\partial_{\mu}\psi)}\psi\right)\ + \ldots \tag{12}$$

‖

C **due to the Euler-Lagrange equation!**

Symmetries and conservation laws

$$L = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi$$

To fulfill $\delta L = 0$ ⇒

$$i\alpha\,\partial_{\mu}\left(\frac{\partial L}{\partial(\partial_{\mu}\psi)}\psi - \bar{\psi}\frac{\partial L}{\partial(\partial_{\mu}\bar{\psi})}\right) \equiv \partial_{\mu}j^{\mu} = 0 \qquad (13)$$

$$\frac{\partial L}{\partial(\partial_{\mu}\psi)} = i\bar{\psi}\gamma^{\mu}$$

substitute in (13) =>

$$\frac{\partial L}{\partial(\partial_{\mu}\bar{\psi})} = 0$$

$$i\alpha\,\partial_{\mu}\left(\frac{\partial L}{\partial(\partial_{\mu}\psi)}\psi - \cancel{\bar{\psi}\frac{\partial L}{\partial(\partial_{\mu}\bar{\psi})}}\right) = \partial_{\mu}\left(-\alpha\bar{\psi}\gamma^{\mu}\psi\right) = -\partial_{\mu}(e\bar{\psi}\gamma^{\mu}\psi) = \partial_{\mu}j^{\mu} = 0$$

using e.g. *e* for the real constant $\alpha \rightarrow e$

Introduce **the charge 4-current:** $\boxed{j^{\mu} = -e\bar{\psi}\gamma^{\mu}\psi}$

Symmetries and conservation laws

- charge 4-current: $j^\mu : (j^0, \vec{j})$ with $j^\mu = -e\bar{\psi}\gamma^\mu\psi$

(13) = current conservation law : $\boxed{\partial_\mu j^\mu = 0}$ (13)

$$\partial_\mu j^\mu = \partial_t j^0 + \vec{\nabla}\vec{j} = 0$$

Noether Theorem:

A global invariance leads to the existance of a conserved current!

☐ **Global invariance means that the phase α is a real constant, which is unique for all space and time points**

➔ one cannot measure the phase α itself !

Electric charge conservation law

- **Electric charge:** $Q = \int d^3x \; j^0(x)$ $\qquad$ $j^0 = -e\bar{\psi}\gamma^0\psi$

Integrate (13) over d³x ➔

$$\partial_t \underbrace{\int d^3x\, j^0(x)}_{Q} + \underbrace{\int d^3x\, \vec{\nabla}\vec{j}(x)}_{=0} = 0 \qquad \text{(14)}$$

Gauss (divergence) theorem $\int_V dV \; \vec{\nabla}\vec{j} = \oint_S d\vec{s}\;\vec{j}$

From (14) follows $\dfrac{dQ}{dt} = 0$ | ⇒ $\boxed{Q = const}$

! Charge should be conserved due to the global U(1) invariance !

Symmetries and conservation laws

Noether Theorem:
A global invariance leads to the existance of a conserved current!

Global invariance means that the phase α is a real constant, which is unique for all space and time points
➔ one cannot measure the phase α itself !

II) Let‘s now consider symmetry transformations where α can depend on the space-time coordinate

$$\alpha \Rightarrow \alpha(x) \tag{17}$$

- **Local U(1) invariance:**
  i.e. invariance of the Lagrangian under the local gauge group U(1):

$$U(\alpha(x)) \equiv e^{i\alpha(x)} \quad \Rightarrow \quad \psi(x) \rightarrow e^{i\alpha(x)}\psi(x) \tag{18}$$

and $$\bar{\psi}(x) \rightarrow e^{-i\alpha(x)}\bar{\psi}(x) \tag{19}$$

Symmetries and conservation laws

$$\partial_\mu \psi \rightarrow e^{i\alpha(x)} \partial_\mu \psi + \underline{i e^{i\alpha(x)} \psi \, \partial_\mu \alpha(x)} \qquad (20)$$

This term violates the invariance of the Lagrangian under local U(1) transformations!

- In order to construct a Lagrangian, which is invariant under local U(1) transformations, one has to introduce the covariant derivative:

$$D_\mu \equiv \partial_\mu - ieA_\mu \qquad (21)$$

where $A_\mu(x)$ follows the transformation:

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x) \qquad (22)$$

$A_\mu(x)$ is called a gauge vector field , which interacts with the dirac field $\psi(x)$.

Thus,

$$\begin{aligned} L &= i \bar{\psi} \gamma^\mu D_\mu \psi - m \bar{\psi} \psi \\ &= \bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi + e \bar{\psi} \gamma^\mu \psi \, A_\mu \end{aligned} \qquad (23)$$

Symmetries and conservation laws

The request for local gauge invariance leads to the introduction of a gauge vector field $A_\mu(x)$

In order to identify the field $A_\mu(x)$ with real particles (photons), one has to add the kinetic energy term:

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ **- field strength tensor**

- **The local gauge invariance of L requires a massless vector field $A_\mu(x)$, since a mass term $\frac{1}{2}m^2 A_\mu A^\mu$ violates the local gauge invariance of the Lagrangian**

Abelian and nonabelian local symmetries

Local symmetry: **QED** ⟺ **QCD**

local U(1) gauge transformation

$$\psi(x) \rightarrow e^{i\alpha(x)}\psi(x)$$

Introduce one gauge vector field

$A_\mu(x)$ (photons)

$A_\mu(x)$ – massless

QED – abelian theory

$$[U(\alpha_1),U(\alpha_2)]=0$$

no self-interaction of the field $A_\mu(x)$ since the photons do not have a charge

local SU(3) color gauge transformtion

$$\psi(x) \rightarrow e^{i\alpha_a(x)t^a}\psi(x)$$

Introduce 8 gauge vector fields

$A_\mu^a(x)$ (gluons)

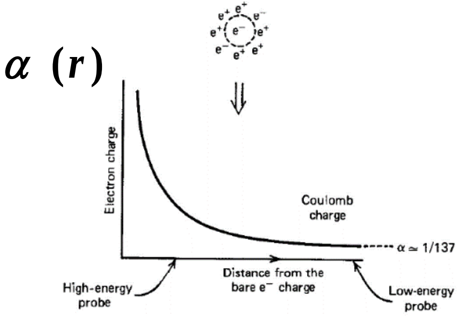
$A_\mu^a(x)$ – massless (a=1,…,8)

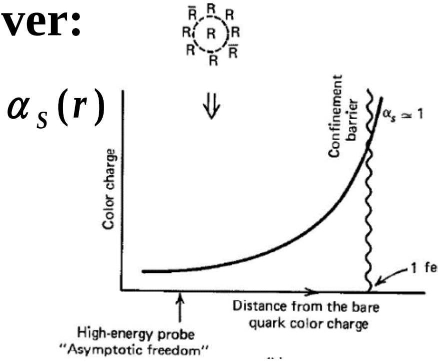
QCD – nonabelian theory:

$$[t_a,t_b]=if_{abc}t_c$$

self-interaction of field $A_\mu^a(x)$ since the gluons do have a charge

Infinite range of interaction, however:





Thus: Local invariance ➔ introduction of gauge fields!

Symmetries of QCD

QCD Lagrangian:

$$L_{QCD}(x) = \bar{\psi}(x)\left(i\gamma^{\mu}\left[\partial_{\mu} - igt^{a}A^{a}_{\mu}\right] - \hat{M}^{0}\right)\psi(x) - \frac{1}{4}G^{a}_{\mu\nu}(x)G^{\mu\nu a}(x)$$

Gluonic field strength tensor:

$$G^{a}_{\mu\nu}(x) = \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} + gf^{abc}A^{b}_{\mu}(x)A^{c}_{\nu}$$

$\psi(x)$ - **quark field**

flavor space	*Dirac space*	*color space*
$q = u, d, s$	$\mu = 0,1,2,3$	$c = r, b, g$

In flavor space (3 flavors):

$$\psi(x) = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

Mass term:

$$\hat{M}^{0} = \begin{pmatrix} m^{0}_{u} & 0 & 0 \\ 0 & m^{0}_{d} & 0 \\ 0 & 0 & m^{0}_{s} \end{pmatrix}$$

3x3 diagonal matrix in flavor space with the bare quark masses on the diagonal

Symmetries in flavor space

☐ Symmetries in flavor space:

From the decay of particles we know that the flavor is conserved, i.e. there is the invariance of the Lagrangian under rotations in flavor space (for the strong interaction)

e.g. $\Delta^{++}(uuu) \rightarrow p(uud) + \pi^{+}(\bar{d}u)$

implies the conservation of flavor currents
➔ no need to introduce gauge fields in order to construct a conserved current in flavor space.

Thus: the flavor symmetry is a global symmetry

3 flavors u,d,s ➔ SU(3) flavor group
for 6 flavors ➔ SU(6) flavor group

Symmetries in flavor space

- The global flavor symmetry

➔ The QCD Lagrangian is invariant under SU(3) flavor transformations:

$$\psi(x) \to \exp\left(-i\sum_{b=1}^{8} a_b t_b\right)\psi(x) \qquad (24)$$

flavor space

t^a are 8 generators of the SU(3)$_{\text{flavor}}$ group $\quad t^a = \dfrac{\lambda^a}{2}$

The parameters a_b in (24) are constants, but an arbitrary vector in flavor space

Since the transformation (24) is global, it holds for massless and massive quarks

Symmetries in flavor space

- **Furthermore we consider the transformations:**

$$\psi(x) \to \exp\left(-i\sum_{b=1}^{8} a_b t_b\, \gamma^5\right)\psi(x) \qquad (25)$$

act on the flavor, Dirac components

The transformation (25) mixes the upper and lower component of the Dirac spinor

$$u_s(\mathrm{p}) = \frac{1}{\sqrt{2M(E_p+M)}}\begin{pmatrix}(E_p+M)\chi_s\\ (\vec{\sigma}\cdot\mathrm{p})\chi_s\end{pmatrix}, \quad v_s(\mathrm{p}) = \frac{1}{\sqrt{2M(E_p+M)}}\begin{pmatrix}(\vec{\sigma}\cdot\mathrm{p})\chi_s\\ (E_p+M)\chi_s\end{pmatrix}$$

➔ The QCD lagrangian is invariant under the transformation (25) only in case of massless quarks !

Symmetries in flavor space

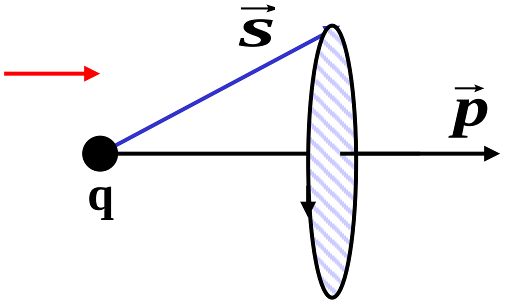
- **Now define right and left-handed quarks by the linear combination:**

$$\psi_{R/L} = \frac{1}{2}\left(1 \pm \gamma^5\right)\psi \qquad \overline{\psi}_{R/L} = \overline{\psi}\,\frac{1}{2}\left(1 \mp \gamma^5\right) \qquad (26)$$

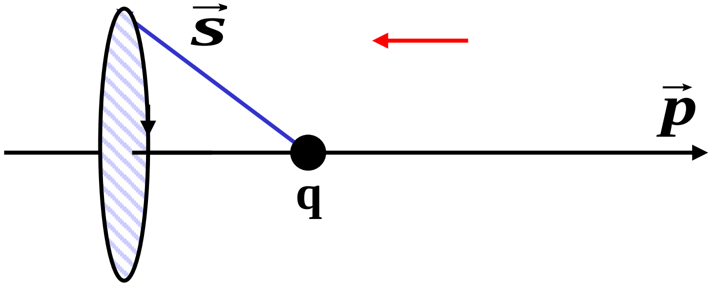
$$\psi = \psi_R + \psi_L \,, \qquad \overline{\psi} = \overline{\psi}_R + \overline{\psi}_L \qquad (27)$$

- **Handedness, or chirality – projection of the spin *s* on the momentum *p* (direction of motion)**

right-handed quarks



left-handed quarks



spin projection of fermions : s=$\underline{+}$1/2

Symmetries in flavor space

- For $\hat{M}^0 = 0$ the left- and right-handed quarks are not mixed dynamically and conserve their handedness (chirality).

➔ chiral invariance of QCD in the limit $\hat{M}^0 = 0$

$$SU(3)_{flavor} \Rightarrow SU(3)_R \times SU(3)_L$$

It expresses the physical effect that the sign of projection of the fermion spin on its momentum direction can not be changed by dynamics if $\hat{M}^0 = 0$,

i.e. the interaction in QCD $ig\overline{\psi}(x)\,\gamma^\mu t^a A^a_\mu\,\psi(x)$
does not break the left-right symmetry

- Mass term $\hat{M}^0 \neq 0$ breaks explicitly the chiral symmetry of the QCD Lagrangian:

$$\begin{aligned}\hat{M}^0\overline{\psi}\psi &= \hat{M}^0(\overline{\psi}_L + \overline{\psi}_R)(\psi_L + \psi_R) \\ &= \hat{M}^0(\overline{\psi}_L\psi_L + \overline{\psi}_R\psi_R) + \underset{\nearrow}{\hat{M}^0(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L)}\end{aligned}$$

the mass term plays the role of an interaction that mixes left- and right-handed quarks

Symmetries in flavor space

- Thus, the global chiral transformations

$$\psi(x) \rightarrow \exp\left(-i\sum_{b=1}^{8} a_b t_b\, \gamma^5\right)\psi(x)$$

$$SU(3)_{flavor} \Rightarrow SU(3)_R \times SU(3)_L$$

are valid only for massless quarks (or fermions as neutrinos) !

- A mass term $\hat{M}^0 \neq 0$ ➔ phenomenon of symmetry breaking

As a consequence of chiral symmetry breaking the bound states of QCD, i.e. mesons and baryons with opposite parity do not show the same mass ! A prominent example: the vector meson ρ and the axial vector meson a_1 differ in mass by more than 500 MeV !

Useful liturature:

Claude Itzykson, “Quantum Field Theory” (Dover Books on Physics)

M. Peskin, “Introduction To Quantum Field Theory”

Michael Le Bellac “Thermal field theory”

Joseph I. Kapusta and Charles Gale, “Finite-Temperature Field Theory Principles and Applications”

F.J. Yndurain, “Quantum Chromodynamics: An Introduction to the Theory of Quarks and Gluonos”