Dynamical description of fluctuations in heavy-ion collisions

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EMMI workshop
November 2015, GSI
Transport models vs. grand-canonical ensemble

In a grand-canonical ensemble, the system is
- in thermal equilibrium (= long-lifed),
- in equilibrium with a particle heat bath,
- spatially infinite
- and static.

Systems created in heavy-ion collisions are
- short-lived,
- spatially small,
- inhomogeneous,
- and highly dynamical!

- Transport models take the microcanonical nature of individual particle scatterings into account.
Baryon-number conservation limits fluctuations of net-baryon number.

\[ P_\mu(N, C) = \mathcal{N}(\mu, C) e^{-\mu} \frac{\mu^N}{N!} \text{ on } [\mu - C, \mu + C] \]

\( \mu \): the expectation value of the original Poisson distribution, \( \mathcal{N}(\mu, C) \): normalization factor, \( C > 0 \): cut parameter

\[ C = \alpha \sqrt{\mu} \left( 1 - \left( \frac{\mu}{N_{\text{tot}}} \right)^2 \right). \]

\( \alpha = 3, N_{\text{tot}} = 416. \)

- An increase of the average net-baryon number does not lead to stronger fluctuations.
- At the upper limit of \( N_{\text{tot}} = 416 \) the distribution changes to a \( \delta \)-function \( (K_{\text{eff}} = 0)\).
Net-baryon number distribution in UrQMD

- central Pb+Pb collisions at $E_{\text{lab}} = 20$ AGeV
- fit to a Poisson distribution
- shoulders are enhanced
- tails are cut

$\Rightarrow$ decrease from $K_{\text{Poisson}}^{\text{eff}} = 1$ to $K_{\text{UrQMD}}^{\text{eff}} = -22.2$

ratio of UrQMD to Poisson distribution
Rapidity window dependence of the effective kurtosis

- Same qualitative behavior of the net-baryon kurtosis as expected from the toy model.
- \( E_{\text{lab}} = 158\text{AGeV} \)
- The net-proton kurtosis only slightly follows this trend.
- The net-charge kurtosis is not affected, but error bars are larger.
Energy dependence of the effective kurtosis

- adapting the rapidity window to fix the mean net-baryon number
- net-baryon effective kurtosis does not show an energy dependence
- for lower $\sqrt{s}$ $K_{\text{eff}}$ becomes increasingly negative
- at $E_{\text{lab}} = 2A\text{GeV}$: $\langle N_{B-B} \rangle \simeq 240$
Difficulties in the transport calculations

- large statistics are needed (more than in our old analysis)
- very involved experimental analysis - crucial to repeat (some of) the same procedures
- error estimates in most theory calculations not very accurate
  + Skewness?
  + $p_T$-range dependence?
  + Fluctuations of strangeness? (Why do they not show the conservation laws?)

- There is no phase transition in UrQMD...
Fluctuations at the phase transition

At a critical point

- correlation length of fluctuations of the order parameter diverges $\xi \to \infty$
- fluctuations of the order parameter diverge: $\langle \Delta \sigma^n \rangle \propto \xi^\alpha$ with higher powers of divergence for higher moments
- mean-field studies in Ginzburg-Landau theories, beyond mean-field: renormalization group
- relaxation time diverges $\Rightarrow$ critical slowing down! $\Rightarrow$ fluctuations in equilibrated systems!

... and a first-order PT:

- at $T_c$ coexistence of two stable thermodynamic phases
- metastable states above and below $T_c$ $\Rightarrow$ supercooling and -heating
- nucleation and spinodal decomposition in nonequilibrium
- domain formation and large inhomogeneities
  $\Rightarrow$ fluctuations in nonequilibrium!

... but also at the crossover:

- remnant of the $O(4)$ universality class in the chiral limit.
  $\Rightarrow$ fluctuations in equilibrated systems!

Nonequilibrium chiral fluid dynamics (NχFD)

IDEA: combine the dynamical propagation of fluctuations at the phase transition with fluid dynamical expansion!

(model-independent is nice, but in the end some real input is needed...)

- Langevin equation for the sigma field: damping and noise from the interaction with the quarks
  \[ \partial_\mu \partial^\mu \sigma + \frac{\delta U}{\delta \sigma} + g \rho_s + \eta \partial_t \sigma = \zeta \]

- Phenomenological dynamics for the Polyakov-loop
  \[ \eta_\ell \partial_t \ell T^2 + \frac{\partial V_{\text{eff}}}{\partial \ell} = \zeta_\ell \]

- Fluid dynamical expansion of the quark fluid = heat bath, including energy-momentum exchange
  \[ \partial_\mu T^{\mu \nu}_q = S^\nu = -\partial_\mu T^{\mu \nu}_\sigma, \quad \partial_\mu N^\mu_q = 0 \]
  ⇒ includes a stochastic source term!

MN, S. Leupold, I. Mishustin, C. Herold, M. Bleicher, PRC 84 (2011); PLB 711 (2012); JPG 40 (2013)
Dynamical slowing down

Phenomenological equation: $\frac{d}{dt} m_\sigma(t) = -\Gamma[m_\sigma(t)](m_\sigma(t) - \frac{1}{\xi_{eq}(t)})$

with input from the dynamical universality class $\Rightarrow \xi \sim 1.5 - 2.5$ fm

B. Berdnikov and K. Rajagopal, PRD 61 (2000))

More advanced: Expanded Fokker-Planck dynamics by S. Mukherjee et al. 1506.00645

$G(r) = \int d^3 x d^3 y \langle \sigma(x) - \sigma_0 \rangle \langle \sigma(y) - \sigma_0 \rangle$

$\sim \exp(-r/\xi)$

Assume $\sigma_0$ is the volume averaged field.

From the curvature of $V_{\text{eff}}$:

$\langle \xi^2 \rangle = \langle 1 / m_\sigma^2 \rangle = \left\langle \left( \frac{d^2 V_{\text{eff}}}{d\sigma^2} \right)^{-1} \right\rangle$

C. Herold, MN, I. Mishustin, M. Bleicher PRC 87 (2013)

Definition of $\xi$ in inhomogeneous systems involves averaging!

$\Rightarrow$ Similar magnitude of $\xi \sim 1.5 - 3$ fm!
PQM vs. QH model: stability of droplets

See also: J. Steinheimer, J. Randrup, V. Koch PRC89 (2014)

PQM EoS

QH EoS

- Dynamical and stochastic droplet formation at the phase transition and subsequent decay in the hadronic phase.
Define normalized moments of the net-baryon density distribution as:

\[
\langle n^N \rangle = \int d^3x n(x)^N P_n(x) \quad \text{with} \quad P_n(x) = \frac{n(x)}{\int d^3x n(x)}
\]

- Infinite increase in the PQM.
- Increase in the HQ model around the phase transition followed by a rapid decrease due to pressure in the hadronic phase!
- REMEMBER: We started with smooth initial conditions and all inhomogeneities are formed dynamically!
From densities fluctuations to particle fluctuations

- From densities to particle via Cooper-Frye particlization:
  \[ \frac{dN_i}{d^3p} = \int d\sigma^\mu \rho_\mu (f_i^{eq}(p) + \delta f) \]

- Sampled from a single-particle distribution function \( \Rightarrow \) (partially) destroys correlations.

- Modify the single-particle distribution functions or the entire particlization procedure?

For single-particle distribution: \( m_i \rightarrow m_{i,0} + g\sigma(x) \)
First net-proton fluctuations from $N\chi$FD

- Here: Only couple the densities of the order parameter field with the fluid dynamical densities.

UrQMD initial conditions rescaled to the EoS of the effective model.
The critical mode

- At $\mu_B \neq 0$ $\sigma$ mixes with the net-baryon density $n$ (and $e$ and $\bar{m}$)
- In a Ginzburg-Landau formalism:
  \[
  V(\sigma, n) = \int d^3 x \left( \sum_m (a_m \sigma^m + b_m n^m) + \sum_{m,l} c_{m,l} \sigma^m n^l \right) - h\sigma - jn
  \]
- $V(\sigma, n)$ is flat in $(a\sigma, bn)$ direction
- Equations of motion (including symmetries in $V(\sigma, n)$):
  \[
  \partial_t^2 \sigma = -\Gamma \delta V / \delta \sigma + ...
  \]
  \[
  \partial_t n = \gamma \vec{\nabla}^2 \delta V / \delta n + ...
  \]
- two time scales (with diffusion $D \rightarrow 0$ at the critical point)
  \[
  \omega_1 \propto -i\Gamma a
  \]
  \[
  \omega_2 \propto -i\gamma D \bar{q}^2
  \]
- The diffusive mode becomes the critical mode in the long-time dynamics. These fluctuations need to be included at the critical point!

Fluid dynamical fluctuations

Conventional fluid dynamics propagates thermal averages of the energy density, pressure, velocities, charge densities, etc.

However, ...

- ... already in equilibrium there are thermal fluctuations
- ... the fast processes, which lead to local equilibration also lead to noise!

Conventional ideal fluid dynamics:

\[ T^{\mu\nu} = T^{\mu\nu}_{\text{eq}} \]
\[ N^{\mu} = N^{\mu}_{\text{eq}} \]

Fluid dynamical fluctuations

Conventional fluid dynamics propagates thermal averages of the energy density, pressure, velocities, charge densities, etc.

However, ...

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Conventional viscous fluid dynamics:

\[
T^{\mu\nu} = T^{\mu\nu}_{eq} + \Delta T^{\mu\nu}_{visc}
\]

\[
N^\mu = N^\mu_{eq} + \Delta N^\mu_{visc}
\]

Fluid dynamical fluctuations

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Fluctuating viscous fluid dynamics:

\[ T_{\mu\nu} = T_{\mu\nu}^{\text{eq}} + \Delta T_{\mu\nu}^{\text{visc}} + \Xi_{\mu\nu} \]

\[ N^\mu = N^\mu_{\text{eq}} + \Delta N^\mu_{\text{visc}} + I^\mu \]

Fluid dynamical fluctuations

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However, ...

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**Fluctuating viscous fluid dynamics:**

\[
T^{\mu\nu} = T_{\text{eq}}^{\mu\nu} + \Delta T_{\text{visc}}^{\mu\nu} + \Xi^{\mu\nu} \\
N^{\mu} = N_{\text{eq}}^{\mu} + \Delta N_{\text{visc}}^{\mu} + I^{\mu}
\]

- \( \langle T^{\mu\nu} T_{\mu\nu} \rangle \) give viscosities (Kubo-formula), consistently with dissipation-fluctuation theorem fluctuations need to be included as well!

Fluid dynamical fluctuations

The noise terms are such that averaged quantities exactly equal the conventional quantities:

\[
\langle T_{\mu\nu} \rangle = T_{\text{eq}}^{\mu\nu} + \Delta T_{\text{visc}}^{\mu\nu} \quad \text{with} \quad \langle \Phi^{\mu\nu} \rangle = 0
\]

\[
\langle N^\mu \rangle = N_{\text{eq}}^\mu + \Delta N_{\text{visc}}^\mu \quad \text{with} \quad \langle \Gamma^\mu \rangle = 0
\]

The two formulations will, however, differ when one calculates correlation functions:

\[
\langle T_{\mu\nu}(x) T_{\mu\nu}(x') \rangle
\]

\[
\langle N_{\mu}(x) N_{\mu}(x') \rangle
\]
Fluid dynamical fluctuations

In linear response theory the retarded correlator

\[ \langle T^{\mu \nu}(x) T^{\mu \nu}(x') \rangle \]
gives the viscosities and

\[ \langle N^\mu(x) N^\mu(x') \rangle \]
the charge conductivities

via the dissipation-fluctuation theorem (Kubo-formula)!

It means that when dissipation is included also fluctuations need to be included!

CAUTION: If nonlinearities are included fluid dynamical fluctuations contribute to the transport coefficients!

\[ \Rightarrow \text{ absolut lower limit for the effective viscosity!} \]

\[ \Rightarrow \text{ non-analytic contribution to } \tau_\pi, \text{ breakdown of gradient expansion!} \]

Fluid dynamical fluctuations

- Linearized fluid dynamical equations: small fluctuations $\bar{e} + \delta e$, $\bar{p} + \delta p$ and $\delta v^i$ with: $\delta T^{00} = \delta e$ and $\delta T^{ij} = m^i = (\bar{e} + \bar{p}) v^i = \bar{\omega} v^i$

  $$\partial_t m_\perp + \eta / \bar{\omega} k^2 m_\perp = 0$$

  $$\partial_t \delta e + i k \cdot m_\parallel = 0$$

  $$\partial_t m_\parallel + i v_s^2 k \delta e + \gamma v k^2 m_\parallel = 0$$

- retarded Green's function for $\delta e$ and $m_\parallel$:

  $$G_{ab}^{ret}(\omega, k) = \frac{\bar{\omega}}{\omega^2 - v^2_s k^2 + i \omega \gamma_s k^2} \begin{pmatrix} \frac{k^2}{\omega |k|} \\ \omega |k| \end{pmatrix} \begin{pmatrix} k^2 & \omega |k| \\ v^2_s k^2 - i \omega \gamma_s k^2 \end{pmatrix}$$

- including the transverse momentum density:

  $$G_{m_i,m_j}^{ret}(\omega, k) = \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \frac{\eta k^2}{i \omega - \gamma_\eta k^2} + \frac{k_i k_j}{k^2} \frac{\bar{\omega} (v^2_s k^2 - i \omega \gamma_s k^2)}{\omega^2 - v^2_s k^2 + i \omega \gamma_s k^2}$$

- Kubo-formulas for viscosities:

  $$\eta = -\frac{\omega}{2k^2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \Im G_{m_i,m_j}^{ret}(\omega, k \to 0)$$

  $$\zeta + \frac{4}{3} \eta = -\frac{\omega^3}{k^4} \Im G_{ee}^{ret}(\omega, k \to 0)$$
Fluid dynamical fluctuations

\[
\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x'^\mu} \langle \Xi^\mu_0(x) \Xi^\mu_0(x') \rangle^S = - \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x'^\mu} \langle T^{\mu_0}(x) T^{\mu_0}(x') \rangle^S
\]

\[
= \int \frac{d\omega}{2\pi} \int \frac{d^3 k}{(2\pi)^3} e^{ik(x-x')} e^{-i\omega(t-t')} \times
\]

\[
\times \left( \omega^2 G^S_{ee}(\omega, k) - 2\omega |k| G^S_{em}(\omega, k) + k^2 G^S_{m\parallel m\parallel}(\omega, k) \right)
\]

\[
G^S_{ab}(\omega, k) = -\frac{2T}{\omega} \Im G^\text{ret}_{ab}(\omega, k)
\]

\[
= 0
\]

\[
\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x'^\mu} \langle \Xi^{\mu i}(x) \Xi^{\mu j}(x') \rangle^S = - \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x'^\mu} \langle T^{\mu i}(x) T^{\mu j}(x') \rangle^S
\]

\[
= 2T \left[ \left( \zeta + \frac{4}{3} \eta \right) \partial_i \partial_j + \eta (\delta_{ij} \nabla^2 - \partial_i \partial_j) \right] \delta^4(x - x')
\]

Then boost to arbitrary frame:
Fluid dynamical fluctuations

\[ T^{\mu \nu} = T_{eq}^{\mu \nu} + \Delta T^{\mu \nu}_{\text{visc}} + \Xi^{\mu \nu} \]
\[ N^\mu = N_{eq}^\mu + \Delta N^\mu_{\text{visc}} + I^\mu \]

with

\[ \langle \Xi^{\mu \nu}(x) \Xi^{\alpha \beta}(x') \rangle = 2T \left[ \eta (\Delta^{\mu \alpha} \Delta^{\nu \beta} + \Delta^{\mu \beta} \Delta^{\nu \alpha}) + (\zeta - 2/3 \eta) \Delta^{\mu \nu} \Delta^{\alpha \beta} \right] \delta^4(x - x') \]

- In second-order fluid dynamics there are relaxation equations for \( \Xi^{\mu \nu} \):\
  \[ u^\gamma \partial_\gamma \langle \nu \nu \rangle = -\frac{\Xi^{\mu \nu} - \zeta^{\mu \nu}_{\text{gauss}}}{\tau_\pi} \]

- In white noise approximation and ignoring bulk viscosity (\( \zeta = 0 \)):\
  \[ \langle \zeta^{\mu \nu}_{\text{gauss}}(x) \zeta^{\alpha \beta}_{\text{gauss}}(x') \rangle = 4T \eta \Delta^{\mu \nu \alpha \beta} \delta^{(4)}(x - x') \]
Fluid dynamical fluctuations

- In a numerical treatment $\rightarrow$ discretization: $\langle \zeta^2 \rangle \propto \frac{1}{\Delta V}$
- $\Rightarrow$ large fluctuations from cell to cell $\Rightarrow$ coarse-graining, smearing, etc. compare to expectations from equilibrium and MC kinetic theory!

- Example: non-relativistic Navier-Stokes + fluctuations
- 1d, dilute gas, periodic boundary conditions

- Different algorithms treat fluctuations differently, third-order methods seem to work best.

J. Bell, A. Garcia, S. Williams, PRE76 (2007)
Fluid dynamical fluctuations

- Static box with periodic boundary conditions in relativistic 3 + 1d fluid dynamics
  based on 3 + 1d viscous fluid dynamical code by Y. Karpenko.
- Noise correlated over 1 fm

Time evolution of the variance $\langle \delta e^2 \rangle$:

\[
\langle \delta e(x) \delta e(x + dx) \rangle \text{ correlation function:}
\]

- Initial increase of the variance $\rightarrow$ saturation (thermalization?) at later times.
- Noise-correlation length recovered.

work in progress with M. Bluhm and Th. Schäfer (NCSU)
Fluid dynamical fluctuations

- Reduction of the pressure due to the nonlinearities in the fluctuations.
- Variance of energy density fluctuations approximately $30 - 40\%$ of what is expected in a grandcanonical ensemble.
- NEXT: include net-baryon number density, diffusion and fluctuations.
Conclusions

- How to propagate fluctuations at the phase transition?
- How to propagate fluctuations at the phase transition consistently with an expanding, inhomogeneous medium?
- How to connect fluctuations/correlations at each stage consistently? Preequilibrium initial state $\rightarrow$ fluid dynamics $\rightarrow$ particles.
- Plea to experiments: please provide $p_T$-spectra, etc. for model tuning!
The Kurtosis

The kurtosis is a measure of the deviation of fluctuations from Gaussian statistical fluctuations.

\[ \langle \Delta X_i \Delta X_j \Delta X_k \Delta X_l \rangle \sim \langle \Delta X_i \Delta X_j \rangle \langle \Delta X_k \Delta X_l \rangle + \langle \Delta X_i \Delta X_k \rangle \langle \Delta X_j \Delta X_l \rangle + \langle \Delta X_i \Delta X_l \rangle \langle \Delta X_j \Delta X_k \rangle \]

\[ \Rightarrow \langle \Delta X^4 \rangle - 3\langle \Delta X^2 \rangle^2 = 0 \text{ in the Gaussian approximation.} \]

compare to Binder cumulant for eg. 2d Ising model:

\[ U = 1 - \frac{\langle M^4 \rangle}{\langle M^2 \rangle^2} \]

\[ = 0 + O(1/V) \text{ in symmetric phase} \]

\[ = U^* = 2/3 \text{ at } T = T_c \]

\[ = 2/3 + O(1/V) \text{ in the broken phase} \]

T. Preis et al., JCP228 (2009)
Kurtosis in thermal models (confined phase)

**Hadron Resonance Gas model:**

contains hadrons as relevant d.o.f. & interactions that result in resonance formation

- good description of lattice QCD equation of state
- rather well description of continuum-extrapolated lattice QCD susceptibility results, cf. 1507.04627

in Boltzmann approximation net-numbers of conserved charges follow a Skellam distribution, i.e. $\kappa_B \sigma_B^2 = 1$

$\rightarrow$ cannot capture non-thermal fluctuations

early comparison with STAR net-proton number fluctuations F. Karsch, K. Redlich, PLB 695, 136 (2011)

but net-proton $\neq$ net-baryon ...

STAR, PRL 105, 22302 (2010); STAR, PRL 112, 032302 (2014)
Kurtosis in thermal models (confined phase)

- resonance decays affect net-distributions of identified particles:
  average influence (thermal fluctuations of resonance numbers) $\leftrightarrow$ HRG in partial chemical equilibrium
  in reality multinomial distribution
  $\rightarrow$ probabilistic effect erasing non-Gaussianities

- regeneration of resonances:
  $p(n) + \pi \rightarrow \Delta \rightarrow n(p) + \pi$
  can lead to (complete) isospin randomization, M. Kitazawa, M. Asakawa, PRC 86, 024904 (2012)
  $\rightarrow$ similar effect on non-Gaussianities

... (global) conservation of baryon number, A. Bzdak et al., PRC87, 014901 (2013); M. Sakaida et al., PRC90, 064911 (2014), ...
Dynamics versus equilibration

- Static box with temperature quench to $T < T_c$.
- Fluctuations of the order parameter:

![Graph](image)

- Strong enhancement of the intensities for a first-order phase transition during the evolution.
- Strong enhancement of the intensities for a critical point scenario after equilibration.

C. Herold, MN, I. Mishustin, M. Bleicher PRC 87 (2013)