# Relativistic quantum mechanics

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# 1 Introduction

When we are dealing with microscopic problems at relativistic energies, we have to cope with relativistic kinematics and with particle production in a quantum mechanical description. These problems cannot be dealt with consistently in non-relativistic quantum mechanics (i.e. by solving the Schrödinger equation). The consistent framework for dealing with such problems is a *relativistic quantum field theory* (QFT). The formalism of QFT leads to a perturbation theory, which can be phrased in terms of Feynman diagrams. The full formalism of QFT is outside the scope of these lectures, so we will take a less formal approach. We will motivate the Feynman rules and Feynman diagrams starting from relativistic wave equations. This approach lacks mathematical rigor, but is more intuitive. We start by reviewing the most important principles of non-relativistic quantum mechanics.

# 2 Schrödinger equation

The wave equations cannot be derived from the laws of classical mechanics. One can at most provide plausibility arguments for the form of the equations.

Consider the simplest possible physical system, namely that of an isolated free particle. The non-relativistic hamiltonian is

$$H = \frac{p^2}{2m}\,. \tag{1}$$

In quantum mechanics every *physical observable* is represented by a *linear, hermitian operator*, which act on the wavefunction $\Psi$. Thus. e.g. in coordinate representation

$$H \to i\hbar\frac{\partial}{\partial t} \tag{2}$$

$$\vec{p} \to \frac{\hbar}{i}\vec{\nabla} \tag{3}$$

which leads to the non-relativistic Schrödinger equation

$$i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} = \frac{-\hbar^2\vec{\nabla}^2}{2m}\Psi(\vec{r},t)\,. \tag{4}$$

It is assumed that the energy and momentum operators remain also in the presence of interactions. From the non-relativistic form of the total hamiltonian

$$H = \frac{p^2}{2m} + V(\vec{r}, t) \tag{5}$$

one then arrives at the general from of the Schrödinger equation

$$i\hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t} = \left( \frac{-\hbar^2 \vec{\nabla}^2}{2m} + V(\vec{r}, t) \right) \Psi(\vec{r}, t) \, . \tag{6}$$

In most cases the potential is independent of time, in which case the energy is conserved. For a stationary problem, one can then separate the time and space coordinates, and one arrives at the time independent Schrödinger equation

$$\Psi(\vec{r}, t) = \psi(\vec{r}) \exp(-i \frac{Et}{\hbar}) \Rightarrow H\psi(\vec{r}) = E\psi(\vec{r}) \tag{7}$$

The wave function is a complex probability amplitude. Only changes of the phase and relative phases are observable, not the absolute value of the phase. The probability density is given by

$$\rho(\vec{r}) = |\psi(\vec{r})|^2. \tag{8}$$

while the probability to find the particle in the volume element $d^3r$ is

$$|\psi(\vec{r})|^2 d^3r \, . \tag{9}$$

We also need the probability current density, $\vec{j}$. The conservation of probability implies

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \, , \tag{10}$$

the equation of continuity.

The derivation of the current corresponding to the Schrödinger equation

$$\vec{j}(\vec{r}, t) = \frac{\hbar}{2im} \left( \Psi^\star \vec{\nabla} \Psi - (\vec{\nabla} \Psi) \Psi \right) \tag{11}$$

is left as an exercise.

## 3 Klein-Gordon equation

For a free particle the relativistic energy-momentum relation is[1]

$$E^2 = \vec{p}^{\,2}c^2 + m^2c^4 \tag{12}$$

or

$$E = \sqrt{\vec{p}^{\,2}c^2 + m^2c^4} \tag{13}$$

In analogy to the Schrödinger equation, one may try to start from the latter equation, which would yield

$$i\hbar\frac{\partial\phi}{\partial t} = \left(\sqrt{-\hbar^2c^2\vec{\nabla}^2 + m^2c^4}\right)\phi\,. \tag{14}$$

However, this form is problematic, since it involves gradients of the wave function of arbitrary order, as seen by expanding the square root. This implies that the equation is non-local. Furthermore, because the time- and space-coordinates are not treated on the same footing, the form of the equation depends on the reference frame. Therefore, one tried the first relation, which is quadratic in $E$ and leads to

$$-\hbar^2\frac{\partial^2\phi}{\partial t^2} = (-\hbar^2c^2\vec{\nabla}^2 + m^2c^4)\phi\,. \tag{15}$$

This is the Klein-Gordon equation. Note that by requiring relativistic invariance and by refuting the square root form, we arrived at a differential equation which is second order in $\partial/\partial t$. Using the notation

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \qquad \partial^\mu = \frac{\partial}{\partial x_\mu}\,, \tag{16}$$

where $x^\mu = (t, x^1, x^2, x^3)$ and $x_\mu = (t, x_1, x_2, x_3) = (t, -x^1, -x^2, -x^3)$, one can rewrite the Klein-Gordon equation in explicitly covariant form[2]

$$(\partial_\mu\partial^\mu + m^2)\phi = 0 \tag{17}$$

The solutions of the free Klein-Gordon equation are of the form

$$\phi = \frac{1}{\sqrt{V}}e^{(i(\vec{p}\cdot\vec{r}-\omega t))}\,. \tag{18}$$

[1] We employ the notation $\vec{p}^{\,2} = \sum_{i=1}^{3} p^i$ for the three momentum squared and $p^2 = E^2 - \vec{p}^{\,2}$ for the four momentum squared.

[2] From now on we (most of the time) use natural units, where $\hbar = c = 1$.

Inserting this solution into the Klein-Gordon equation, one finds $\omega^2 = \vec{p}^2 + m^2$, which implies that there are solutions with positive as well as with negative energy, $E = \omega = \pm\sqrt{\vec{p}^2 + m^2}$.

Clearly the Klein-Gordon equation is covariant (invariant under Lorentz transformations). However, by using the squared form of the energy-momentum relation, we have also introduced the solutions with negative energy. Consequently, the spectrum of the Klein-Gordon equation is not bounded from below, which leads to problems with stability. Later, we will see that the negative energy solutions can be reinterpreted in terms of antiparticles.

For the Klein-Gordon equation one can also define a probability density and a probability current:

$$\rho = \frac{i}{2m}\left[\phi^\star \frac{\partial \phi}{\partial t} - \left(\frac{\partial \phi^\star}{\partial t}\right)\phi\right] \tag{19}$$

$$\vec{j} = \frac{1}{2\,i\,m}\left[\phi^\star \vec{\nabla}\phi - \left(\vec{\nabla}\phi^\star\right)\phi\right] . \tag{20}$$

The equation of continuity may be brought into covariant form by defining the current four vector $j^\mu = (\rho, \vec{j}) = (i/2m)\,[\phi^\star \partial^\mu \phi - (\partial^\mu \phi^\star)\phi]$:

$$\partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0\,. \tag{21}$$

Clearly the probability density is not positive definite. As an example, consider a non-interacting particle. The corresponding probability density equals

$$\rho = \frac{E}{mV}\,, \tag{22}$$

which is positive for positive energy solutions and negative for negative energy solutions. Thus, a negative probability density is possible, which would give problems with the interpretation of the wave function.

## 4 Dirac equation

The second order time derivative in the Klein-Gordon equation leads to the negative energy solutions. In order to avoid this problem, Dirac tried to find an equation, which is first order in the time derivative, like the Schrödinger equation, but at the same time relativistically invariant.

It turns out that although the probability density is positive for the Dirac equation, it also has negative energy solutions. These were interpreted by Dirac. Based on this he predicted the existence of antiparticles. This was certainly one of the most important contributions to modern physics. The Dirac equation describes spin-$\frac{1}{2}$ particles. It thus is the wave equation for electrons, muons, neutrinos, quarks, and also for the composite nucleons.

A relativistically covariant equation, which is first order in the time derivative, must also be first order in the spatial derivative. Dirac made the following *Ansatz*

$$i\frac{\partial\Psi}{\partial t} = H\Psi = \frac{1}{i}\sum_{i=1}^{3}\alpha^i\frac{\partial\Psi}{\partial x^i} + \beta m\Psi \;= \left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\Psi\,, \tag{23}$$

where $\alpha^i$ and $\beta$ are constants, that are to be determined. By acting with $\partial/\partial t$ on the equation, we obtain

$$i\frac{\partial^2\Psi}{\partial t^2} = -i\sum_{i=1}^{3}\alpha^i\frac{\partial}{\partial t}\frac{\partial\Psi}{\partial x^i} + \beta m\frac{\partial\Psi}{\partial t}\,, \tag{24}$$

which by using the Dirac equation yields

$$\frac{\partial^2\Psi}{\partial t^2} = \sum_{i=1}^{3}(\alpha^i)^2\frac{\partial^2\Psi}{\partial(x^i)^2} - \beta^2 m^2\Psi \tag{25}$$

$$+ \sum_{j\neq k}\frac{1}{2}(\alpha^j\alpha^k + \alpha^k\alpha^j)\frac{\partial^2\Psi}{\partial x^j\partial x^k} - i\,m\sum_{j=1}^{3}(\alpha^j\beta + \beta\alpha^j)\frac{\partial\Psi}{\partial x^j}\,. \tag{26}$$

This equation reduces to the Klein-Gordon equation

$$\frac{\partial^2\Psi}{\partial t^2} = (\vec{\nabla}^2 - m^2)\Psi\,, \tag{27}$$

which yields the desired relativistic energy-momentum relation if

$$(\alpha^i)^2 = 1 \qquad \alpha^i\alpha^j + \alpha^j\alpha^i = 0 \quad (i\neq j) \tag{28}$$

$$\beta^2 = 1 \qquad \alpha^i\beta + \beta\alpha^i = 0 \tag{29}$$

or in other words if the coefficients satisfy the anticommutation relations

$$\left\{\alpha^i,\alpha^j\right\} = \alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij} \qquad \left\{\alpha^i,\beta\right\} = 0 \tag{30}$$

and $\beta^2 = 1$. These conditions cannot be fulfilled by real or complex numbers. Thus, one has to consider matrices. The matrices must i) be hermitian, since

the hamiltonian is hermitian, ii) have eigenvalues $\pm 1$ since $(\alpha^i)^2 = \beta^2 = 1$ and iii) be traceless, since e.g. $Tr(\alpha^i) = Tr(\beta\beta\alpha^i) = Tr(\beta\alpha^i\beta) = -Tr(\alpha^i)$.

The properties ii) and iii) imply that the dimension $N$ of the matrices must be even, i.e. $N = 2,4,6\ldots$. $N = 2$ is not enough, since there are only three independent, traceless hermitian matrices of dimension 2. These can be chosen as the Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{31}$$

Thus, the smallest possible dimension is $N = 4$.

One choice is

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{32}$$

where $\mathbf{1}$ the $2 \times 2$ unit matrix. Thus, e.g.

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{33}$$

This implies that the wavefunction is a 4 component vector:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \qquad \Psi^\dagger = \begin{pmatrix} \psi_1^\star & \psi_2^\star & \psi_3^\star & \psi_4^\star \end{pmatrix} \tag{34}$$

The corresponding probability density and current are

$$\rho = \Psi^\dagger\Psi \qquad \vec{j} = \Psi^\dagger\vec{\alpha}\Psi\,. \tag{35}$$

The density is positive definite, since it is a sum of squares:

$$\rho = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2\,. \tag{36}$$

## 5 Non-relativistic correspondence

Before constructing the general solutions of the free Dirac equation, we explore the non-relativistic limit, to see that the equation makes sense physically. Con-

sider an electron at rest, i.e. $\vec{\nabla}\Psi = 0$. The Dirac equation then reduces to

$$i\frac{\partial \Psi}{\partial t} = m\beta\Psi \Rightarrow i\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = m\begin{pmatrix}1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}. \tag{37}$$

For each component one thus finds

$$i\frac{\partial \psi_1}{\partial t} = m\psi_1\,, \qquad i\frac{\partial \psi_2}{\partial t} = m\psi_2\,, \tag{38}$$

$$i\frac{\partial \psi_3}{\partial t} = -m\psi_3\,, \qquad i\frac{\partial \psi_4}{\partial t} = -m\psi_4\,. \tag{39}$$

The solutions are

$$\Psi_1 = Ce^{-imt}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad \Psi_2 = Ce^{-imt}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \tag{40}$$

$$\Psi_3 = Ce^{imt}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad \Psi_4 = Ce^{imt}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \tag{41}$$

and the corresponding energy eigenvalues are $E_{1,2} = m$ and $E_{3,4} = -m$. Consequently, the Dirac equation also has negative energy solutions. The solutions $\Psi_1$ and $\Psi_2$ can be interpreted as wave functions of particles, while $\Psi_3$ and $\Psi_4$ cannot. The interpretation of the negative energy solutions is postponed for a while.

We now consider a slow electron, which can be treated non-relativistically. Thus, we can expand the energy

$$E = \sqrt{\vec{p}^{\,2} + m^2} \simeq m + \frac{\vec{p}^{\,2}}{2m}\,. \tag{42}$$

By inserting the following *Ansatz*

$$\Psi(\vec{r}, t) = \begin{bmatrix}\tilde{\phi}\\ \tilde{\chi}\end{bmatrix}, \tag{43}$$

where $\tilde{\phi}$ and $\tilde{\chi}$ are two-spinors, into the Dirac equation, one finds

$$i\frac{\partial}{\partial t}\left[\begin{array}{c}\tilde{\phi}\\ \tilde{\chi}\end{array}\right] = \frac{1}{i}\vec{\sigma}\cdot\vec{\nabla}\left[\begin{array}{c}\tilde{\chi}\\ \tilde{\phi}\end{array}\right] + m\left[\begin{array}{c}\tilde{\phi}\\ -\tilde{\chi}\end{array}\right]. \tag{44}$$

In the non-relativistic limit, the rest energy dominates and we take care of the fast variation with time by defining new fields $\phi$ and $\chi$

$$\left[\begin{array}{c}\tilde{\phi}\\ \tilde{\chi}\end{array}\right] = e^{-imt}\left[\begin{array}{c}\phi\\ \chi\end{array}\right], \tag{45}$$

where now $\phi$ and $\chi$ are relatively slowly varying functions of time. They are solutions of the coupled equations

$$i\frac{\partial}{\partial t}\left[\begin{array}{c}\phi\\ \chi\end{array}\right] = \frac{1}{i}\vec{\sigma}\cdot\vec{\nabla}\left[\begin{array}{c}\chi\\ \phi\end{array}\right] + 2m\left[\begin{array}{c}0\\ -\chi\end{array}\right]. \tag{46}$$

The coupled equations are in explicit form

$$i\frac{\partial\phi}{\partial t} = \frac{1}{i}\vec{\sigma}\cdot\vec{\nabla}\chi\,, \tag{47}$$

$$i\frac{\partial\chi}{\partial t} = \frac{1}{i}\vec{\sigma}\cdot\vec{\nabla}\phi - 2m\chi\,. \tag{48}$$

In the latter equation one can neglect the time derivative compared to the mass term, since the time dependence of the two-spinors is slow. Thus,

$$\chi = \frac{1}{2\,i\,m}\vec{\sigma}\cdot\vec{\nabla}\phi\,, \tag{49}$$

which we can insert into the first equation:

$$i\frac{\partial\phi}{\partial t} = -\frac{1}{2m}(\vec{\sigma}\cdot\vec{\nabla})(\vec{\sigma}\cdot\vec{\nabla})\phi\,. \tag{50}$$

Note that the lower components are small ($\sim (p/m)\phi$) compared to ther upper components. Thus, the upper components dominate for particle states. Now, using the identity

$$\sigma^i\sigma^j = \delta^{ij} + i\,\epsilon^{ijk}\sigma^k\,, \tag{51}$$

where $\epsilon^{ijk}$ is the completely antisymmetric tensor[3] of rank 3, one recovers the Schrödinger equation for the upper components

$$i\frac{\partial\phi}{\partial t} = -\frac{\vec{\nabla}^{\,2}\phi}{2m} \tag{52}$$

[3] $\epsilon^{ijk} = 1$ for even permutations of 123, $-1$ for odd permutations and 0 otherwise

Thus, in the non-relativistic limit, the upper components can be identified with the Schrödinger wave function.

We note that one can perform the same analysis for an electron in an electromagnetic potential (scalar potential $\Phi$ and vector potential $\vec{A}$). The corresponding Dirac equation is obtained by minimal substitution ($\vec{\nabla} \to \vec{\nabla} - ie\vec{A}$). One finds ($e = -|e|$)

$$i\frac{\partial \Psi}{\partial t} = \left(\vec{\alpha} \cdot \left(\frac{1}{i}\vec{\nabla} - e\vec{A}\right) + \beta m + e\Phi\right)\Psi\,. \tag{53}$$

After some algebra, which we leave as an exercise (see e.g. Bjorken & Drell, vol. 1), one finds for a slow electron

$$i\frac{\partial \phi}{\partial t} = \left[\frac{(-i\vec{\nabla} - e\vec{A})^2}{2m} - \frac{e}{2m}\vec{\sigma}\cdot\vec{B} + e\Phi\right]\phi\,, \tag{54}$$

where $\vec{B} = \vec{\nabla} \times \vec{A}$ is the magnetic field. This is the so called Pauli equation, which describes a non-relativistic particle, with spin 1/2. We compare the $\vec{\sigma}\cdot\vec{B}$ term with the energy of a particle with a magnetic moment $\mu$ in a magnetic field

$$E_{magn} = -\vec{\mu}\cdot\vec{B}\,. \tag{55}$$

For a particle with spin $\vec{S}$ and mass $m$

$$\vec{\mu} = \frac{e}{2m}g\vec{S} = \frac{e}{2m}\frac{g}{2}\vec{\sigma} \tag{56}$$

where $g$ is the gyromagnetic ratio. We thus find that the Dirac equation describes a spin-1/2 particle with gyromagnetic ratio $g = 2$. Experimentally $(g-2)/2$ is $\simeq 10^{-3}$. Hence, the Dirac equation reproduces the correct value of the electron (and muon) magnetic moment to a very good approximation. The difference is understood in QED, the quantum field theory of the electromagnetic interactions of spin-1/2 particles.

## 6 Dirac equation in covariant form

The Dirac equation can be written in a form, which is explicitly Lorentz covariant. Since any 4-vector product $A_\mu B^\mu$ is invariant under Lorentz transformations, we want to bring the Dirac equation

$$i\frac{\partial \Psi}{\partial t} = \left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\Psi\,, \tag{57}$$

into such a form, which explicitly shows the symmetry between time and space. To this end we define the $\gamma$ matrices

$$\gamma^0 = \beta \qquad \gamma^i = \beta\alpha^i \tag{58}$$

Multiplying the Dirac equation by $\beta$, we then find

$$\left[i\left(\gamma^0\frac{\partial}{\partial t} + \sum_{i=1}^{3}\gamma^i\frac{\partial}{\partial x^i}\right) - m\right]\Psi = \left[i\gamma^\mu\frac{\partial}{\partial x^\mu} - m\right]\Psi = 0 \tag{59}$$

or in compact form

$$\left(i\gamma^\mu\partial_\mu - m\right)\Psi = 0. \tag{60}$$

An even more compact form can be obtained by introducing the Feynman slash notation $\not{a} = \gamma^\mu a_\mu$

$$\left(i\not{\partial} - m\right)\Psi = 0\,. \tag{61}$$

The new matrices satisfy the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,, \tag{62}$$

where

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{63}$$

The fact that the Dirac equation is Lorentz covariant means that in a different Lorentz frame, the Dirac equation has the same form, i.e.,

$$\left[i\gamma^\mu\frac{\partial}{\partial x^\mu} - m\right]\Psi(x) = 0 \tag{64}$$

in the Lorentz frame $\{x^\mu\}$ and

$$\left[ i\gamma^\mu \frac{\partial}{\partial (x')^\mu} - m \right] \Psi(x') = 0 \tag{65}$$

in the frame $\{(x')^\mu\}$, which is connected with the original frame by a Lorentz transformation. The mathematical proof that the Dirac equation is Lorentz covariant is outside the scope of these lectures, but can be found in textbooks, like Bjorken & Drell vol.1 and Schmüser.

In order to obtain the Dirac equation for the conjugate field, we need the properties of the $\gamma$ matrices under conjugation

$$(\gamma^0)^\dagger = \gamma^0 \qquad (\gamma^i)^\dagger = -\gamma^i \qquad \Rightarrow \qquad \gamma^0 (\gamma^\mu)^\dagger \gamma^0 = \gamma^\mu \,. \tag{66}$$

The conjugate wave equation is

$$\Psi^\dagger \left[ -i(\gamma^\mu)^\dagger \frac{\overleftarrow{\partial}}{\partial x^\mu} - m \right] = 0 \,. \tag{67}$$

Multiplying this equation from the left with $\gamma^0$, using the properties of the $\gamma$ matrices and defining $\bar{\Psi} \equiv \Psi^\dagger \gamma^0$, one finds

$$\bar{\Psi}(i \overleftarrow{\partial\!\!\!/} + m) = 0 \,. \tag{68}$$

One can also write the current explicitly in 4-vector form

$$j^\mu = \bar{\Psi}\gamma^\mu \Psi = (\Psi^\dagger \Psi, \Psi^\dagger \vec{\alpha} \Psi) \tag{69}$$

$$\partial_\mu j^\mu = 0 \,. \tag{70}$$

The proof that the current is conserved is left as an exercise.

The current $j^\mu$ transforms like a 4-vector (e.g. $x^\mu, p^\mu$). This is just one example for so called bilinear forms, which are important because they enter the coupling of fermions to mesons of different quantum numbers. A list of the most important forms are

| form | name | $J^P$ | Lorentz transf. $(\vec{x} \to -\vec{x})$ | |
|---|---|---|---|---|
| $S = \bar{\Psi}\Psi$ | scalar | $0^+$ | invariant | $(+)$ |
| $P = \bar{\Psi}\gamma_5\Psi$ | pseudoscalar | $0^-$ | invariant | $(-)$ |
| $V^\mu = \bar{\Psi}\gamma^\mu\Psi$ | vector | $1^-$ | 4-vector | $(-)$ |
| $A^\mu = \bar{\Psi}\gamma^\mu\gamma_5\Psi$ | axial vector | $1^+$ | 4-vector | $(+)$ |
| $T^{\mu\nu} = \bar{\Psi}\sigma^{\mu\nu}\Psi$ | tensor | $2^+$ | $x^\mu p^\nu$ | $(+)$ |

(71)

Here we have introduced two new combinations of $\gamma$ matrices

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{72}$$

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] \tag{73}$$

where

$$\gamma_5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \tag{74}$$

$$\sigma^{ij} = \sum_{k=1}^{3} \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \qquad i,j \in \{1,2,3\} \tag{75}$$

$$\sigma^{0i} = i\alpha^i = i \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}. \tag{76}$$

## 7 Solutions of the free Dirac equation

Consider a free particle described by the Dirac equation

$$\left[i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right] \Psi(x) = 0\,. \tag{77}$$

Since the momentum is a constant of motion for a free particle, the solution must be a plane wave

$$\Psi_p(x) = \begin{pmatrix} \phi(p) \\ \chi(p) \end{pmatrix} e^{-ipx}\,, \tag{78}$$

where $px$ is a short form for the four-vector product $p^\mu x_\mu = p^0x^0 - \vec{p}\cdot\vec{x} = Et - \vec{p}\cdot\vec{x}$. The notation is chosen so that $E = \sqrt{\vec{p}^2 + m^2}$ throughout. By evaluating the derivatives, one finds

$$(\not{p} - m) \begin{pmatrix} \phi(p) \\ \chi(p) \end{pmatrix} = 0\,, \tag{79}$$

which is a matrix equation for the four components. One finds 2 positive energy solutions of the form

$$\Psi_i(x) = u_i(p) e^{-ipx}\,, \tag{80}$$

where

$$u_1(p) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \varphi_\uparrow \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\varphi_\uparrow \end{pmatrix} \tag{81}$$

and

$$u_2(p) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \varphi_\downarrow \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\varphi_\downarrow \end{pmatrix} \tag{82}$$

are so called four-spinors or Dirac spinors. Furthermore

$$\varphi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \varphi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{83}$$

are two-spinors[4].

There are also 2 negative energy solutions of the form

$$\Psi_i(x) = v_{5-i}(p)e^{ipx} \qquad i \in 3,4\,, \tag{84}$$

where the four-vector $p$ is the same as above and

$$v_1(p) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\varphi_\downarrow \\ \varphi_\downarrow \end{pmatrix} \tag{85}$$

and

$$v_2(p) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\varphi_\uparrow \\ \varphi_\uparrow \end{pmatrix} \tag{86}$$

are the corresponding spinors. Note that in the negative energy solutions, the lower components are large. In the next section we will provide an interpretation for these solutions.

For a particle at rest these solutions agree with the ones obtained in section 5. The spinors are normalized such that

$$u_i^\dagger(p)u_j(p) = \frac{E}{m}\delta_{ij} \qquad v_i^\dagger(p)v_j(p) = \frac{E}{m}\delta_{ij} \tag{87}$$

$$u_i^\dagger(p)v_j(\tilde{p}) = v_i^\dagger(p)u_j(\tilde{p}) = 0\,, \tag{88}$$

where $\tilde{p} = (E, -\vec{p})$. The factor $E/m$ is due to the Lorentz contraction of the volume element along the direction of motion. Furthermore, for $\bar{u} = u^\dagger\gamma^0$ etc.,

$$\bar{u}_i(p)u_j(p) = \delta_{ij} \qquad \bar{v}_i(p)v_j(p) = -\delta_{ij} \tag{89}$$

$$\bar{u}_i(p)v_j(p) = \bar{v}_i(p)u_j(p) = 0\,. \tag{90}$$

Thus, $\bar{u}u$ is a Lorentz invariant quantity, in agreement with eq. 71. The verification of these solutions is left as an exercise.

[4]When there is no risk for confusion one often refers to the Dirac spinors as spinors for simplicity.

## 8 The negative energy solutions

If we use the standard quantum mechanics interpretation of one-body wave equations for the Dirac and Klein-Gordon equations we run into deep trouble. The reason is the existence of the negative energy solutions. Because the spectrum is not bounded from below, this lead to an instability of any state with one or more electrons, like an atom. Nothing prevents an electron in a positive energy state from making a transition to any one of the infinite number of negative energy states, thereby emitting photons. In fact, the rate for an electron in the ground state of the hydrogen atom to make a transition into the energy interval $-2m < E < -m$ is on the order of $10^8 \text{sec}^{-1}$. The lifetime of the hydrogen atom goes to zero as all negative energy states are included. Clearly this cannot be!

In order to avoid this catastrophe we must abandon the one-body interpretation of the Dirac equation. The first solution, presented by Dirac in 1927, is to postulate that all the negative energy states are filled with electrons, with one electron in each state. Since the Pauli principle allows only one electron per state, the positive energy states are now stable. This is analogous to the stability of the last filled electron orbit in an atom, which cannot decay, because all lower lying states are filled. Because of the analogy with a filled Fermi sea, the filled negative energy states are called the Dirac sea.

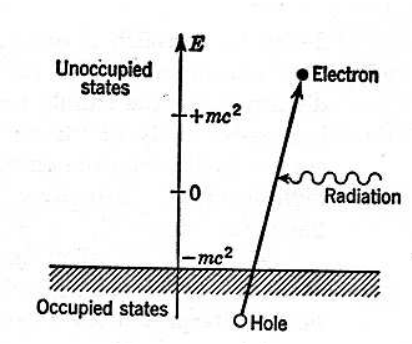


Figure 1: Pair production in the hole theory. (Figure from J.D. Bjorken & S.D. Drell, Relativistic Quantum Mechanics)

In order not to have infinite energy and charge, one *renormalizes* these quantities in vacuum. In other words, the energy and charge is measured relative to the filled Dirac sea. We have now arrived at a many-body theory of the

electron. The vacuum is filled with electrons in the Dirac sea, which become detectable only if we excite a negative energy electron into a positive energy state (see Fig. 1). This process results in an empty state in the Dirac sea, a hole. Relative to the filled Dirac sea, the hole carries the opposite quantum numbers of the original electron, e.g. the charge is positive. The hole is interpreted as an antiparticle. Based on these considerations, Dirac postulated the existence of antiparticles and in particular the positron in 1930-31. The positron was found in cosmic rays by Anderson in 1932. Our treatment of the hole theory is very brief. A more detailed discussion can be found in textbooks, e.g. Bjorken & Drell vol 1.

Diracs interpretation was revolutionary. Remember that at the time he made his prediction there were no experimental results that required the existence of antiparticles. Nevertheless, it is not satisfactory because it works only for fermions but not for bosons and it requires the vacuum to be filled with an infinitely charged unobservable sea of electrons.

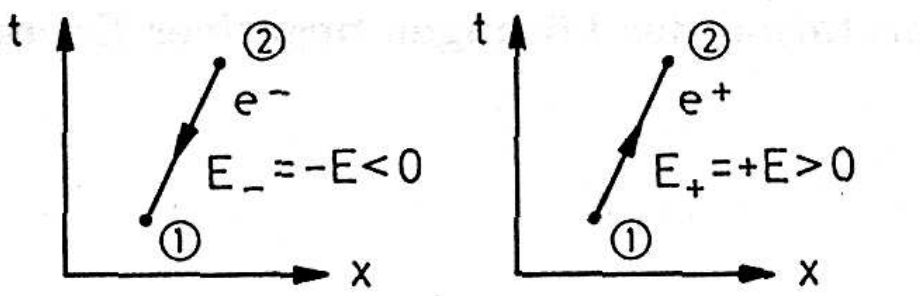


Figure 2: The interpretation of an electron moving backwards in time with negative energy as a positron with positive energy. (Figure from P. Schmüser, Feynman Graphen und Eichtheorien für Experimentalphysiker)

The presently accepted interpretation of the negative energy states is due to Stückelberg and Feynman. In their interpretation the negative energy states make sense when one lets them propagate backwards in time. A negative energy solution propagating backwards in time describes an antiparticle propagating forwards in time, as illustrated in Fig. 2. Thus, an electron with negative energy propagating backwards in time from point (2) to point (1) is equivalent to a positron propagating forwards in time from point (1) to point (2). Within this picture, which works also for bosons, one can describe

all scattering processes of particles and antiparticles as well as the annihilation and production of particle-antiparticle pairs. The Stückelberg-Feynman interpretation leads to the following statements:

- The emission of an antiparticle with four-momentum $p^\mu$ is equivalent to the absorption of a particle with four-momentum $-p^\mu$.
- The absorption of an antiparticle with four-momentum $p^\mu$ is equivalent to the emission of a particle with four-momentum $-p^\mu$.

Consider the scattering of charged pions off a time-dependent electromagnetic potential. We call the $\pi^+$ particles and $\pi^-$ antiparticles. The time dependence of the potential is assumed to be of the simple form $V(t) = V_0 \exp(-i\omega t)$. The sign in the exponent means that the potential gives energy to the pion, i.e. the pion absorbs gamma quanta. We consider three cases:

1. $\pi^+$ scattering (Fig. 3a). The transition matrix element is given by

$$M \propto \int \phi^\star_{\rm out} V(t) \phi_{\rm in} {\rm dt}\,, \tag{91}$$

where

$$\phi_{\rm in} \propto \exp(-iE_{\rm in}t) \qquad \phi^\star_{\rm out} \propto \exp(iE_{\rm out}t)\,. \tag{92}$$

The time integration yields

$$M \propto \delta(E_{\rm out} - E_{\rm in} - \omega) \Rightarrow E_{\rm out} = E_{\rm in} + \omega\,. \tag{93}$$

Thus the $\pi^+$ meson has absorbed a photon of energy $\omega$.

2. $\pi^-$ scattering (Fig. 3b). The incoming $\pi^-$ with energy $E_1 > 0$ corresponds to an outgoing $\pi^+$ with negative energy $E_{\rm out} = -E_1 < 0$, while the outgoing $\pi^-$ with energy $E_2 > 0$ corresponds to an incoming $\pi^+$ with energy $E_{\rm in} = -E_2 < 0$. The transition matrix element is computed for the backwards running particle with negative energy

$$M \propto \int \phi^\star_{\rm out} V(t) \phi_{\rm in} {\rm dt} \propto \int \exp\left(i\left(E_{\rm out} - E_{\rm in} - \omega\right)t\right) {\rm dt}\,. \tag{94}$$

Expressed in terms of the $\pi^-$ energies

$$M \propto \int \exp\left(i\left(E_2 - E_1 - \omega\right)t\right) {\rm dt} = 2\pi\delta(E_2 - E_1 - \omega)\,. \tag{95}$$

Thus the energy of the $\pi^-$ meson is also increased by the amount $\omega$.

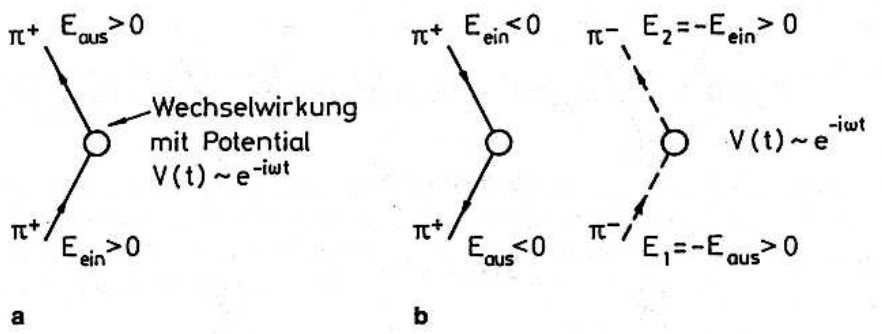


Figure 3: a) $\pi^+$ scattering off a time dependent potential. b) Scattering of a backwards propagating $\pi^+$ with negative energy is equivalent to a forward propagating $\pi^-$ with positive energy. (Figure from P. Schmüser, Feynman Graphen und Eichtheorien für Experimentalphysiker)

3. Creation of a $\pi^+\pi^-$ pair (Fig. 4a). In the Stückelberg-Feynman interpretation this corresponds to a negative energy $\pi^+$ propagating backwards in time, which through the interaction with the potential is converted into a positive energy $\pi^+$ propagating forwards in time.

$$M \propto \int \exp\left(i\left(E_{\text{out}} - E_{\text{in}} - \omega\right)t\right) \mathrm{dt} = 2\pi\delta(E_2 + E_1 - \omega)\,, \qquad (96)$$

which implies $E_1 + E_2 = \omega$. Thus, the energy of the absorbed photon equals the total energy of the created pair.

For the absorption of a pion pair (Fig. 4b) one must choose a potential of the form $V(t) = V_0 \exp(i\omega t)$, in order that energy can be absorbed by the potential. One finds that the energy absorbed by the potential equals the total energy of the absorbed pair.

These considerations are only qualitative, but they illustrate the utility of the Stückelberg-Feynman approach, where four different processes can be handled in the same formalism.

We have found that a relativistic quantum field theory necessarily involves antiparticles. These lead to new processes, not present in non-relativistic quantum mechanics. We have discussed the lowest order processes, the creation and annihilation of particles and antiparticles. In processes that are of higher order in the sense of a perturbation expansion, these processes give rise to new contributions to the scattering of particles. Consider electron scattering off a

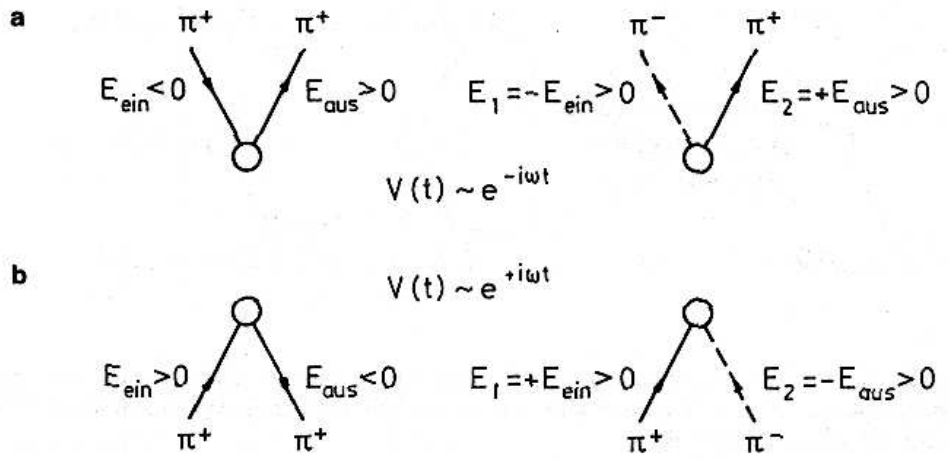


Figure 4: a) $\pi^+\pi^-$ creation by a time dependent potential. b) $\pi^+\pi^-$ annihilation. (Figure from P. Schmüser, Feynman Graphen und Eichtheorien für Experimentalphysiker)

potential. To second order in the interaction two processes are possible, as shown in Fig. 5.

1. For $t_2 > t_1$ one obtains the standard double scattering term, which is present also in ordinary second order perturbation theory of the Schrödinger type.

2. For $t_2 < t_1$, a new type of contribution arises, where at $t = t_2$ an $e^+e^-$ pair is created by the potential and at $t = t_1$ the positron and the incoming electron is annihilated by the potential. This process illustrates the many-body nature of QFT, where in an intermediate state there are three particles present, two electrons and a positron.

Because the initial and final states are identical, the two processes must be added coherently! Thus, there will be interference between the two contributions. At low energies the processes of the second type are generally expected to be small, due to the large energy of the intermediate state. However, for electromagnetic interactions they cannot be neglected, since they are required by gauge invariance. In fact at threshold, such a term dominates scattering of photons off electrons and gives rise to the famous Thomson term, obtained also using classical considerations.

We conclude that the Dirac "wavefunction", due to the interpretation of the

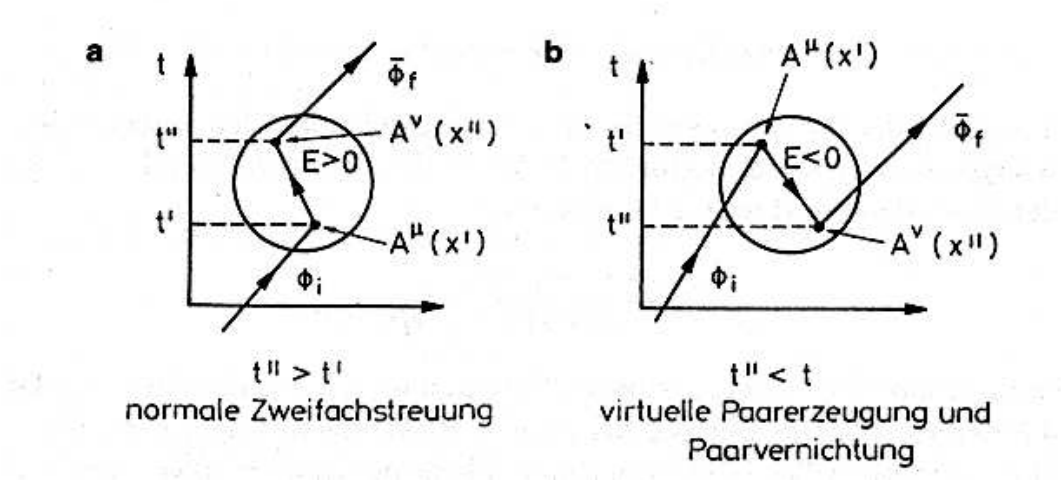


Figure 5: Electron scattering to second order in the interaction. a) Ordinary double scattering. b) production and destruction of a virtual electron-positron pair. (Figure from P. Schmüser, Feynman Graphen und Eichtheorien für Experimentalphysiker.)

negative energy states, describes many-body processes in a complicated vacuum.

# 9 Feynman diagrams

So far we have dealt only with non-interacting particles, except for the qualitative arguments at the end of section 8. For the more interesting case of interacting particles one is not in general able to find the exact solution. Therefore one has developed strategies to find approximate answers to problems one is interested in. The most important approximate method for dealing with quantum mechanical problems is perturbation theory. In QFT the perturbation series can be phrased in terms of the Feynman diagrams, which is the subject of this section. Before we come to the Feynman diagrams we need some preliminaries, in particular the Green's functions or propagators.

## 9.1 Green's functions

The Dirac equation for an electron in an electromagnetic field is most simply obtained by minimal substitution $\partial^\mu \to \partial^\mu + ieA^\mu$, a procedure which preserves gauge invariance[5]. One thus finds

$$\left[i\gamma^\mu \left(\partial_\mu + ieA_\mu(x)\right) - m\right] \Psi(x) = 0\,, \tag{97}$$

where we have introduced the four-dimensional vector potential $A^\mu = (\Phi, \vec{A})$, which transforms like a four-vector. By moving the interaction term to the right hand side, we obtain the free Dirac equation on the left hand side with an inhomogeneity on the right hand side

$$\left[i\partial\!\!\!/ - m\right] \Psi(x) = eA\!\!\!/(x)\Psi(x)\,. \tag{98}$$

This equation is not in general analytically solvable. In order to understand the technique for solving this equation, we need to understand the Green's function technique for solving differential equations. To recall this technique, we consider the Poisson equation for the electric field generated by a charge distribution $\rho(\vec{x})$

$$\vec{\nabla}^{\,2}\Phi(\vec{x}) = -\rho(\vec{x}) \tag{99}$$

One first solves the equation for a point charge at $\vec{x}_0$, $\rho_p(\vec{x}) = q\delta^{(3)}(\vec{x} - \vec{x}_0)$. This is easily done by a Fourier transform to momentum space

$$\Phi(\vec{x}) \;\; = \;\; \int \frac{d^3k}{(2\pi)^3}\, \Phi(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} \tag{100}$$

[5]This is the Lorentz covariant version of the substitution used in non-relativistic problems, see also eq. 53

$$\rho_p(\vec{x}) \;=\; \int \frac{d^3k}{(2\pi)^3}\,\rho_p(\vec{k})\,e^{i\vec{k}\cdot(\vec{x}-\vec{x}_0)}\,, \tag{101}$$

where $\rho_p(\vec{k}) = q$. Each Fourier component satisfies the equation

$$-\vec{k}^{\,2}\Phi_p(\vec{k}) = -q\,e^{-i\vec{k}\cdot\vec{x}_0} \tag{102}$$

or

$$\Phi_p(\vec{k}) = \frac{q}{\vec{k}^{\,2}}\,e^{-i\vec{k}\cdot\vec{x}_0}\,. \tag{103}$$

Inserting this into the Fourier integral (101) we find

$$\Phi_p(\vec{x}) \;=\; q\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\vec{k}^{\,2}}\,e^{i\vec{k}\cdot(\vec{x}-\vec{x}_0)} = \frac{q}{4\,\pi}\frac{1}{|\vec{x}-\vec{x}_0|} \tag{104}$$

$$\;=\; \int d^3x'\,G(\vec{x},\vec{x}')\,\rho_p(\vec{x}'). \tag{105}$$

Using the last equation we can identify the Green's function

$$G(\vec{x},\vec{x}') \equiv G(|\vec{x}-\vec{x}'|) = \frac{1}{4\,\pi}\frac{1}{|\vec{x}-\vec{x}'|}\,. \tag{106}$$

In a translationally invariant system the Green's function depends only on the difference in the coordinates. With the help of the Green's function one can construct the solution to the original problem (99)

$$\Phi(\vec{x}) = \int d^3x'\,G(|\vec{x}-\vec{x}'|)\,\rho(\vec{x}')\,. \tag{107}$$

In summary, The Green's function for the Poisson equation, which is given by eq. (106), satisfies the equation

$$\vec{\nabla}_x^{\,2}\,G(|\vec{x}-\vec{x}'|) = -\delta^{(3)}(\vec{x}-\vec{x}')\,. \tag{108}$$

In momentum space the Green's function is simply $G(\vec{k}) = 1/\vec{k}^{\,2}$.

Now we return to the Dirac equation for an electron in an electromagnetic field (98). In analogy with the Green's function for the Poisson equation, we define a Green's function $S(x,y)$ for the free Dirac equation, which satisfies the differential equation[6]

$$\Big[i\partial\!\!\!/ - m\Big]\,S(x,y) = \mathbf{1}\delta^{(4)}(x-y)\,. \tag{109}$$

[6]For simplicity we use the following convention: a function of a 4-vector (e.g. $x^\mu$) is denoted by $f(x)$, while a function of only the spatial components is denoted by $f(\vec{x})$. We will also sometimes for simplicity drop the indices in products of four vectors: e.g. $xp \equiv x^\mu p_\mu$.

Here **1** is the $4 \times 4$ unit matrix. The Green's function is also a $4 \times 4$ matrix. It follows from invariance under time reversal, space inversion and translational invariance that $S(x,y) = S(y,x) = S(x-y)$. The 4-dimensional $\delta$ function is defined by $\delta^{(4)}(x-y) = \delta(x^0 - y^0)\delta^{(3)}(\vec{x} - \vec{x}')$, where $x^0$ is the time coordinate of the four vector $x = (x^0, \vec{x})$ etc..

A formal solution of the (98) is then given by

$$\Psi(x) = \phi(x) + e \int d^4y \, S(x-y) \not{A}(y) \Psi(y) \tag{110}$$

where $\phi(x)$ is a solution of the free Dirac equation, i.e.,

$$\left(i \not{\partial} - m\right) \phi(x) = 0 \tag{111}$$

and $d^4x = dt d^3x$. This is determined by fixing the boundary conditions. For instance in a scattering problem one has to fix the current of incident particles. However, (110) is not really a solution to the problem, since it is not in closed form, i.e. the unknown function $\Psi$ appears also on the right hand side of (110).

However, since the electromagnetic coupling constant is small, the second term in (110) can be treated as a perturbation. The relevant parameter is the fine structure constant $\alpha = e^2/(4\pi) \simeq 1/137$. Thus, we try to solve the problem by iteration:

$$\Psi^{(0)}(x) = \phi(x) \tag{112}$$

$$\Psi^{(1)}(x) = \phi(x) + e \int S(x-y) \not{A}(y) \Psi^{(0)}(y) d^4y \tag{113}$$

$$\begin{aligned} \Psi^{(2)}(x) &= \phi(x) + e \int S(x-y) \not{A}(y) \Psi^{(1)}(y) d^4y \\ &= \phi(x) + e \int S(x-y) \not{A}(y) \phi(y) d^4y \\ &+ e^2 \int \int S(x-y) \not{A}(y) S(y-z) \not{A}(z) \phi(z) d^4y d^4z \end{aligned} \tag{114}$$

The three terms in the second order expression correspond to unperturbed motion, single scattering and double scattering off the potential $A^\mu$.

### 9.2 The Dirac propagator

In quantum field theory the Green's functions are most often referred to as propagators, because they "propagate" a solution in time. The form of the

propagators for free particles are rather simple in momentum space but fairly complicated in configuration space. It is therefore usually easier to work in momentum space. The free Dirac propagator satisfies the equation (109). Its Fourier transform is defined by

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4}\, S(p)\, e^{-ip(x-y)}\,, \tag{115}$$

where $d^4p = dp^0 d^3p = dE d^3p$. Similarly,

$$\delta^{(4)}(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\,, \tag{116}$$

We thus obtain the equation satisfied by the momentum space propagator

$$(\not{p} - m)\, S(p) = \mathbf{1}\,. \tag{117}$$

One then obtains the propagator by multiplying from the left by $\not{p} + m$ and using $(\not{p} - m)(\not{p} + m) = p^2 - m^2$, which follows from the anticommutation rule for $\gamma$ matrices (62). One thus finds

$$S(p) = \frac{\not{p} + m}{p^2 - m^2}. \tag{118}$$

or in short hand form

$$S(p) = \frac{1}{\not{p} - m}\,, \tag{119}$$

where the matrix inversion is not explicitly performed. Thus, the propagator in configuration space is given by

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{\not{p} + m}{p^2 - m^2}\, e^{-ip(x-y)}\,. \tag{120}$$

For the propagator to be well defined one must specify how the poles at $p^2 = m^2$, i.e., at $p^0 = \pm\sqrt{\vec{p}^{\,2} + m^2} = \pm E$ are to be circumvented. This corresponds to a boundary condition on the propagator in configuration space. The physically sensible boundary condition, that *positive energy solutions propagate forwards and negative energy solutions propagate backwards in time*, is implemented by adding an infinitesimal imaginary part $i\delta$ to the denominator

$$S_F(p) = \frac{\not{p} + m}{p^2 - m^2 + i\delta} = \frac{1}{\not{p} - m + i\delta}\,, \tag{121}$$

This propagator is called the Feynman propagator. A corresponding Feynman propagator exists for all particle species in a QFT. The Feynman propagator in configuration space is then given by

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{\not{p}+m}{p^2-m^2+i\delta} e^{-ip(x-y)} . \tag{122}$$

Let us now discuss how solutions of the Dirac equation are propagated in time by the Feynman propagator $S_F$. We use the identity

$$\frac{\not{p}+m}{p^2-m^2+i\delta} = \frac{m}{E}\left(\Lambda_+(\vec{p})\frac{1}{p^0-E+i\delta} - \Lambda_-(\vec{p})\frac{1}{p^0+E-i\delta}\right) , \tag{123}$$

where

$$\Lambda_+(\vec{p}) = \frac{\gamma^0 E - \vec{\gamma}\cdot\vec{p}+m}{2m} = \sum_{s=1,2} u(\vec{p},s)\bar{u}(\vec{p},s) \tag{124}$$

$$\Lambda_-(\vec{p}) = \frac{-\gamma^0 E + \vec{\gamma}\cdot\vec{p}+m}{2m} = -\sum_{s=1,2} v(\vec{p},s)\bar{v}(\vec{p},s) , \tag{125}$$

where $\Lambda_+$ and $\Lambda_-$ project onto positive and negative energy states, respectively. The integral over $p^0$ is then easily performed

$$\begin{aligned} S_F(x-x') &= -i\int \frac{d^3p}{(2\pi)^3}\frac{m}{E}\left[e^{-ip(x-x')}\,\Lambda_+(\vec{p})\,\theta(t-t')\right. \\ &+ \left. e^{ip(x-x')}\,\Lambda_-(\vec{p})\,\theta(t'-t)\right] . \end{aligned} \tag{126}$$

Note that here the zeroth component of $p^0 = E = \sqrt{\vec{p}^{\,2}+m^2}$.

Let's now consider a positive energy solution at time $t$ ($x=(t,\vec{x})$)

$$\Psi_k^{(+)}(x) = Nu(k)\,e^{-ikx} . \tag{127}$$

The solution at another time $t'$ ($x'=(t',\vec{x}')$) is given by

$$\theta(t'-t)\Psi_k^{(+)}(x') = i\int d^3x S_F(x'-x)\gamma^0\Psi_k^{(+)}(x) . \tag{128}$$

To prove this, one uses (126), the identity

$$\int d^3x e^{ipx}\,e^{-ikx} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{k}) , \tag{129}$$

as well as

$$\Lambda_+(\vec{p})\gamma^0 u(p) = \frac{E}{m}u(p) \qquad \Lambda_-(\vec{p})\gamma^0 u(p) = 0 \tag{130}$$

For the conjugate solution one finds similarly

$$\theta(t'-t)\bar{\Psi}_k^{(+)}(x') = i\int d^3x \bar{\Psi}_k^{(+)}(x)\,\gamma^0\,S_F(x-x') \tag{131}$$

Analogously one can show that a negative energy solution is propagated only backwards in time, i.e.,

$$\theta(t-t')\Psi_k^{(-)}(x') = -i\int d^3x S_F(x'-x)\gamma^0\Psi_k^{(-)}(x)\,. \tag{132}$$

We will not expand further on this subject here. For a more detailed discussion see the textbooks by Schmüser and Bjorken & Drell vol. 1.

We close this subsection by discussing the interacting propagator $S'_F$, which includes the effects of the interactions. For the problem at hand, the interacting propagator is defined by

$$\left[i\gamma^\mu\frac{\partial}{\partial x^\mu} - e\not{A} - m\right]S'_F(x-y) = \delta^{(4)}(x-y)\,, \tag{133}$$

i.e., it describes the propagation of an electron including the interaction effects. It is straightforward to show that the solution of the Dirac equation can now be written as

$$\Psi(x) = \phi(x) + e\int S'_F(x-y)\not{A}(y)\phi(y)d^4y\,, \tag{134}$$

where the unknown function $\Psi$ no longer appears on the right hand side, i.e., it is an explicit equation for $\Psi$. However, the interacting propagator $S'_F$ must be determined before this equation can be used. By taking the difference of the defining equations for the interacting and free Dirac propagator

$$\left[i\not{\partial} - e\not{A} - m\right]S'_F(x-y) = \delta^{(4)}(x-y) \tag{135}$$

$$\left[i\not{\partial} - m\right]S_F(x-y) = \delta^{(4)}(x-y)\,, \tag{136}$$

one finds

$$\left[i\not{\partial} - m\right](S'_F(x-y) - S_F(x-y)) = e\not{A}(x)S'_f(x-y)\,. \tag{137}$$

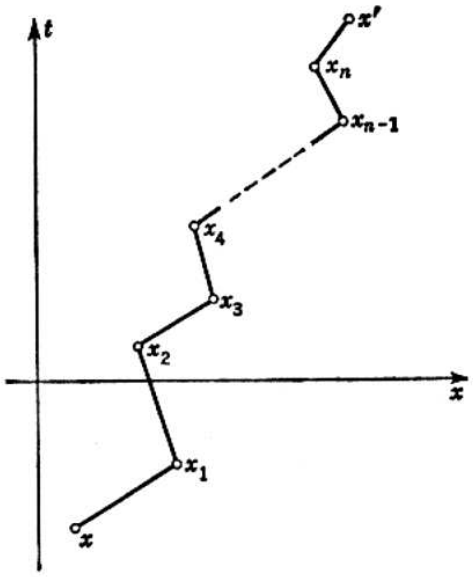


Figure 6: Space time diagram illustrating the $n$th order contribution to the Feynman propagator. (Figure from J.D. Bjorken & S.D. Drell, Relativistic Quantum Mechanics.)

Now we can use the Green's function technique to formally solve this equation. We then arrive at the Dyson equation, which determines the interacting propagator in terms of the free one for a given interaction

$$S'_F(x-y) = S_F(x-y) + e\int S_F(x-z)\,\not{A}(z)\,S'_F(z-y)d^4z\,. \tag{138}$$

Clearly this is an implicit equation for $S'_F$. Consequently, we have just moved the difficulties to a different level.

We can generate the perturbation series for the propagator by iteration of the Dyson equation (138)

$$S_F^{(0)}(x-x') = S_F(x-x') \tag{139}$$

$$S_F^{(1)}(x-x') = S_F(x-x') + e\int S_F(x-x_1)\,\not{A}(x_1)\,S_F(x_1-x')d^4x_1 \tag{140}$$

$$S_F^{(2)}(x-x') = S_F(x-x') + e\int S_F(x-x_1)\,\not{A}(x_1)\,S_F(x_1-x')d^4x_1$$

$$+ \quad e^2 \int\int S_F(x-x_1)\, \not{A}(x_1)\, S_F(x_1-x_2) \not{A}(x_2) S_F(x_2-x') d^4x_1 d^4x_2 \,. \tag{141}$$

A typical term in the series is diagrammatically illustrated in Fig. 6. Each line corresponds to a propagator $S_F$ and each intermediate dot to an interaction with the electromagnetic field $A^\mu$. All intermediate points are integrated over all of space-time. This is a prototype Feynman diagram! Given the Feynman rules there is a one-to-one correspondence between the Feynman diagrams and the mathematical expression for the corresponding contribution to the propagator. This will be discussed in more detail in the next section.

### 9.3 Rutherford scattering

As an example, which we use to illustrate the Feynman rules we consider the scattering of electrons off a Coulomb potential, Rutherford scattering. The scattering process is shown schematically in Fig. 7. At time $t = t_1$ the incident electron, denoted by a wave packet, is far away from the potential and does not experience any interaction. At time $t = t'$ the electron arrives at the target, and a scattered radially outwards traveling wave is created. Finally, at time $t = t_2$, the scattered wave reaches the detector, which is located at an angle $\theta$ relative to the incident beam and covers a small solid angle $\Delta\Omega$. Only that part of the scattered wave, which runs in the direction of $\vec{p}_f$ is measured. Thus, one must project out a final state $\phi_f$ with momentum direction $\vec{p}_f$ from the scattered wave. In order to compute the necessary matrix element, we define the scattering matrix, S-matrix, through

$$\psi_{scatt} = S\phi_i \,. \tag{142}$$

We obtain the matrix element for the transition $i \to f$, by projecting the scattering state onto the final state $\phi_f$

$$S_{fi} = \int d^3x_2\, \phi_f^\dagger(x_2)\, \underbrace{S\,\phi_i(x_2)}_{\psi_{scatt}(x_2)} \,. \tag{143}$$

By inserting the perturbation expansion for the scattered wave (112-114) we obtain a perturbation expansion for the S-matrix: $S_{fi} = \delta fi + S_{fi}^{(1)} + S_{fi}^{(2)} + \ldots$. To lowest order in the interaction, the scattered wave is then given by (113)

$$\psi_{scatt}^{(1)}(x_2) = \phi_i(x_2) + e \int S_F(x_2 - x') \not{A}(x') \phi_i(x') d^4x' \,, \tag{144}$$

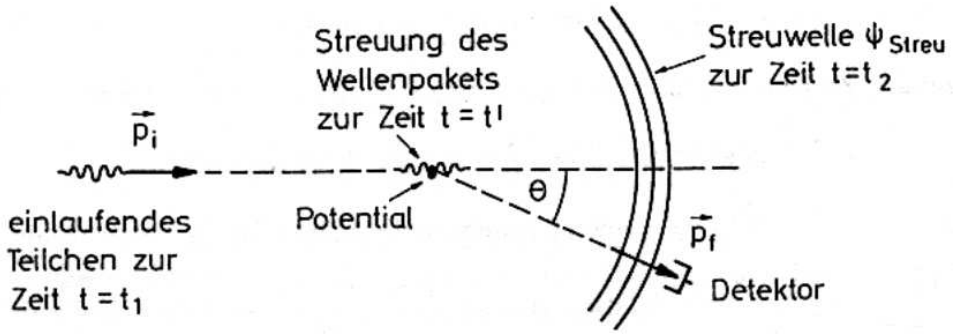


Figure 7: The scattering of an electron off a potential. (Figure from P. Schmüser, Feynman Graphen und Eichtheorien für Experimentalphysiker.)

and the corresponding S-matrix by

$$S_{fi}^{(1)} = e \int d^4x' \underbrace{\int d^3x_2 \phi_f^\dagger(x_2) S_F(x_2 - x')}_{-i\bar{\phi}_f(x')} \not{A}(x') \phi_i(x') \tag{145}$$

Thus, the resulting S-matrix is

$$S_{fi}^{(1)} = -ie \int d^4x' \bar{\phi}_f(x') \not{A}(x') \phi_i(x') \,. \tag{146}$$

For the incoming and outgoing states we use the plain wave states[7]

$$\begin{aligned} \phi_i(x) &= \sqrt{\frac{m}{E_i V}}\, u(p_i, s_i)\, e^{-ip_i x} \\ \bar{\phi}_f(x) &= \sqrt{\frac{m}{E_f V}}\, \bar{u}(p_f, s_f)\, e^{ip_f x} \,. \end{aligned} \tag{147}$$

and for the electromagnetic field we assume a Coulomb potential[8]

$$A^0(x) = \frac{-Ze}{4\,\pi\,|\vec{x}|} \qquad \vec{A}(x) = 0 \,. \tag{148}$$

Using this input, the S-matrix becomes

$$S_{fi}^{(1)} = \frac{iZe^2}{4\pi} \frac{1}{V} \frac{m}{\sqrt{E_i E_f}} \bar{u}(p_f, s_f) \gamma^0 u(p_i, s_i) \int d^4x' \frac{1}{|\vec{x}'|} e^{i(p_f - p_i)x'} \tag{149}$$

[7]The normalization is chosen such that $\phi^\dagger \phi = 1$.

[8]Remember that for the electron $e = -|e|$, so that (148) corresponds to the Coulomb field generated by a positive charge $Z$.

The final integral is easily performed

$$\int d^4x' \frac{1}{|\vec{x}'|} e^{i(p_f - p_i)x'} = 2\pi\delta(E_f - E_i)\frac{4\pi}{\vec{q}^{\,2}}\,, \tag{150}$$

which yields

$$S_{fi}^{(1)} = \frac{2\pi i Z e^2}{V}\frac{m}{\sqrt{E_i E_f}}\frac{\bar{u}(p_f, s_f)\gamma^0 u(p_i, s_i)}{\vec{q}^{\,2}}\delta(E_f - E_i)\,. \tag{151}$$

The transition probability per particle for $p_i s_i \to p_f s_f$ is

$$\begin{aligned} |S_{fi}|^2 \quad \overbrace{\frac{V d^3 p_f}{(2\pi)^3}}^{\text{density of states}} \quad &= \quad \frac{Z^2 e^4 m^2}{V E_i}\,\frac{|\bar{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2}{(\vec{q}^{\,2})^2} \\ &\times \quad \frac{d^3 p_f}{(2\pi)^3 E_f}(2\pi\delta(E_f - E_i))^2\,. \end{aligned} \tag{152}$$

For transitions in the time interval $[-T/2, T/2]$, one of the factors $2\pi\delta(E_f - E_i)$ can be replaces by a factor $T$. The transition probability per unit time is then

$$\begin{aligned} R &= \frac{1}{T}|S_{fi}|^2\,\frac{V d^3 p_f}{(2\pi)^3} \\ &= \frac{Z^2 e^4 m^2}{V E_i}\,\frac{|\bar{u}\gamma^0 u|^2}{(\vec{q}^{\,2})^2}\frac{d^3 p_f}{(2\pi)^3 E_f}\,2\pi\delta(E_f - E_i)\,. \end{aligned} \tag{153}$$

Furthermore, the cross section is equal to $R/J$, where $J$ is the flux of incident particles

$$\vec{J} = \bar{\phi}_f \vec{\gamma}\phi_i = \frac{1}{V}\frac{\vec{p}}{E_i} = \frac{1}{V}\vec{v_i} \tag{154}$$

Furthermore, using $d^3 p_f = d\Omega p_f^2 dp_f$ and $E_f dE_f = p_f dp_f$ we find

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \int \frac{Z^2 e^4 m^2}{|\vec{v}_i| E_i}\,\frac{|\bar{u}\gamma^0 u|^2}{(\vec{q}^{\,2})^2}\frac{p_f E_f dE_f}{(2\pi)^3 e_f}2\pi\delta(E_f - E_i) \\ &= \frac{Z^2 e^4 m^2}{4\pi^2}\,\frac{|\bar{u}\gamma^0 u|^2}{(\vec{q}^{\,2})^2}\,. \end{aligned} \tag{155}$$

In an experiment, where the beam is unpolarized and the polarization of the final particles is not measured, we must average over the initial polarizations and sum over those in the final state. Thus the spin-averaged cross section is

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{Z^2 e^4 m^2}{8\pi^2}\sum_{s_i s_f}\frac{|\bar{u}\gamma^0 u|^2}{(\vec{q}^{\,2})^2}\,. \tag{156}$$

The spin sum can be converted into a trace, which we evaluate using the trace theorems

*The trace of an odd number of $\gamma$ matrices vanishes* (157)

$$\begin{aligned}
Tr\mathbf{1} &= 4 && (158)\\
Tr\not{a}\not{b} &= 4ab = 4a^\mu b_\mu && (159)\\
Tr\not{a}\not{b}\not{c}\not{d} &= 4[(ab)(cd) - (ac)(bd) + (ad)(bc)] && (160)\\
Tr\gamma_5\not{a}\not{b} &= 0 && (161)\\
Tr\gamma_5\not{a}\not{b}\not{c}\not{d} &= 4i\epsilon_{\alpha\beta\gamma\delta}a^\alpha b^\beta c^\gamma d^\delta\,. && (162)
\end{aligned}$$

The spin sum can be written as

$$\sum_{s_i s_f} \bar{u}(p_f,s_f)\gamma^0 u(p_i,s_i)\underbrace{u^\dagger(p_i,s_i)(\gamma^0)^\dagger(\gamma^0)^\dagger u(p_f s_f)}_{\bar{u}(p_i,s_i)\gamma^0 u(p_f,s_f)} \tag{163}$$

Using the identity (124)

$$\sum_s u(p,s)_\alpha \bar{u}(p,s)_\beta = \left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta} \tag{164}$$

we can rewrite the spin sum

$$\begin{aligned}
&\textstyle\sum_{\alpha\beta\gamma\delta} \quad \sum_{s_f} \bar{u}(p_f,s_f)_\alpha\,(\gamma^0)_{\alpha\beta}\left(\frac{\not{p}_i+m}{2m}\right)_{\beta\gamma}(\gamma^0)_{\gamma\delta}\,u(p_f,s_f)_\delta\\
= &\textstyle\sum_{\alpha\beta\gamma\delta} \quad (\gamma^0)_{\alpha\beta}\left(\frac{\not{p}_i+m}{2m}\right)_{\beta\gamma}(\gamma^0)_{\gamma\delta}\left(\frac{\not{p}_f+m}{2m}\right)_{\delta\alpha}\,. \qquad (165)
\end{aligned}$$

Thus, the trace, which we have to evaluate is

$$\frac{1}{4m^2}\,Tr\,\left(\gamma^0(\not{p}_i+m)\gamma^0(\not{p}_f+m)\right) \tag{166}$$

Using the trace theorems we find

$$\frac{1}{m^2}\left(2E_iE_f - (p_ip_f) + m^2\right)\,. \tag{167}$$

For elastic scattering $|\vec{p_i}| = |\vec{p_f}| \equiv p$ and $E_f = E_i \equiv E$. Furthermore

$$\begin{aligned}
(p_ip_f) &= E_iE_f - |\vec{p_i}||\vec{p_f}|\cos\theta\\
&= E^2 - p^2\cos\theta\\
&= m^2 + p^2\sin^2\theta/2 && (168)
\end{aligned}$$

Thus, the trace reduces to

$$2\frac{E^2}{m^2}(1-\beta^2\sin^2\theta/2)\,, \tag{169}$$

where $\beta = p/E$, the velocity. Finally $\vec{q}^{\,2} = (\vec{p}_i - \vec{p}_f)^2 = 2p^2(1-\cos\theta) = 4p^2\sin^2\theta/2$. We are now ready to write down the final form of the spin averaged cross section

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{Z^2\alpha^2}{4\beta^2p^2\sin^4\theta/2}(1-\beta^2\sin^2\theta/2)\,. \tag{170}$$

This is the so called Mott cross section. We note that the factor $(1-\beta^2\sin^2\theta/2)$ is due to spin. In the limit of slow electrons ($\beta \to 0$) the Mott cross section reduces to

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{Z^2\alpha^2m^2}{4p^4\sin^4\theta/2}\,. \tag{171}$$

This is the Rutherford cross section. In the ultrarelativistic limit again, $\beta \to 1$, one finds

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{Z^2\alpha^2}{4\beta^2E^2\sin^4\theta/2}(1-\sin^2\theta/2)\,. \tag{172}$$

Note that in the for an ultrarelativistic spin-1/2 particle, the spin factor is $(1-\sin^2\theta/2) = (1/2)(1+\cos\theta)$. Exercise: Why does the spin factor vanish for $\theta = \pi$ in this case?

### 9.4 Photon propagator

In the previous section we discussed the interaction of an electron with an external potential $A^\mu$. Such a calculation can be used for scattering off a heavy nucleus, which is almost not affected by the scattered electron. However, for scattering off a lighter object, like another electron or a muon, we must compute the electromagnetic field generated by the target particle. We consider the reaction $e^- + \mu^- \to e^- + \mu^-$. The wave equation for the electromagnetic field is the Maxwell equation

$$\partial_\mu\partial^\mu A^\nu(x) = ej^\nu_{muon}(x)\,, \tag{173}$$

where $j^\nu_{muon}(x)$ is the muon current and $A^\nu = (\Phi, \vec{A})$ is the vector potential. We solve the wave equation by using the Green's function, which is defined by

$$\partial_\mu\partial^\mu D^{\alpha\beta}(x-y) = g^{\alpha\beta}\delta^{(4)}(x-y)\,. \tag{174}$$

Then, the solution is given by

$$A_\mu(x) = e \int d^4y \, D_{\mu\nu}(x-y) j^\nu_{muon}(y) \, . \tag{175}$$

To determine the Green's function we change to momentum space

$$D^{\mu\nu}(x-y) = \int \frac{d^4q}{(2\pi)^4} D^{\mu\nu}(q) e^{-iq(x-y)} \, . \tag{176}$$

One finds

$$D_F^{\mu\nu}(q) = -\frac{g^{\mu\nu}}{q^2 + i\delta} \tag{177}$$

where the imaginary infinitesimal $i\delta$ is due to the Feynman boundary condition, where positive energies propagate forwards and negative backwards.

The Dirac equation for the electron in the electromagnetic field is

$$\left[ i\partial\!\!\!/ - m \right] \Psi^{(e)}(x) = eA\!\!\!/(x) \Psi^{(e)}(x) \, , \tag{178}$$

while the electromagnetic field is given by

$$A_\mu(x) = e \int d^4y \, [D_F(x-y)]_{\mu\nu} j^\nu_{muon}(y) \, . \tag{179}$$

Inserting this into the S-matrix (146), one finds

$$S_{fi} = -ie^2 \int\int \bar{\phi}_f^{(e)}(x') \gamma^\alpha \phi_i^{(e)}(x') [D_F(x'-x)]_{\alpha\beta} j^\beta_{muon}(x) d^4x d^4x' \, . \tag{180}$$

The electron current is

$$j^{(e)}_\alpha(x) = \bar{\phi}_f^{(e)}(x) \gamma_\alpha \phi_i^{(e)}(x). \tag{181}$$

Since the muon also satisfies the Dirac equation, its current is of the same form

$$j^{(\mu)}_\alpha(x) = \bar{\phi}_f^{(\mu)}(x) \gamma_\alpha \phi_i^{(\mu)}(x) \, . \tag{182}$$

We thus find

$$S_{fi} = -ie^2 \int\int \bar{\phi}_f^{(e)}(x') \gamma^\alpha \phi_i^{(e)}(x') [D_F(x'-x)]_{\alpha\beta} \bar{\phi}_f^{(\mu)}(x) \gamma_\alpha \phi_i^{(\mu)}(x) d^4x d^4x' \, . \tag{183}$$

Using

$$\phi_i^{(e)}(x) = \sqrt{\frac{m_e}{E_i^{(e)}V}}u^{(e)}(p_1)e^{-ip_1x}\,, \tag{184}$$

$$\bar{\phi}_f^{(e)}(x) = \sqrt{\frac{m_e}{E_f^{(e)}V}}\bar{u}^{(e)}(p_3)e^{ip_3x}\,, \tag{185}$$

$$\phi_i^{(\mu)}(x) = \sqrt{\frac{m_\mu}{E_i^{(\mu)}V}}u^{(\mu)}(p_2)e^{-ip_2x}\,, \tag{186}$$

$$\bar{\phi}_f^{(\mu)}(x) = \sqrt{\frac{m_\mu}{E_f^{(\mu)}V}}\bar{u}^{(\mu)}(p_4)e^{ip_4x} \tag{187}$$

and (176-177) we then obtain

$$\begin{aligned} S_{fi} &= -ie^2\frac{m_em_\mu}{V^2\sqrt{E_i^{(e)}E_f^{(e)}E_i^{(\mu)}E_f^{(\mu)}}}\bar{u}^{(e)}(p_3)\gamma^\alpha u^{(e)}(p_1)(-g_{\alpha\beta})\bar{u}^{(\mu)}(p_4)\gamma^\beta u^{(\mu)}(p_2) \\ &\times \int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+i\delta}\underbrace{\int\int d^4xd^4x'e^{i(p_3-p_1-q)x}e^{i(p_4-p_2+q)x'}}_{(2\pi)^4\delta^{(4)}(p_3-p_1-q)(2\pi)^4\delta^{(4)}(p_4-p_2+q)} \end{aligned} \tag{188}$$

$$\begin{aligned} &= ie^2\frac{m_em_\mu}{V^2\sqrt{E_i^{(e)}E_f^{(e)}E_i^{(\mu)}E_f^{(\mu)}}}\frac{\bar{u}^{(e)}\gamma^\alpha u^{(e)}\bar{u}^{(\mu)}\gamma_\alpha u^{(\mu)}}{(p_3-p_1)^2+i\delta} \\ &\times (2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\,. \end{aligned} \tag{189}$$

We could now write down the cross section in terms of the S-matrix, like we did for the Rutherford cross section. However, it is more convenient to extract the kinematical factors which allways occur and deal with those once and for all, and define a reduced object, the invariant matrix element, which for $e\mu$ scattering is given by

$$\mathcal{M} = ie^2\frac{\bar{u}^{(e)}(p_3)\gamma^\alpha u^{(e)}(p_1)\bar{u}^{(\mu)}(p_4)\gamma_\alpha u^{(\mu)}(p_2)}{q^2+i\delta}\,, \tag{190}$$

where $q = p_3 - p_1$. One then has Feynman rules for calculating $\mathcal{M}$. In terms of the invariant matrix element, the cross section for the general reaction involving only fermions $12 \to 345\ldots n$ is

$$d\sigma = \frac{1}{|v_1-v_2|}\frac{m_1}{E_1}\frac{m_2}{E_2}\left|\mathcal{M}\right|^2\frac{m_3d^3p_3}{E_3(2\pi)^3}\cdots\frac{m_nd^3p_n}{E_n(2\pi)^3}$$

$$\times \quad (2\pi)^4\delta^{(4)}(p1+p2-\sum_{i=3}^{n}p_i)S\,, \tag{191}$$

where $E_i = \sqrt{\vec{p}_i^{\,2}+m_i^2}$ and $S$ is a symmetry factor, which is obtained by including a factor $1/m!$ if there are $m$ identical particles in the final state, i.e.,

$$S = \prod \frac{1}{m_i!}\,. \tag{192}$$

For bosons the normalization is different, so for a bosons in the initial or final state, the factor $m_i/E_i$ in (191) is replaced by $1/2E_i$.

For $e\mu$ scattering we then obtain

$$\begin{aligned} d\sigma &= \frac{1}{|v_e - v_\mu|}\frac{m_e}{E_i^{(e)}}\frac{m_\mu}{E_i^{(\mu)}}\,|\mathcal{M}|^2\,\frac{m_e d^3p_3}{E_f^{(e)}(2\pi)^3}\frac{m_\mu d^3p_4}{E_f^{(\mu)}(2\pi)^3} \\ &\times \quad (2\pi)^4\delta^{(4)}(p1+p2-p_3-p_4)\,, \end{aligned} \tag{193}$$

where the invariant matrix element is given by (190). The corresponding Feynman diagram is shown in fig. 8. The Feynman rules relevant for this diagram are indicated.

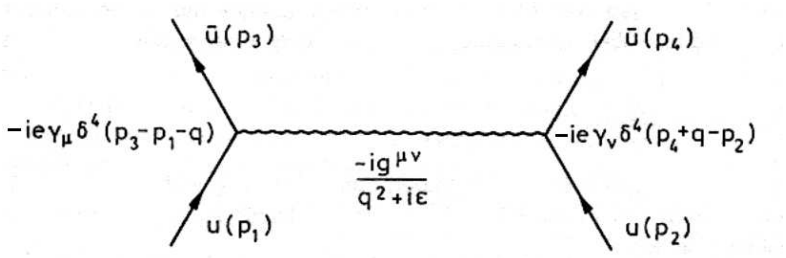


Figure 8: Feynman diagram for electron-muon scattering. (Figure from P. Schmüser, Feynman Graphen und Eichtheorien für Experimentalphysiker.)

### 9.5 Feynman rules

With the help of the Feynman rules one can put together the invariant amplitude for a scattering problem, a Green's function or some other amplitude one may need. The ingredients are the interaction vertices and the propagators. Which vertices are relevant depends on the process of interest and on the

theory one considers. In the Feynman diagram for $e\mu$-scattering the relevant vertices and propagator are indicated. The general Feynman rules for a theory describing spin-1/2 fermions, spin-0 bosons and photons are (for details see e.g. Bjorken & Drell)

1. Draw all connected diagrams for the process in question.

2. For each external spin-1/2 fermion line entering the graph, a factor $u(p,s)$ if the line is in the initial state and $v(p,s)$ if it is in the final state. Likewise, a factor $\bar{u}(p,s)$ or $\bar{v}(p,s)$ for each line leaving the graph[9].

3. For each external photon line, a polarization vector $\epsilon^\mu$.

4. For each internal spin-1/2 fermion line with momentum $p$, a factor

$$iS_F(p) = \frac{i}{\not{p} - m + i\delta} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\delta}\,. \tag{194}$$

5. For each internal meson line of spin zero with momentum $q$ a factor

$$i\Delta_F(q) = \frac{i}{q^2 - m^2 + i\delta}\,. \tag{195}$$

6. For each internal photon line with momentum q a factor[10]

$$iD_F^{\mu\nu}(q) = -\frac{ig^{\mu\nu}}{q^2 + i\delta} \tag{196}$$

7. For each internal momentum $k$ not fixed by momentum conservation, a factor

$$\int \frac{d^4k}{(2\pi)^4} \tag{197}$$

8. For each closed fermion loop, a factor $-1$

9. A factor $-1$ between graphs which differ only be an interchange of two external identical fermions.

[9] In these lectures we do not consider so called loop diagrams. If such diagrams are included, one must also include so called renormalization factors $Z$ for the external lines.

[10] For an internal line of a vector meson (e.g. $\rho$ or $\omega$), add an $m^2$ term to the denominator.

Rules that are specific to a certain type of interaction are e.g.[11]

10. For the electrodynamics of spin-1/2 particles, a factor $-ie\gamma^\mu$ for each vertex of the type shown in Fig. 8[12].

11. For the electrodynamics of spin-0 bosons, a factor $-ie(p+p')^\mu$ for each vertex of the type shown to the left in Fig. 9 and a factor $2ie^2 g^{\mu\nu}$ for the type of vertices shown to the right.

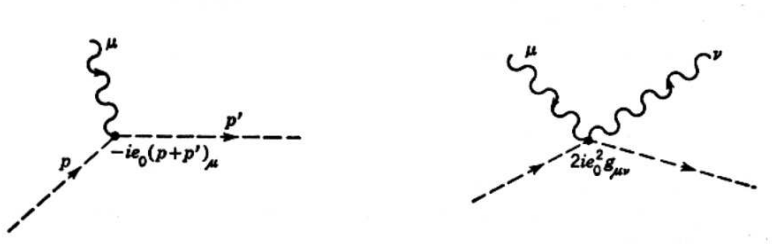


Figure 9: Feynman diagrams for scalar electrodynamics. (Figure from J.D. Bjorken & S.D. Drell, Relativistic Quantum Mechanics.)

For more examples, consult textbooks on QFT.

[11]Because we do not consider loop diagrams, we also ignore the counter terms for mass renormalization etc..

[12]For the interactions of vector mesons with nucleons, a factor of the form $-ig\gamma^\mu$ enters, but also other types of vertices are possible.