Introduction to the physics of hot and dense hadronic matter

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Content:

- Kinematics (Lorentz Transformation/ phase space)
- Elements of scattering theory
- Thermodynamics of strongly interacting matter
- Transport in theory and practice
- Chiral symmetry in vacuum and at finite density/temperature
- The properties of hadrons in dense/hot matter and possible signatures

1 Kinematics

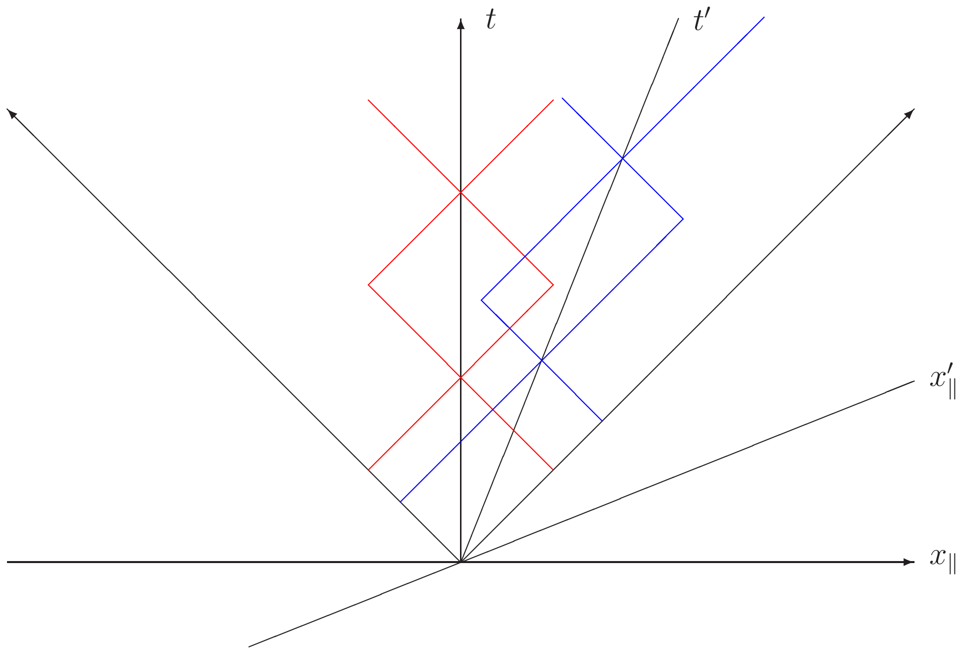
1.1 Lorentz transformation (LT)

Four-vector notation ($t = x_0$):

$$x = \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_\parallel \\ \vec{x}_\perp \end{pmatrix}, \quad x^2 = t^2 - \vec{x}^{\,2} = x_0^2 - (x_1^2 + x_2^2 + x_3^2) \tag{1}$$

Transfm.	Coordinates	transformations matrix	Det	invariances
rotation	$x_1 = x_1' \cos\varphi - x_2' \sin\varphi$ $x_2 = x_1' \sin\varphi + x_2' \cos\varphi$	$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$	1	angles $x_1^2 + x_2^2$
Galiei	$t = t'$ $x_\parallel = \beta t' + x_\parallel'$ $\vec{x}_\perp = \vec{x}\,'_\perp$	$\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$	1	t d^3x
Lorentz	$t = (t' + \frac{1}{c^2}\beta x_\parallel')\,\gamma$ $x_\parallel = (\beta t' + x_\parallel')\,\gamma$ $\vec{x}_\perp = \vec{x}\,'_\perp$	$\gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{ch}\, y & \mathrm{sh}\, y \\ \mathrm{sh}\, y & \mathrm{ch}\, y \end{pmatrix}$	1	$\mathrm{d}^4x = \mathrm{d}t\mathrm{d}^3x$ $\tau^2 = t^2 - \vec{x}^{\,2}$

(boost in $\parallel$ direction) (c=1)



Lorentz transformation for oscillating strings (yo-yo mode)

The boost is taken in $x_{\|}$-direction, the transverse coordinates $\vec{x}_{\perp}$ are unchanged. Condition $\det(\ldots) = 1$ leads to:

$$\gamma^2 - \beta^2\gamma^2 = \mathrm{ch}^2\, y - \mathrm{sh}^2\, y = 1 \tag{2}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \text{(time-dilatation factor)}, \tag{3}$$

$$\frac{\mathrm{sh}\, y}{\mathrm{ch}\, y} = \mathrm{th}\, y = \beta \quad \text{or} \quad y = \frac{1}{2}\ln\frac{1+\beta}{1-\beta} \quad \text{(rapidity)} \tag{4}$$

LT leaves the square of the four length $x^2 = x_0^2 - (x_1^2 + x_2^2 + x_3^2) = \tau^2$ (τ = proper time) and the space-time volume $\mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z = \mathrm{d}^4x$ invariant.
Successive transformations form groups. Rotations and LT are non-commutative in general. Rotations around the same axis commute and the corresponding rotation angles are additive: $\varphi = \varphi_1 + \varphi_2$. LT in the *same boost direction* are also commutative and the corresponding rapidities are additive: $y = y_1 + y_2$. Experiments: prefered direction $\|$ to the beam direction!

All 4-vectors transform according to LT-matrix given above.

1.2 Particle kinetics

Four vector notation ($\varepsilon = p_0$):

$$p = \begin{pmatrix} \varepsilon \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_0 \\ p_\parallel \\ \vec{p}_\perp \end{pmatrix}, \quad p^2 = \varepsilon^2 - \vec{p}^{\,2} = p_0^2 - (p_1^2 + p_2^2 + p_3^2) \tag{5}$$

Transfm.	4-momenta	transformations matrix	Det	invariances
Lorentz	$\varepsilon = (\varepsilon' + \frac{1}{c^2}\beta p'_\parallel)\,\gamma$ $p_\parallel = (\beta\varepsilon' + p'_\parallel)\,\gamma$ $\vec{p}_\perp = \vec{p}\,'_\perp$	$\gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{ch}\, y & \mathrm{sh}\, y \\ \mathrm{sh}\, y & \mathrm{ch}\, y \end{pmatrix}$	1	$\mathrm{d}^4 p = \mathrm{d}\varepsilon \mathrm{d}^3 p$ $p^2 = \varepsilon^2 - \vec{p}^{\,2}$

For boosts $\vec{\beta}$ in a general direction the LT can be summarized as follows:

$$\vec{p} = \vec{p}\,' + \gamma\vec{\beta}\left(\frac{\gamma}{1+\gamma}\vec{\beta}\,\vec{p}\,' + \varepsilon'\right) \tag{6}$$

$$\varepsilon = \gamma\left(\varepsilon' + \vec{\beta}\,\vec{p}\,'\right) \tag{7}$$

For on-shell particles of mass m:

$$p^2 = \varepsilon^2 - \vec{p}^{\,2} = m^2 \quad \longrightarrow \quad \varepsilon \mathrm{d}\varepsilon = \vec{p}\,\mathrm{d}\vec{p} \tag{8}$$

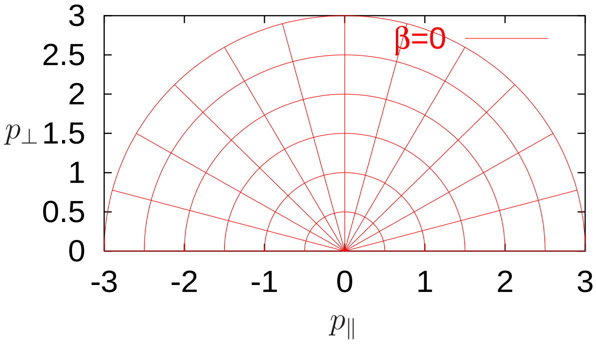
$$\longrightarrow \quad \text{LT:} \quad \mathrm{d}p_\parallel = (\beta \mathrm{d}\varepsilon' + \mathrm{d}p'_\parallel)\,\gamma = (\beta\frac{p'_\parallel}{\varepsilon'} + 1)\,\gamma\,\mathrm{d}p'_\parallel = \frac{\varepsilon}{\varepsilon'}\,\mathrm{d}p'_\parallel \tag{9}$$

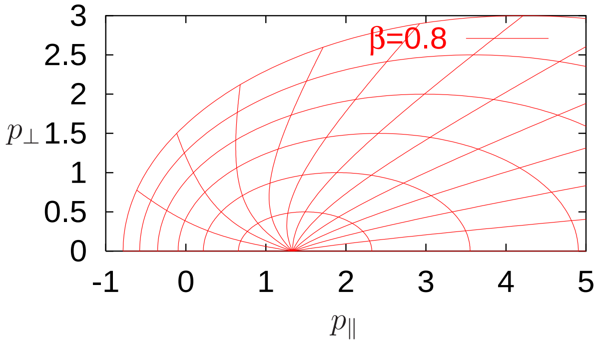
$$\longrightarrow \quad \frac{\mathrm{d}p_\parallel}{\varepsilon} \text{ and } \frac{\mathrm{d}^3 p}{\varepsilon} \text{ are invariant} \tag{10}$$

The velocity $\vec{\beta}$ and γ-value of the particle are

$$\gamma = \frac{\varepsilon}{m}, \quad \vec{\beta} = \frac{\vec{p}}{\varepsilon}, \quad \vec{\beta}\gamma = \frac{\vec{p}}{m} \tag{11}$$

Isotropic momentum distributions in a certain rest frame become longitudinally elongated and non isotropic, if boosted.





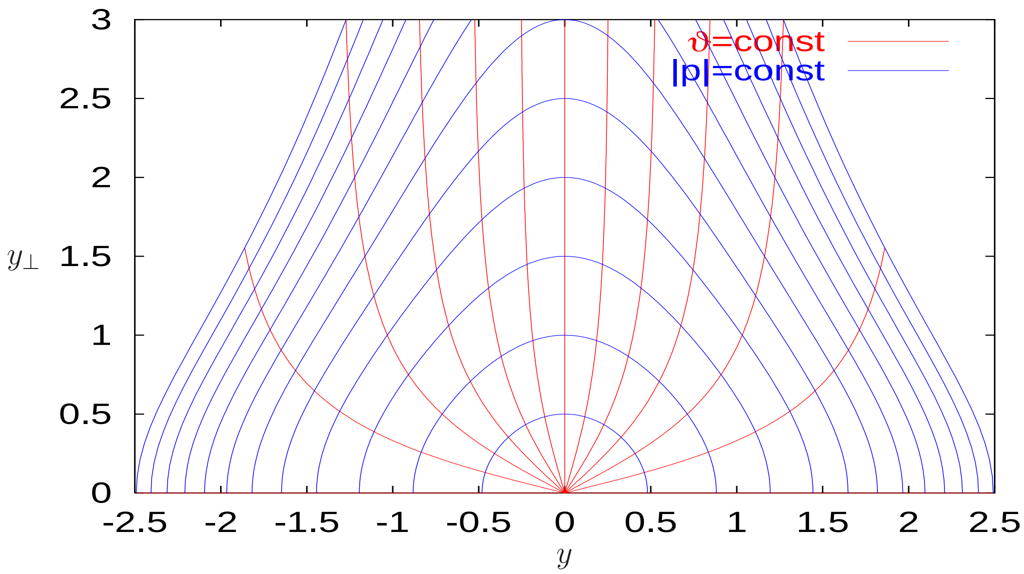
isotropic momentum distribution in its rest frame ($\beta = 0$) and viewed from a boosted frame with $\beta = 0.8$; momenta in units of m.

Rapidity and transverse mass:

The rapidity can also be used to describe the kinematics of on-shell particles. One defines

$$y = \frac{1}{2}\ln\frac{1+\beta_\parallel}{1-\beta_\parallel} = \frac{1}{2}\ln\frac{\varepsilon+p_\parallel}{\varepsilon-p_\parallel} \tag{12}$$

$$\vec{y}_\perp = \vec{p}_\perp/m \tag{13}$$



lines of constant angle (red) and constant $|\vec{p}|$ in rapidity space $(y, y_\perp)$

A Lorentz-boost in $\parallel$ direction just amounts to shift the plot by y_{boost} (use two transparencies of this plot overlayed to illustrate the Lorentz transformation from one frame to another).

For all boost in paralell direction the transverse mass

$$m_\perp = \sqrt{m^2 + \vec{p}_\perp^{\,2}} \tag{14}$$

is invariant. It permits to express energy and momenta in terms of rapidity

$$\varepsilon = m_\perp \,\mathrm{sh} y, \quad \vec{p}_\parallel = m_\perp \,\mathrm{ch} y \tag{15}$$

$$\vec{p}_\perp = m\vec{y}_\perp \tag{16}$$

For $|\vec{p}_\perp| \gg m$ the rapidity become solely a function of angle and merges the so-called pseudo rapidity

$$y \to \eta = \ln\cot\frac{\vartheta}{2}, \tag{17}$$

where ϑ is the angle of $\vec{p}$ with respect to the $\parallel$ direction. The invariant momentum-space volume becomes

$$\frac{1}{\varepsilon}\mathrm{d}^3p = m^2\mathrm{d}y\mathrm{d}^2y_\perp = m^2\mathrm{d}^3y \tag{18}$$

The invariant single particle cross section in different forms:

$$\varepsilon \frac{\mathrm{d}\sigma}{\mathrm{d}^3 p} = \varepsilon \frac{\mathrm{d}\sigma}{\mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z} \quad \text{(cartesian representation)} \tag{19}$$

$$= \varepsilon \frac{\mathrm{d}\sigma}{\mathrm{d}^2 p_\perp \mathrm{d}p_\parallel} = \varepsilon \frac{\mathrm{d}\sigma}{p_\perp \mathrm{d}p_\perp \mathrm{d}p_\parallel \mathrm{d}\varphi} \quad \text{(cylindric representation)} \tag{20}$$

$$= \varepsilon \frac{\mathrm{d}\sigma}{\vec{p}^{\,2} \mathrm{d}p \mathrm{d}^2\Omega} = \varepsilon \frac{\mathrm{d}\sigma}{\vec{p}^{\,2} \mathrm{d}p \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi} \quad \text{(spherical representation)} \tag{21}$$

$$= \frac{\mathrm{d}\sigma}{|\vec{p}| \mathrm{d}\varepsilon \mathrm{d}^2\Omega} = \frac{\mathrm{d}\sigma}{|\vec{p}| \mathrm{d}\varepsilon \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi} \quad \text{(spherical energy representation)} \tag{22}$$

$$\tag{23}$$

1.3 Multi-particle systems

$$p_i = \begin{pmatrix} \varepsilon_i \\ \vec{p}_i \end{pmatrix}; \quad \text{total four-momentum} \quad P = \begin{pmatrix} E \\ \vec{P} \end{pmatrix} = \sum_{i=1}^{N} p_i = \sum_{i=1}^{N} \begin{pmatrix} \varepsilon_i \\ \vec{p}_i \end{pmatrix} \tag{24}$$

The corresponding invariant mass given by

$$M^2 = s = E^2 - \vec{P}^{\,2} \tag{25}$$

It determines the c.m. energy $\sqrt{s}$ of the N-particle system. The c.m. system is defined by that inertial system in which the total 3-momentum vanishes, i.e. $\vec{P} = 0$. In that frame $M^2 = s = E^2_{c.m.}$. The Lorentz-boost which leads to the c.m. frame is given by

$$\vec{\beta}_{\text{boost}} = -\frac{\vec{P}}{E} \tag{26}$$

Excercise: show that this leads to the fact that in the boosted frame the total momentum $\vec{P}_{c.m.} = 0$. Splitting such an N-particle system into two groups (clusters) of k and $N-k$ particles, In the c.m. frame these two cluster have opposite three-momentum $\vec{P}_k = -\vec{P}_{N-k}$ (omitting the label *c.m.*). The size of the respective c.m. momenta of both clusters can be calculated from the total c.m. energy and the invariant masses of both sub-clusters

$$M^2 = s = (E_k + E_{N-k})^2 = M_k^2 + M_{N-k}^2 + 2\sqrt{M_k^2 + \vec{P}_k^{\,2}}\sqrt{M_{N-k}^2 + \vec{P}_k^{\,2}} + 2\vec{P}_k^{\,2} \tag{27}$$

$$4s\, P^2_{c.m.} = (s - M_k^2 - M_{N-k}^2)^2 - 4M_k^2 M_{N-k}^2 \tag{28}$$

$$P_{c.m.} = \frac{1}{2\sqrt{s}} \sqrt{\lambda(s, M_k^2, M_{N-k}^2)} \tag{29}$$

with the completely symmetric Källen-function of three arguments

$$\lambda(x^2, y^2, z^2) = x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2 \tag{30}$$

$$= (x - y - z)(x - y + z)(x + y - z)(x + y + z) \tag{31}$$

Special cases are

$$\lambda(s, m^2, m^2) = s(s - 4m^2), \qquad \lambda(s, s', 0) = (s - s')^2 \tag{32}$$

This λ-function will be of quite importance for the phase-space distributions to be discussed next.

1.4 Non-relativistic limit

In the non-relativistic limit $E_{kin} = \sqrt{s} - \sum_i m_i \ll \sum_i m_i$ one has the following relations

$$E_{kin} = \sqrt{s} - \sum_i m_i = \sum_i \frac{\vec{p}_i^{\,2}}{2m_i} \tag{33}$$

$$\gamma = 1 \tag{34}$$

$$\frac{\mathrm{d}^3 p}{\varepsilon} = \frac{\mathrm{d}^3 p}{m} \tag{35}$$

$$\lambda(s, m_1^2, m_2^2) = 8E_{kin} m_1 m_2 (m_1 + m_2) = \underbrace{4(m_1 + m_2)^2}_{4s} \vec{P}^{\,2}_{c.m.} \tag{36}$$

$$\tag{37}$$

1.5 Multi-particle Phase-space

Starting point is the general cross section for a transition from some initial state I to a final multi-particle state

$$\mathrm{d}\sigma = \frac{2\pi}{\hbar|v_P - v_T|} \left|T(I \to p_1, \ldots, p_N)\right|^2 \mathrm{d}\tilde{\Phi}_{1,\ldots,N}(P_I), \quad \text{where} \tag{38}$$

$$\mathrm{d}\tilde{\Phi}_{1,\ldots,N}(P_I) = \frac{\mathrm{d}^3 p_1}{2(2\pi\hbar)^3 \varepsilon_1} \cdots \frac{\mathrm{d}^3 p_N}{2(2\pi\hbar)^3 \varepsilon_N} \delta^4(P_I - \sum_k p_k) \tag{39}$$

$$= \frac{1}{2^N (2\pi\hbar)^{3N}} \mathrm{d}\Phi_{1,\ldots,N}(P_I) \tag{40}$$

where v_P and v_T are the velocities of projectile and target, respectively. The quantity

$$\mathrm{d}\Phi(P_I; m_1^2, \ldots, m_N^2) = \frac{\mathrm{d}^3 p_1}{\varepsilon_1} \cdots \frac{\mathrm{d}^3 p_N}{\varepsilon_N} \delta^4(P_I - \sum_k p_k) \tag{41}$$

is called the phase-space distribution of the N particles with masses m_i and total four-momentum P_I. Normalized by the inverse of the *phase-space integral*

$$\Phi(P_I; m_1^2, \ldots, m_N^2) = \int \frac{\mathrm{d}^3 p_1}{\varepsilon_1} \cdots \frac{\mathrm{d}^3 p_N}{\varepsilon_N} \delta^4(P_I - \sum_k p_k) \tag{42}$$

it determines the probability distribution in the N-particle momentum space

$$W_I(\vec{p}_1, \ldots, \vec{p}_N) \frac{\mathrm{d}^3 p_1}{\varepsilon_1} \cdots \frac{\mathrm{d}^3 p_N}{\varepsilon_N} = \frac{1}{\Phi(P_I; m_1^2, \ldots, m_N^2)} \frac{\mathrm{d}^3 p_1}{\varepsilon_1} \cdots \frac{\mathrm{d}^3 p_N}{\varepsilon_N} \delta^4(P_I - \sum_k p_k) \tag{43}$$

solely under the constraint of conservation of energy and three-momentum. The distribution W_I depends on the initial four momentum P_I and the masses m_i of the N particles in the final state. A reaction is phase-space dominated, if the details of the T-matrix do not matter and thus $|T|^2$ can be approximated by a constant. This will not be true in general, however the more inclusively the reaction is observed, the more one intergrates over the details of $|T|^2$ and available phase-space may be a good approximation.

The inclusive probabiltity distribution to observe particle N with momentum $\vec{p}_N$ is then given by

$$W_I(\vec{p}_N) = \int \frac{\mathrm{d}^3 p_1}{\varepsilon_1} \cdots \frac{\mathrm{d}^3 p_{N-1}}{\varepsilon_{N-1}} W(P_I \to \vec{p}_1, \ldots, \vec{p}_N) \tag{44}$$

$$= \frac{\Phi((P_I - p_N)^2; m_1^2, \ldots, m_{N-1}^2)}{\Phi(P_I^2; m_1^2, \ldots, m_N^2)} \quad \text{with} \quad \int \frac{\mathrm{d}^3 p_N}{\varepsilon_N} W_I(\vec{p}_N) = 1 \tag{45}$$

The latter relation implies a recursion relation for the phase-space integrals $\Phi_{1,\ldots,N}(P)$ in the number N of particles.

$$\Phi(P_I^2; m_1^2, \ldots, m_N^2) = \int \frac{\mathrm{d}^3 p_N}{\varepsilon_N} \Phi((P_I - p_N)^2; m_1^2, \ldots, m_{N-1}^2) \tag{46}$$

For the evaluation of such relations it is important to realize that the $\Phi_{1,\ldots,N}(P^2)$ are invariant under LT, i.e. they solely depend on P through the invariant quantity $s = P^2$. This permits to go to appropriately choosen frames, e.g. where $\vec{P} = 0$. Since in that frame $s = P^2 = E^2$ one obtains $s' = (P - p_N)^2 = s + m_N^2 - 2\varepsilon_N\sqrt{s}$, and therefore the integrand $\Phi_{1,\ldots,N-1}((P - p_N)^2)$ is isotropic in three-momentum $\vec{p}_N$. Thus

$$\Phi(s; m_1^2, \ldots, m_N^2) = 4\pi \int \mathrm{d}\varepsilon_N \, |\vec{p}_N| \, \Phi(s + m_N^2 - 2\varepsilon_N\sqrt{s}; m_1^2, \ldots, m_{N-1}^2) \tag{47}$$

$$= \frac{2\pi}{\sqrt{s}} \int \mathrm{d}s' \, |\vec{p}_N| \, \Phi(s'; m_1^2, \ldots, m_{N-1}^2) \tag{48}$$

We see the N-particle phase-space intergrals are recursively contructed from those with less particles. The folding weight is given by the relative momentum $|\vec{p}_N|$ of the newly added particle with the $N-1$-particle cluster. Above $\vec{p}_N$ is the momentum in the over all c.m. frame ($\vec{P} = 0$) and therefore $\vec{p}_N$ is just the corresponding relative momentum which in turn is a function of s, s' and the mass m_N. From $s' = s + m_N^2 - 2\varepsilon_N\sqrt{s}$ one obtains

$$|\vec{p}_N| = \frac{1}{2\sqrt{s}} \sqrt{(s' - s - m_N^2)^2 - 4sm_N^2} \tag{49}$$

$$= \frac{1}{2\sqrt{s}} \sqrt{\lambda(s, s', m_N^2)} \tag{50}$$

with the λ-function defined in (30). In terms of this λ-function the phase-space recursion (47) becomes

$$\Phi(s; m_1^2, \ldots, m_N^2) = \frac{\pi}{s} \int_{s_l}^{s_u} \mathrm{d}s' \, \sqrt{\lambda(s, s', m_N^2)} \, \Phi(s'; m_1^2, \ldots, m_{N-1}^2) \tag{51}$$

$$\text{with} \quad s_l = \left(\sum_{i=1}^{N-1} m_i \right)^2, \quad s_u = \left(\sqrt{s} - m_N \right)^2 \tag{52}$$

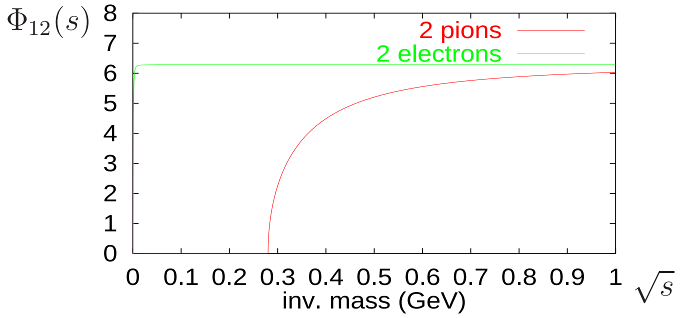
This is just a special case of a general relation decomposing the N-particle system into two clusters of particles $\{1, \ldots, k\}$ and $\{k+1, \ldots, N\}$

$$\Phi(s; m_1^2, \ldots, m_N^2) = \int \frac{\mathrm{d}s_1}{2} \frac{\mathrm{d}s_2}{2} \Phi(s; s_1, s_2) \; \Phi(s_1; m_1^2, \ldots, m_k^2) \; \Phi(s_2; m_{k+1}^2, \ldots, m_N^2) \tag{53}$$

$$\text{with} \quad \Phi(s; m^2) = 2\delta(s - m^2) \tag{54}$$

$$\Phi(s; m_1^2, m_2^2) = \frac{2\pi}{s} \sqrt{\lambda(s, m_1^2, m_2^2)} \tag{55}$$

which can be generalized to arbitry cluster decompositions. *Excercise:* derive (53) and determine the limits of integration.



Two-particle phase-space integrals for
two pions and two electrons

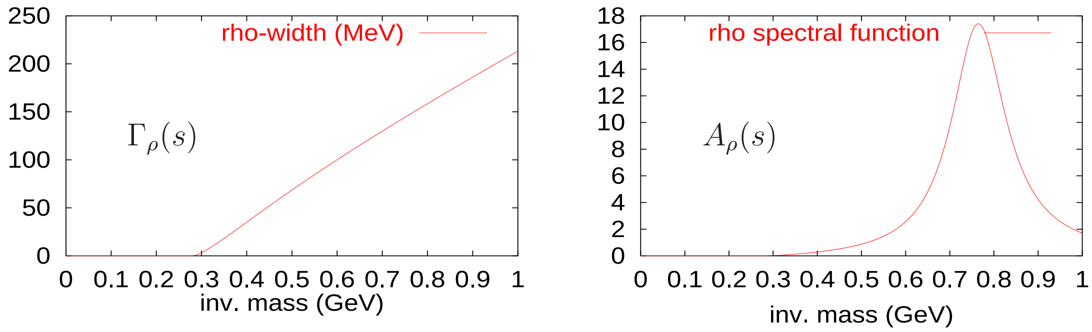
Applications: the decay of a ρ-meson into two pions. We need the square of the coupling matrix element $|M|$, which due to p-wave coupling is proportional to the square three-momentum of the two pions in the c.m. frame $p_\pi = \sqrt{\lambda(s, m_\pi^2, m_\pi^2)}/(2\sqrt{s})$ times the phase-space $\Phi(s; \pi, \pi) = 2\pi\sqrt{\lambda(s, m_\pi^2, m_\pi^2)}/s$. Thus the decay rate becomes

$$2\sqrt{s}\,\Gamma_{\rho\to\pi\pi}(s) = |M|^2\ \Phi(s; \pi\pi), \qquad \text{with} \quad |M|^2 \propto (p_\pi)^2 \tag{56}$$

$$= c\frac{1}{s^2}\sqrt{\lambda(s, m_\pi^2, m_\pi^2)}^3 \tag{57}$$

$$\Gamma_{\rho\to\pi\pi}(s) = \Gamma_R \frac{m_R^2}{s}\left(\frac{s - 4m_\pi^2}{m_R^2 - 4m_\pi^2}\right)^{3/2} \tag{58}$$

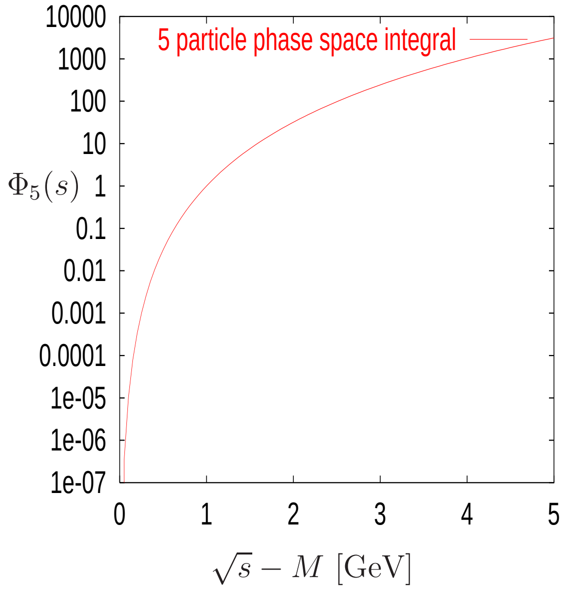
where $\Gamma_R \approx 150$ MeV is the vacuum width at resonance where $m_R = 770$ MeV.



rho-meson decay width into two pions   rho-meson spectral function

Ignoring changes of the real part of the self energy, the spectral function is simply given by (form for relativistic bosons)

$$A_\rho(s) = \frac{2m_R\Gamma(s)}{(s - m_R^2)^2 + m_R^2\Gamma^2(s)} \tag{59}$$



$\Phi(s; 1, \ldots, 5)$ (in non-rel approximation, arbitrary units)

In two limits explicit forms of the phase-space integrals can be derived. These are the non-relativistic limit $\sqrt{s}/(\sum m) - 1 \ll 1$ and the limit of mass-less particles. In both cases the analytic form can be derived from scaling arguments. In the $m = 0$ case the only scale entering is s and one finds

$$\Phi_N(\sqrt{s}) = c_N \; s^{N-2} \tag{60}$$

$$\text{for all } m_i = 0; \; i = 1, \ldots, N$$

$$c_N = 2\frac{\pi^{N-1}}{(N-1)!(N-2)!} \tag{61}$$

where the explicit result for the c_N follows from the recursion (51) with $c_2 = 2\pi$.

In the non-relativistic limit the energy conservation becomes a quadratic form in momenta. Going to Jokobi-coordinates this quadratic form is expressed as

$$E - \sum_i \varepsilon_i = E - M - \frac{\vec{P}^{\,2}}{2M} - \sum_{i=2}^{N} \frac{\vec{\pi}_i^{\,2}}{2\mu_i}, \qquad M = \sum_i m_i \tag{62}$$

Here π_i and μ_i are the relative momentum and reduced mass of particle i with the sub-cluster of particles $1, \ldots, i-1$. The momentum conservation touches only the $\vec{P}^{\,2}/(2M)$ term determining the recoil energy $\sqrt{s} = E - \vec{P}^{\,2}/(2M)$. The remaining integration over all Jakobi-momenta can be solved by substituting all $\vec{\pi}_i^{\,2}/(2\mu_i) = \vec{q}_i^{\,2}$ such that one obtains

$$\Phi_{1,\ldots,N}(s) \propto \int \mathrm{d}^3\pi_2 \cdots \mathrm{d}^3\pi_N \; \delta\left(\sqrt{s} - M - \sum_{i=2}^{N} \frac{\vec{\pi}_i^{\,2}}{2\mu_i}\right) \tag{63}$$

$$\propto \int \mathrm{d}^{3(N-1)}q \; \delta\left(\sqrt{s} - M - q^2\right), \quad q = (\vec{q}_2, \ldots, \vec{q}_N), \quad \text{i.e.} \tag{64}$$

$$\Phi_{1,\ldots,N}(s) = c_{1,\ldots,N} \; (\sqrt{s} - M)^{(3N-5)/2} \tag{65}$$

where again the analytic form can be obtained from scaling arguments. The explicit result for the coefficient is

$$c_{1,\ldots,N} = \frac{(\prod m_i)^{1/2}}{M^{3/2}} \frac{(2\pi)^{3(N-1)/2}}{\Gamma(\frac{3}{2}(k-1))} \tag{66}$$

For large N the $\Phi^{1,\ldots,N}(s)$ are steeply rizing functions starting at $\sqrt{s} \geq M$. This impies that the invariant single particle distributions in the c.m. frame ($\vec{P} = 0$.) finally become exponential functions of the single particle energies

$$W(\vec{p}) = \frac{\Phi_{1,\ldots,N-1}((E - \varepsilon(\vec{p}))^2 - (\vec{p})^2)}{\Phi_{1,\ldots,N}(E^2)} \tag{67}$$

$$= \frac{\Phi_{1,\ldots,N-1}(s+m_N^2)}{\Phi_{1,\ldots,N}(s)} \exp\left(-2\sqrt{s}\varepsilon_N \underbrace{\frac{\mathrm{d}}{\mathrm{d}s} \ln \Phi_{1,\ldots,N-1}(s)|_{(s+m_N^2)}}_{1/(2\sqrt{s}\,T^*)}\right) \tag{68}$$

$$\propto \begin{cases} (1-\varepsilon/E)^{(3N-5)/2} = \exp(-\varepsilon/T^*) & T^* = 2(E-M)/(3N) \text{ for non-rel. kin.} \\ (1-\varepsilon/E)^{2N-2} = \exp(-\varepsilon/T^*) & T^* = (E-M)/(2N) \text{ for all } m_i \ll T^* \end{cases} \tag{69}$$

Phase-space integrals permit to define differential phase-space distributions with respect to observables constructed from the N momanta $A(\vec{p}_1, \ldots, \vec{p}_N)$

$$\frac{\mathrm{d}\Phi_{1,\ldots,N}}{\mathrm{d}a} = \int \frac{\mathrm{d}^3p_1}{\varepsilon_1} \cdots \frac{\mathrm{d}^3p_N}{\varepsilon_N} \delta^4(P_I - \sum_k p_k)\, \delta\left(a - A(\vec{p}_1, \ldots, \vec{p}_N)\right) \tag{70}$$

$$\text{with} \int \mathrm{d}a \frac{\mathrm{d}\Phi_{1,\ldots,N}(s)}{\mathrm{d}a} = \Phi_{1,\ldots,N}(s). \tag{71}$$

Here the extra δ-function provides the cut onto the observable A. In simple cases like for the single particle distributions it just amounts to omit the corresponding integrations

$$\varepsilon_N \frac{\mathrm{d}\Phi_{1,\ldots,N}}{\mathrm{d}^3p_N} = \varepsilon_N \int \frac{\mathrm{d}^3p'_1}{\varepsilon'_1} \cdots \frac{\mathrm{d}^3p'_N}{\varepsilon'_N} \delta^4(P - \sum_k p'_k)\delta^3\left(\vec{p}'_N - \vec{p}_N\right) \tag{72}$$

$$= \Phi_{1,\ldots,N-1}\left(E-\varepsilon_N)^2 - (\vec{P} - \vec{p}_N)^2\right) \tag{73}$$

1.6 Dalitz representation of three particle phase-space distributions

Dalitz (1954) realized that the 3-body phase has the particular feature that, if cuts are taken with respect to the c.m. energies of two particles, say ε_1 and ε_2, the resulting distribution is constant inside the kinematically allowed region

$$\begin{aligned} \frac{\mathrm{d}\Phi_{1,2,3}}{\mathrm{d}\varepsilon_1\mathrm{d}\varepsilon_2} &= \int \frac{\mathrm{d}^3p'_1}{\varepsilon'_1}\frac{\mathrm{d}^3p'_2}{\varepsilon'_2}\frac{\mathrm{d}^3p'_3}{\varepsilon'_3}\, \delta(\sqrt{s} - \varepsilon'_1 - \varepsilon'_2 - \varepsilon'_3)\, \delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \\ &\quad \times \delta\left(\varepsilon_1 - \varepsilon'_1\right)\, \delta\left(\varepsilon_2 - \varepsilon'_2\right)) \end{aligned} \tag{74}$$

$$= |\vec{p}_1||\vec{p}_2| \int \frac{\mathrm{d}^2\Omega_1\mathrm{d}^2\Omega_2}{\sqrt{m_3^2 + (\vec{p}_1 + \vec{p}_2)^2}} \delta\left(\sqrt{s} - \varepsilon_1 - \varepsilon_2 - \sqrt{m_3^2 + (\vec{p}_1 + \vec{p}_2)^2}\right) \tag{75}$$

$$= \begin{cases} 8\pi^2 & \text{for } x \le 1 \\ 0 & \text{else, } \qquad \text{where} \end{cases} \tag{76}$$

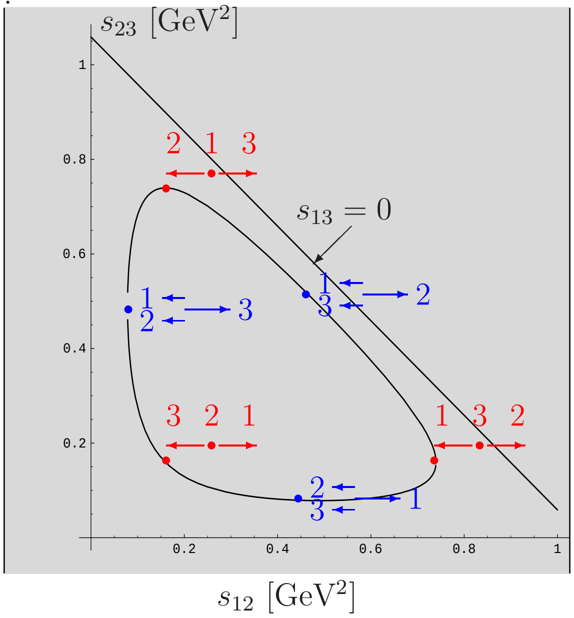
$$x = \cos(\vartheta_{12}) = \frac{1}{2|\vec{p}_1||\vec{p}_2|}\left(s + m_1^2 + m_2^2 - m_3^2 + 2\varepsilon_2\varepsilon_2 - 2\sqrt{s}(\varepsilon_2 + \varepsilon_2)\right). \tag{77}$$

Here ϑ_{12} is the angle between the directions of $\vec{p}_1$ and $\vec{p}_2$ and $\vec{\Omega}_1$ and $\vec{\Omega}_2$ are the respective solid angles. The kinematical boundaries are defined by $x = \pm 1$. The c.m. energies of the particles can be transform to the invariant mass of the cluster of the two complement particles

$$s_{23} = s + m_1^2 - 2\sqrt{s}\varepsilon_1$$

and likewise for the two other pairs. The total sum becomes

$$s_{12} + s_{23} + s_{13} = s + m_1^2 + m_2^2 + m_3^2.$$



Dalitz plot for the decay into three pions. The point with maximum values for s_{ik} correspond to the situation where the other particle rests and the momenta of the pair are opposite, for the minimum the other particle has the maximum momentum

Thus (74) can be transformed to a distribution in any of the two invarant pair masses s_{ik}, e.g.

$$\frac{\mathrm{d}\Phi_{1,\ldots,N}}{(\mathrm{d}s_{12}\,\mathrm{d}s_{23})} = \frac{2\pi^2}{s}$$

which again is constant for a given s. This has the advantage that this distribution depends only an invariant quantities and immediatly displays the features of the two-particle sub clusters. The figure shows the kinematical boundary of the Dalitz plot in s_{12} versus s_{23} for the decay into three pions for $s = 1GeV^2$. In such plots non-uniform distributions directly point towards a dynamical dependence of the matrix element of the decay process, since the phase-space factor in (56) is constant. Thus in this case the ρ-meson shows up as an enhancement around the lines $s_{ik} \approx m_\rho^2$. In the zero mass limit $m_i \to 0$ the kinematically allowed region fills out the full triangle.

1.7 Non-invariant phase-space and statistical equilibrium

In the semi-classical limit of quantum statistics one counts the number of levels per phase-space volume

$$Z(E,N) = \frac{\mathrm{d}\mathcal{N}}{\mathrm{d}E} = \frac{1}{N_1!\cdots N_K!}\int \frac{\mathrm{d}^3x_1\mathrm{d}^3p_1}{(2\pi\hbar)^3}\cdots\frac{\mathrm{d}^3x_N\mathrm{d}^3p_N}{(2\pi\hbar)^3} \times \underbrace{\delta\left(E - \sum_i \varepsilon_i\right)}_{\text{constraint}} \tag{78}$$

$$= \frac{1}{N_1!\cdots N_K!}\frac{V^N}{(2\pi\hbar)^{3N}}\int \mathrm{d}^3p_1\cdots\mathrm{d}^3p_N\;\delta\left(E - \sum_i \varepsilon_i\right). \tag{79}$$

Here the factor $1/(N_1!\cdots N_K!)$ accounts for the proper quantum level counting in case of identical particles, i.e. the system consists of N_1 identical particle of type 1, etc, with $N = \sum_i N_i$ (quantum level counting $\to$ Gibb's paradoxon!). $Z(E,N)$ is then the *micro-canonical* partion sum for such a system of non-interacting classical particles. This can be extended to include momentum conservation

$$Z(E,\vec{P},N) = \frac{\mathrm{d}\mathcal{N}}{\mathrm{d}E\mathrm{d}^3p} = \frac{1}{N_1!\cdots N_K!}\frac{V^N}{(2\pi\hbar)^{3N}} \times \underbrace{\int \mathrm{d}^3p_1\cdots\mathrm{d}^3p_N\,\delta\left(E-\sum_i\varepsilon_i\right)\,\delta^3\left(\vec{P}-\sum_i\vec{p}_i\right)}_{\text{non-invariant phase-space integrals } J_{1,\ldots,N}} \tag{80}$$

The latter relate to the non-invarant phase-space integrals $J_{1,\ldots,N}$. They differ from the invariant ones by the momentum integration measure, which is non-invariant here. In the non-relativistic case both, the invariant and non-invariant phase-space integrals are identical apart from a factor given by the masses

$$\Phi_{1,\ldots,N}(s) = \frac{1}{m_1\cdots m_N}J_{1,\ldots,N}(s) \quad \text{for non-relativistic case.} \tag{81}$$

If one deals with a system with varying number of particles as e.g. a pion gas then the total partition sum is given by a sum over all partial partition sums with a given number of particles, e.g. for a pion gas or for a system of N nucleons and an arbitrary number of pions

$$Z_{\text{total}}(s) = \sum_{N_\pi=1}^{\infty} Z(s,N_pi) \quad \text{(gas of pions);} \tag{82}$$

$$Z_{\text{total}}(s,N) = \sum_{N_\pi=0}^{\infty} Z(s,N,N_pi) \quad \text{(gas of } N \text{ nucleons and pions).} \tag{83}$$

All probability distributions are then normalised with respect to $Z_{\text{total}}(s)$. Thus the partial multiplicity to find a pion or a nucleon with momentum $\vec{p}$ in the system are then

$$f_N(\vec{x},\vec{p}) = \frac{Z_{\text{total}}(s+m_0^2-2\sqrt{s}\varepsilon_N(\vec{p}),N-1)}{Z_{\text{total}}(s,N)} \tag{84}$$

$$f_\pi(\vec{x},\vec{p}) = \frac{Z_{\text{total}}(s+m_\pi^2-2\sqrt{s}\varepsilon_\pi(\vec{p}),N)}{Z_{\text{total}}(s,N)} \tag{85}$$

for all $\vec{x}$ in the volume V. These single particle distributions relate to those used in transport problems. Integration over the space volume gives the momentum distributions

$$F(\vec{p}) = \int \mathrm{d}^3x f(\vec{x},\vec{p}) = V f(\vec{x},\vec{p}) \tag{86}$$

Note that integration over momenta *now* leads to the mean multiplicities of the particles

$$\int \frac{\mathrm{d}^3p}{(2\pi\hbar)^3}\,F_N(\vec{p}) = \langle N\rangle = N, \quad \int \frac{\mathrm{d}^3p}{(2\pi\hbar)^3}\,F_\pi(\vec{p}) = \langle N_\pi\rangle\,. \tag{87}$$

For large particle numbers with $\sqrt{s}/N = const.$ the micro-canonical distributions merge the canonical ones

$$F_N(\vec{p}) = V\underbrace{\frac{Z_{\text{total}}(s+m_0^2,N-1)}{Z_{\text{total}}(s,N)}}_{=\exp(\mu_N/T^*)}\underbrace{\frac{Z_{\text{total}}(s+m_0^2-2\sqrt{s}\varepsilon_N(\vec{p}),N-1)}{Z_{\text{total}}(s+m_0^2,N-1)}}_{\approx\exp(-\varepsilon/T^*)} \tag{88}$$

$$F_\pi(\vec{p}) = V \underbrace{\frac{Z_{\rm total}(s+m_\pi^2,N)}{Z_{\rm total}(s,N)}}_{=\exp(\mu_\pi/T^*)} \underbrace{\frac{Z_{\rm total}(s+m_\pi^2-2\sqrt{s}\varepsilon_\pi(\vec{p}),N)}{Z_{\rm total}(s,+m_\pi^2,N)}}_{\approx\exp(-\varepsilon/T^*)} \tag{89}$$

with chemical potential μ and temperature T^*. The finite particle number correction to the pion chemial potential is $\mu_\pi = m_\pi^2/(2\sqrt{s})$ and vanishes for large systems (note $\sqrt{s}$ scales like the particle number. For the two limits where closed form for the non-invariant phase-space integrals exist the temperatures are given by

$$\begin{aligned} T^* &= 2(\sqrt{s}-M)/(3N-5) \quad \text{for non-rel case} \\ T^* &= (\sqrt{s}-M)/(3N-4) \qquad \text{for rel case} \end{aligned} \tag{90}$$

In the extreme relativistic case significant differences appear between the non-invariant and the invariant phase-space integrals, e.g. for $\sqrt{s}/N \gg m_i$ both integrals behave like

$$\left.\begin{aligned} \Phi_{1,\ldots,N}(s) \propto s^{(2N-4)/2} \\ J_{1,\ldots,N}(s) \propto s^{(3N-4)/2} \end{aligned}\right\} \quad \text{for } \sqrt{s}/N \gg m_i \tag{91}$$

which leads to a significant difference in the slope paprameters T^* in the two cases. Only for the non-invariant case one comes to the proper thermodynamic limit. The invariant phase space integrals are responsible for elementary processes (cross sections), while the non-invariant ones determine equilibrium conditions.