Elements of scattering theory

1 General stuff

We consider non-relativistic, spinless particles interacting through a spherically symmetric potential.

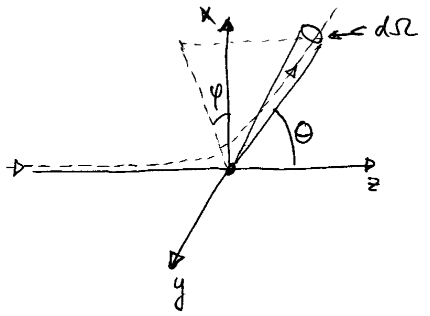


Figure 1: The scattering geometry in the lab system.

First we consider scattering in the Lab system. The current of incident particles is:

$$J_{beam} = \dot{N} = \text{particles per (unit time)}$$

Each of these beam particles scatters with $\mathrm{d}N_{target}$ particles given by

$$\mathrm{d}N = \mathrm{d}\sigma \ \rho \ d \tag{1}$$

where $\mathrm{d}\sigma$ is the diffential cross section for the considered type of reaction, ρ is the density of target atoms and d is the thickness of the target. If this cross section provides events, where the projectile is scattered into a given solid angle area $\mathrm{d}\Omega$, the beam current leads to the following rate of scattered particles into this solid angle area

$$\frac{\mathrm{d}N}{\mathrm{d}t\mathrm{d}\Omega}\mathrm{d}\Omega = J_{beam} \ \rho \ d \ \mathrm{d}\sigma \tag{2}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}N}{\mathrm{d}t\mathrm{d}\Omega}\frac{1}{J_{beam}\rho d}. \tag{3}$$

Here $\frac{d\sigma}{d\Omega}$ is the differential cross section and the product $J_{beam}\rho d$ is also called the luminosity.

2 Partial wave expansion

The Schrödinger equation for the relative motion

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(r,\vartheta,\varphi) + V(r)\psi(r,\vartheta,\varphi) = E\psi(r,\vartheta,\varphi), \tag{4}$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass. The cross section can be determined from the asymptotic form of the wave function ψ in the region where the interaction $V = 0$. The asymptotic form of the wave function can be written in the form

$$\psi(r,\vartheta,\varphi) = \underbrace{\psi_i(r,\vartheta,\varphi)}_{=Ae^{ikz}} + \psi_{\text{scatter}}(r,\vartheta,\varphi) \overset{r\to\infty}{\longrightarrow} A\left[e^{ikz} + \frac{1}{r} f(\vartheta,\varphi) e^{ikr}\right] \tag{5}$$

The first term represents the incoming beam and the second one the radially outgoing scattered particles. Here the beam ψ_i is idealized as a plane wave of infinite extent in the transverse direction. In practice this is not the case; the transverse extent of the beam is limited, so one can neglect the interference between the two terms in the detectors.

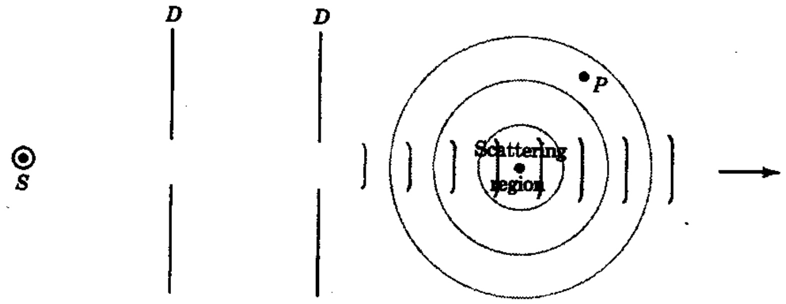


Figure 2: Schematic diagram of an arrangement for the measurement of scattering. (Figure from L.I. Schiff, Quantum Mechanics)

The flux can be expressed in terms of the wave function:

$$\vec{S} = \frac{1}{2im}(\psi^\star \vec{\nabla}\psi - (\vec{\nabla}\psi^\star)\psi) \tag{6}$$

Using the asymptotic form of the wave function given above one finds for the incoming flux $v|A|^2$ and for the scattered flux $v|A|^2|f(\vartheta,\varphi)|^2/r^2 + \ldots$, where the ellipsis stands for terms that vanish faster the $1/r^2$ as $r \to \infty$.

The rate of particles scattered into the solid angle $d\Omega$ is $v|A|^2|f(\vartheta,\varphi)|^2 d\Omega$ By comparison with the expression above, we can identify the differential cross section

$$\frac{d\sigma}{d\Omega} = |f(\vartheta,\varphi)|^2. \tag{7}$$

The quantity $f(\vartheta,\varphi)$ is called the scattering amplitude.

3 Asymptotic behaviour of the wave function

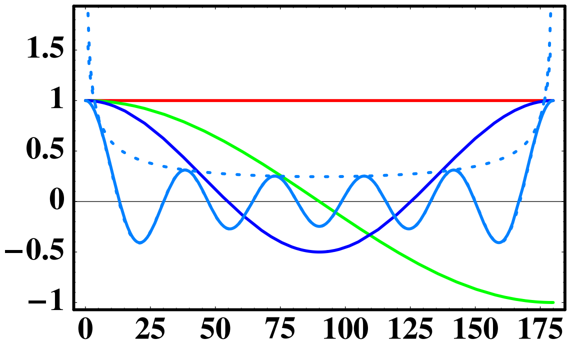
We want to establish the connection between the asymptotic behaviour of the wave function and the cross section. To find its asymptotic behaviour we need to solve the wave equation throughout space.

For a spherically symmetric interaction $V(\vec{r}) = V(|\vec{r}|)$ it is useful to expand the wave function in partial waves, i.e. states of good angular momentum. For such a potential the cross section and thus the scattering amplitude and the wave function are independent of the asimuthal angle φ. Thus we can expand the angular dependence of the wave function in terms of Legendre polynomials

$$\psi(r, \vartheta) = \frac{1}{kr} \sum_{\ell=0}^{\infty} (2\ell + 1) i^{\ell} \chi_{\ell}(r) P_{\ell}(\cos \vartheta), \tag{8}$$

where $\hbar k$ is the asymptotic momentum of the particle and z denotes the beam direction ($\vartheta = 0$) and the overall amplitude $A = 1$ has been chosen. The Legendre functions relate to the eigen functions of the angular momentum operator as

$$P_{\ell}(\cos \vartheta) = \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell 0}(\vartheta, \varphi) \tag{9}$$



Legendre polynoms $P_{\ell}(\cos \vartheta)$ for $\ell = 0, 1, 2$ and 10. Dashed line: asymptotic envelope for $l = 10$.

Special values and the large ℓ behavior are

$$\begin{aligned}
P_0(\cos \vartheta) &= 1 && (10)\\
P_1(\cos \vartheta) &= \cos \vartheta && (11)\\
P_2(\cos \vartheta) &= \frac{1}{2}(3 \cos^2 \vartheta - 1) && (12)\\
P_{\ell}(\cos \vartheta) &\xrightarrow{\ell \gg 1} \sqrt{\frac{4}{(2\ell + 1)\pi \sin \vartheta}} \cos \left((\ell + 1/2)\vartheta - \frac{\pi}{4} \right) && (13)\\
P_{\ell}(1) &= (-1)^{\ell} P_{\ell}(-1) = 1 && (14)\\
&\int \sin \vartheta \mathrm{d}\vartheta \; P_{\ell}(\cos \vartheta) P_{\ell'}(\cos \vartheta) = \frac{2\delta_{\ell\ell'}}{2\ell + 1} && (15)
\end{aligned}$$

The radial wave function $\chi_{\ell}(r)$ satisfies the radial Schrödinger equation

$$\frac{d^2 \chi_{\ell}}{dr^2} + \left[2\mu \left(E - V(r) \right) - \frac{\ell(\ell + 1)}{r^2} \right] \chi_{\ell} = 0 \tag{16}$$

with the boundary condition $\chi_{\ell}(r = 0) = 0$. The $\ell(\ell + 1)/r^2$ term provides the centrifugal potential for the angular momentum ℓ.

For $V = 0$ the properly normalized solutions $\chi_{\ell}(r) = \hat{j}_{\ell}(kr)$, which are the Riccati Bessel functions of order ℓ. They just provide a plane wave $\psi(\vec{r}) = e^{\mathrm{i}kz}$ for the total free wave function. The Riccati Bessel functions $\hat{j}_{\ell}(x)$ and $\hat{n}_{\ell}(x)$ relate to the normal spherical Bessel and Neumann functions $j_{\ell}(x)$ and $n_{\ell}(x)$ by an extra factor x, i.e. $\hat{j}_{\ell}(x) = x j_{\ell}(x)$, and

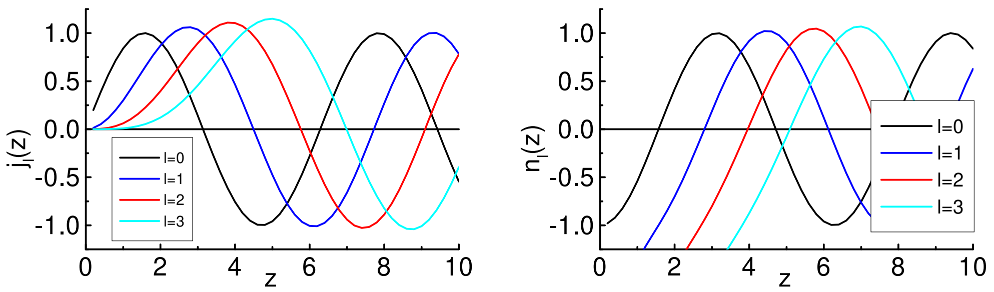


Figure 3: The first few spherical Bessel and Neumann functions

analogously for the irregular solutions the Neummann functions. Thus their asymptotic becomes

$$\hat{j}_\ell(kr) \quad \stackrel{r\to\infty}{\longrightarrow} \quad \sin(kr - \frac{\ell\pi}{2}) \tag{17}$$

$$\hat{n}_\ell(kr) \quad \stackrel{r\to\infty}{\longrightarrow} \quad -\cos(kr - \frac{\ell\pi}{2}). \tag{18}$$

For small arguments x they behave like

$$\hat{j}_\ell(x) \stackrel{r\to 0}{\longrightarrow} \frac{x^{\ell+1}}{(2\ell+1)!!} \tag{19}$$

$$\hat{n}_\ell(x) \stackrel{r\to 0}{\longrightarrow} -\frac{(2\ell-1)!!}{x^\ell} \tag{20}$$

The first few functions are shown in the Fig. 3 together with the corresponding irregular solutions $\hat{n}_\ell(kr)$ of (16).

In the interaction case the potential $V(r)$ is assumed to vanish sufficiently rapid at infinity, i.e., $rV(r) \stackrel{r\to\infty}{\longrightarrow} 0$ (the Coulomb potential deserves particular care). Outside the range of the interaction the wave function must be a linear combination of $\hat{j}_\ell(kr)$ and $\hat{n}_\ell(kr)$. Since the radial wave function is real, the coefficients must be real. Thus, at large distances the wave function can be written in the form

$$A\left[\cos\delta_\ell \hat{j}_\ell(kr) - \sin\delta_\ell \hat{n}_\ell(kr)\right] \to A\sin(kr - \frac{\ell\pi}{2} + \delta_\ell) \tag{21}$$

With $A = e^{\mathrm{i}\delta_\ell}$ the complete wave becomes the asymptotic form

$$\psi(r,\vartheta,\varphi) \quad \stackrel{r\to\infty}{\longrightarrow} \quad \frac{\mathrm{i}}{2k}\sum_\ell (2\ell+1)\left(\underbrace{(-1)^\ell\,\frac{e^{-\mathrm{i}kr}}{r}}_{\text{incoming wave}} - \underbrace{e^{2\mathrm{i}\delta_\ell}\,\frac{e^{\mathrm{i}kr}}{r}}_{\text{outgoing wave}} \right) \tag{22}$$

such that the incoming wave part is identical to that of the plane wave. Thus, the only relevant modification of the asymptotic wave function is a change of the phase shift δ_ℓ. If only elastic scattering is open the phase shifts are real.

By comparing the two asymptotic forms for the wave function (5) and (22) one finds the connection between the scattering amplitude and the phase shift

$$f(\vartheta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(e^{2\mathrm{i}\delta_\ell}-1)P_\ell(\cos\vartheta), \tag{23}$$

where $S_\ell = e^{2\mathrm{i}\delta_\ell}$ is the S-matrix in each partial wave. In case of inelastic processes the phase-shifts generally become complex and one rewrites the $S_\ell = \eta_\ell e^{2\mathrm{i}\delta_\ell}$ with *real* phase-shifts and the "absortption coefficient" η_ℓ with $0 \le \eta_\ell \le 1$.

Thus, one can define a scattering amplitude for each partial wave

$$f_\ell \; = \frac{1}{2\mathrm{i}k}\left(\eta_\ell e^{2\mathrm{i}\delta_\ell}-1\right) \tag{24}$$

with

$$f(\vartheta) = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell P_\ell(\cos\vartheta) \tag{25}$$

which determines the elastically scattered part of the cross section. The differential *elastic* cross section is given by

$$\frac{d\sigma(\vartheta)_{\mathrm{el}}}{d\Omega} = |f(\vartheta)|^2 = \frac{1}{4k^2}\left|\sum_{\ell=0}^{\infty}(2\ell+1)\left(\eta_\ell e^{2\mathrm{i}\delta_\ell}-1\right)P_\ell(\cos\vartheta)\right|^2 \tag{26}$$

The angle integrated cross section can be obtained using the orthogonality relations of the legendre functions

$$\sigma_{el} = \frac{\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\left|\eta_\ell e^{2\mathrm{i}\delta_\ell}-1\right|^2 \tag{27}$$

If $\eta < 1$ one can obtain the reaction cross section by the difference of the angular integrated incomming and outgoing fluxes per incomming flux

$$\sigma_r = \int r^2 \mathrm{d}\Omega\left(|\psi_{in}|^2 - |\psi_{in}|^2\right) = \frac{\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)(1-\eta_\ell^2). \tag{28}$$

The total cross section then becomes

$$\sigma_{tot} = \sigma_{el} + \sigma_r = \frac{2\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)(1-\eta_\ell\cos(2\delta_\ell)) \tag{29}$$

which tekes the same form as the imaginary part of the forward amplitude

$$f(0) \;\; = \;\; \frac{1}{2k}\sum_{\ell=0}^{\infty}(2\ell+1)\left(1-\eta_\ell\cos(2\delta_\ell)\right). \tag{30}$$

This provides the optical theorem

$$\sigma_{tot} = \frac{4\pi}{k} \mathrm{Im} f(0) \tag{31}$$

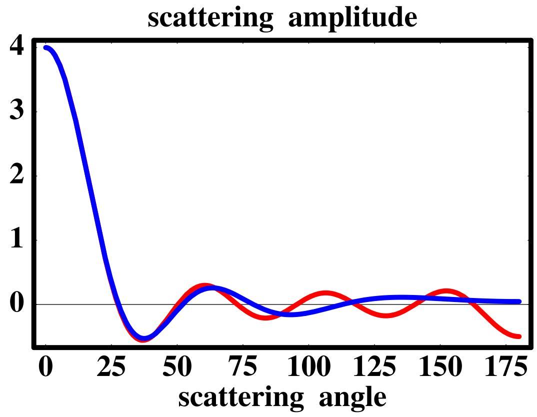
expressing that the loss in forward direction is related to the total cross section.

Note that the equations (5, 23, 26), which refer to the asymptotic behaviour of the wave function remain valid also in the relativistic case, when the momentum k is the relatice momentum in the c.m. frame. They remain true even in the case when the interaction cannot be described by a potential.

4 Elastic cross section

4.1 Black disc absorption model

We consider a schematic model of a totally absorbing black disc, where all partial waves up to a maximum value of L are complete absorbed, i.e $\eta_\ell = 0$ for all $\ell < L$. Then the total elastic and the reaction crossections are identical, i.e.



Elastic scattering amplitude for the black disc with $kR = L = 8$; red (35), blue approximation (36).

$$\sigma_{el} = \sigma_{react} = \frac{\pi}{k^2} \sum_{\ell<L} (2\ell+1) = \pi \frac{L^2}{k^2} = \pi R^2 \qquad \text{with } L = kR \tag{32}$$

$$\sigma_{tot} = \sigma_{el} + \sigma_{react} = 2\pi R^2 \tag{33}$$

Note that the total cross section is twice the geometric one.

The differential scattering amplitude can be calculated expilcitely using the summation relation of the Legendre polynoms[1]

$$f_{\text{el;blackdisc}}(\vartheta) = \frac{\mathrm{i}}{2k} \sum_{\ell<L} (2\ell+1) P_\ell(cos(\vartheta)) \tag{34}$$

$$= \frac{\mathrm{i}L}{2k} \frac{P_L(cos(\vartheta)) - P_{L-1}(cos(\vartheta))}{1 - cos(\vartheta)} \tag{35}$$

$$\approx \mathrm{i}kR^2 \frac{J_1(qR)}{qR} \qquad \text{where} \quad q = 2k \sin\frac{\vartheta}{2} \tag{36}$$

[1] Christoffel Darboux (Abramowitz 22.12.1)

$$\sum_{\ell=0}^{L-1} (2\ell+1) P_\ell(x) P_\ell(y) = L \, \frac{P_L(x)P_{L-1}(y) - P_{L-1}(x)P_L(y)}{x - y}$$

where in the last line an asymptotic formula has been used, which expresses Legendre polynoms of large L by cylindrical Bessel functions.

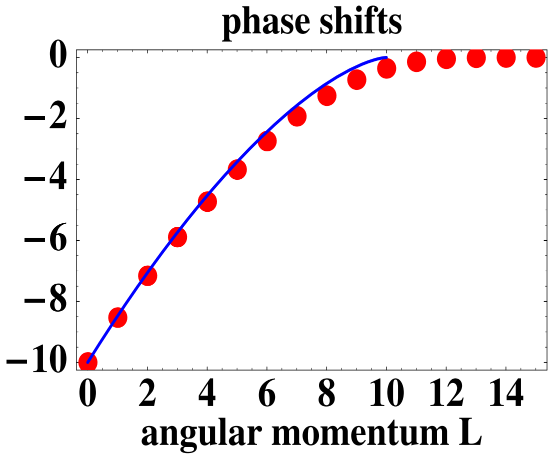
4.2 Elastic phase shifts

As the first illustrative example we discuss the scattering off a hard sphere of radius R. In this case the radial wave functions have to vanish at $r - R$. Since for $r \geq R$ the potential is zero the radial wave function has to be a linear combination of $\hat{j}(kr)$ and $\hat{n}(kr)$, i.e.

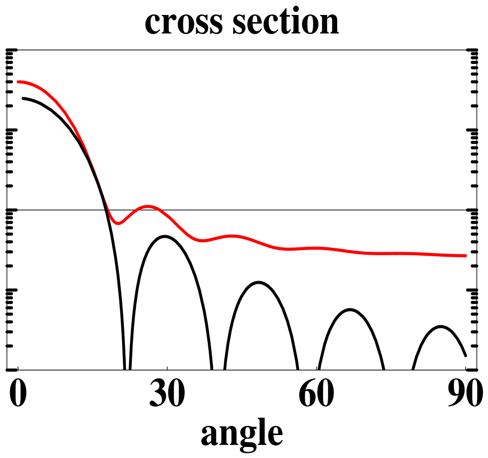
$$\chi_\ell(r) \quad = \quad \cos\delta_\ell \, \hat{j}(kr) - \sin\delta_\ell \, \hat{n}(kr) \tag{37}$$

for $r \geq R$ and with $\chi_\ell(R) = 0$. Hence

$$\tan\delta_\ell \quad = \quad \frac{\hat{j}(kR)}{\hat{n}(kR)} \tag{38}$$



Phase shifts for the hard-sphere with $kR = 10$ (dots), compared to the WKB approximation (full line)



Hard-sphere (red) and black-disk (black) cross sections for $kR = 10$.

For $\ell = 0$ the phase shift is simply given by $\delta_0 = -kR$. For large ℓ the phase shifts drop like

$$\delta_\ell \quad = \quad -\frac{(kR)^{2\ell+1}}{(2\ell+1)!!(2\ell-1)!!} \quad \text{for} \quad \ell \geq kR \tag{39}$$

If phase shifts are taken as continous function in k with $\delta_\ell(k = \infty) = 0$ then repulsive potentials produce negative phase shifts while attractive potential lead to positive $\delta_\ell(k)$. At low energies $k \to 0$ only the $\ell = 0$ term survives which leads to an isotropic finite cross section of

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \stackrel{k\to 0}{\longrightarrow} a^2, \qquad \sigma_{el} \stackrel{k\to 0}{\longrightarrow} 4\pi a^2, \tag{40}$$

where

$$a = \lim_{k\to 0} \frac{\delta_0(k)}{k} \tag{41}$$

is called the scattering length. For the hard sphere case $a = R$.

4.3 How to compute phase shifts

Only in few simples cases the phase shifts can be calculated explicitly. In the general case one has to solve the Schroedinger equation numerically in order to retrieve the phase shift. We are not going to decribe the numerical procedure for the numerical solution of eq. (16). Suppose this to be done for each ℓ, i.e. the wave function $\chi_\ell(r)$ and its derivative $\chi'_\ell(r)$ are known at a radial position R outside the range of the potential ($'$ denoting the derivative with respect to r). The phase shift are obtained by calulating the ratio of the Wronsky determinant of χ with the Riccati Bessel and Neumann functions at that position, i.e

$$\tan\delta_\ell = \frac{1}{k}\frac{\mathcal{W}\{\chi(r),\hat{j}(r)\}}{\mathcal{W}\{\chi(r),\hat{n}(r)\}}\bigg|_{r=R} \tag{42}$$

where the Wronskian is defined as

$$\mathcal{W}\{f(r),g(r)\} = f(r)\frac{\mathrm{d}}{\mathrm{d}r}g(r) - g(r)\frac{\mathrm{d}}{\mathrm{d}r}f(r). \tag{43}$$

It has the properties

$$\begin{aligned}\mathcal{W}\{f(r),g(r)\}_r &= -\mathcal{W}\{g(r),f(r)\}_r & (44)\\ \mathcal{W}\{f(r),f(r)\}_r &= 0. & (45)\end{aligned}$$

For two different solutions of the radial Schr"odinger equation the Wronskian is constant. In particular

$$\mathcal{W}\{\hat{j}(kr),\hat{n}(kr)\} = k. \tag{46}$$

Thus, for the asymptotic representation (21) of the wave function the Wroskians used in (42) project out the coefficients in front of the $\hat{j}_\ell(kr)$ and $\hat{n}_\ell(kr)$ functions.

4.4 WKB[2] phase shifts and connection to classical scattering

There are many different approximate expressions for the phase shifts. The most illustrative and also useful one in the case of a large number of ℓ-values contributing is the WKB approximation.

[2]Wenzel, Kramers, Brillouin

The idea behind this approximation is that the radial wave locally behave like incoming and outgoing plane waves according to the local classical momentum

$$\chi_\ell(r) \approx A_{\mathrm{WKB}}(r) \Big[\underbrace{\mathrm{i} \exp\Big(-\frac{\mathrm{i}}{\hbar} \int_{r_0}^{r} \mathrm{d}r' \, p_{\mathrm{local}}(r') \Big)}_{\text{incoming wave}} \underbrace{- \mathrm{i} \exp\Big(\frac{\mathrm{i}}{\hbar} \int_{r_0}^{r} \mathrm{d}r' \, p_{\mathrm{local}}(r') \Big)}_{\text{outgoing wave}} \Big] \tag{47}$$

Here A_{WKB} is supposed to behave smoothly and

$$p_{\mathrm{local}}(r) = \sqrt{2\mu(E - V(r)) - \hbar^2 \frac{\ell(\ell+1)}{r^2}}, \quad \hbar^2 k^2 = 2\mu E, \quad kb = \ell + \tfrac{1}{2} \tag{48}$$

$$= \hbar k \sqrt{1 - \frac{V(r)}{E} - \frac{b^2}{r^2}} \tag{49}$$

Comparing the accumulated phase in this expression with the corresponding form for the free wave function determines the WKB phase shifts as

$$\delta_\ell \approx \delta_{\mathrm{WKB}}(b) = \lim_{R\to\infty} k \left(\int_{r_0}^{R} \mathrm{d}r \, \sqrt{1 - \frac{V(r)}{E} - \frac{b^2}{r^2}} - \int_{b}^{R} \mathrm{d}r \, \sqrt{1 - \frac{b^2}{r^2}} \right). \tag{50}$$

Here and above r_0 is the outermost classical turning point given by the outermost zero of the classical local momentum, i.e. $p_{local}(r_0) = 0$. The condition for this approximation is that the potential does not vary too rapidly within distances of order $1/k$. The WKB phase shifts are easily to be calculated, see e.g. the phase shifts for the hard sphere (Fig. above) and the are connected to classical scattering properties as deflection angle and delay times, as we shall see next. Inserting into the scattering amplitude, both, the WKB phase shifts and the asymptotic form for the Legendre polynoms leads to

$$f_{\mathrm{WKB}}(\vartheta) = \sqrt{\frac{kb}{2\pi \sin\vartheta}} \int_0^\infty \mathrm{d}b \left(e^{2\mathrm{i}\delta_{\mathrm{WKB}}(b)} - 1 \right) \left(\mathrm{i}e^{-\mathrm{i}(\ell+\frac{1}{2})\vartheta} - \mathrm{i}e^{\mathrm{i}(\ell+\frac{1}{2})\vartheta} \right) \tag{51}$$

Here we have converted the ℓ-summation to an integral over impact parameter using $\ell + \frac{1}{2} = kb$. This is an integral with strongly oscillating integrand if many partial waves contribute. Thus the main contributions are expected to arize from impact paramaters for which the the phase in the exponents are stationary, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}b} 2\delta_{\mathrm{WKB}}(b) = \pm k\theta \tag{52}$$

The explicit evaluation of (50) leads to

$$\pm\vartheta(b) = \pi - 2 \int_{r_0}^{R} \mathrm{d}r \, \frac{b}{r^2 \sqrt{1 - \frac{V(r)}{E} - \frac{b^2}{r^2}}} \tag{53}$$

which nothing else but the classical deflection angle in the potential $V(r)$ as a function of the impact parameter. The partial contribution of each impact parameter leading to the deflection angle ϑ are given by

$$f_{\text{sem.cl}}(\vartheta) = \sum_{b(\pm\vartheta)} \sqrt{\mathrm{i}\underbrace{\frac{2\pi b\mathrm{d}b}{2\pi\sin\vartheta\mathrm{d}\vartheta}}_{\text{classical cross section}}}\; e^{2\mathrm{i}\delta_{WKB}(b)\pm\mathrm{i}kb\vartheta}. \tag{54}$$

While the WKB phase shifts inserted into the discrete partial wave summation may lead to good descriptions of the scattering amplitude, the semi-classical expression (54) based on classical trajectories cannot describe genuine quantum feature like diffraction and tunneling into shadow areas. However Eq. (54) gives quite an intuitive picture of the interplay of classical paths and quantum phases leading to interference structure.

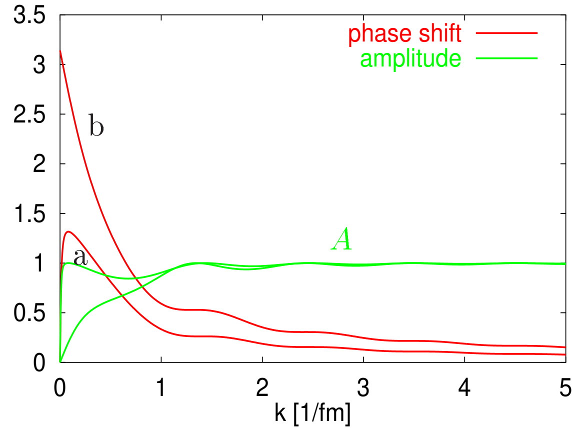
The WKB phase shifts also clarify another semi-classical relation. Taking the derivative of $\delta_{WKB}(\ell, E)$ with respect to the energy E at fixed ℓ leads to

$$\begin{aligned} \frac{\partial}{\partial E}\,\delta_{\text{WKB}}(\ell,E) &= 2\mu\left(\int_{r_0}^{R}\frac{\mathrm{d}r}{\sqrt{2\mu(E-V(r))-\frac{\ell(\ell+\frac{1}{2})}{r^2}}} - \int_{b}^{R}\frac{\mathrm{d}r}{\sqrt{2\mu E-\frac{\ell(\ell+\frac{1}{2})}{r^2}}}\right) && (55)\\ &= \Delta t_{\text{delay}} && (56)\end{aligned}$$

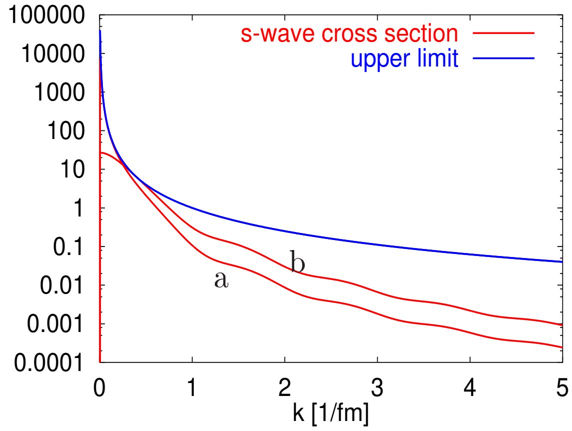
which is identical to the difference between the propagation time along the potential orbit minus that along the free orbit. This is called the delay time due to the interaction, a quatity that reappears in the context of resonances.

4.5 Phase-shifts for attractive potentials

Attractive potentials show much richer structures than purely repulsive potentials, becaus they can for resonances, inparticular for for partial waves with $\ell \neq 0$ since they have a centrifugal potential wich together with the attractive potential provides a barrier. We first will discuss the s-wave behavior for an attractive potential square well potential of radius $R = 3.1$ fm and depth $V(r = 0) = -V_0$. The wave number in the potential is $\kappa = \sqrt{2\mu(E + V_0)}$. Depending on the value of κR at zero energy $\kappa_0 R = \sqrt{2\mu V_0}R$ the potential has n bound statesb with n being the maximal positive integer satifying $(n + \frac{1}{2})\pi \leq \kappa_0 R$. This determines the phase-shifts at zero energy to be $\delta(E = 0) = n\pi$ with the exeption, if there is a boundstate at exactly zero energy, i.e. $(n + \frac{1}{2}) =\leq \kappa_0 R$ then $\delta(E = 0) = (n + \frac{1}{2})\pi$. For each wave number k the phase shifts $\delta_0(k)$ and the amplitude $A(k)$ can be calculated from (42). The explicite formulae are given below in (58) for $V_B = 0$. The plot show the phase shifts and the amplitude A of the wave function $\chi_0(r) = A\sin(\kappa r)$ within the range of the potential for two case, (a) where there is just not bound state ($\kappa_0 R = 1.55$) and (b) where there is just one bound state ($\kappa_0 R = 2.2$)



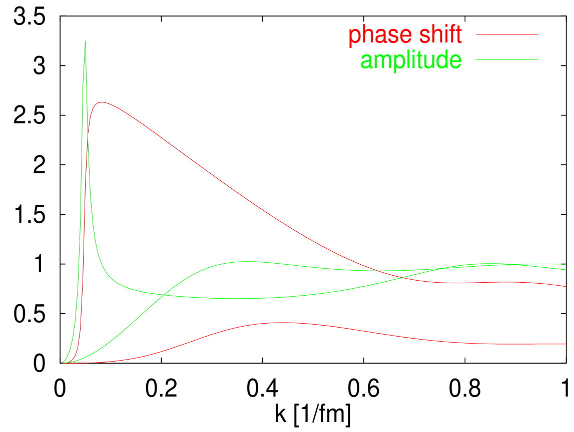
Phase-shift $\delta_0(k)$ (red) and amplitude A (green) for two cases, (a) no bound state yet and (b) one bound state.



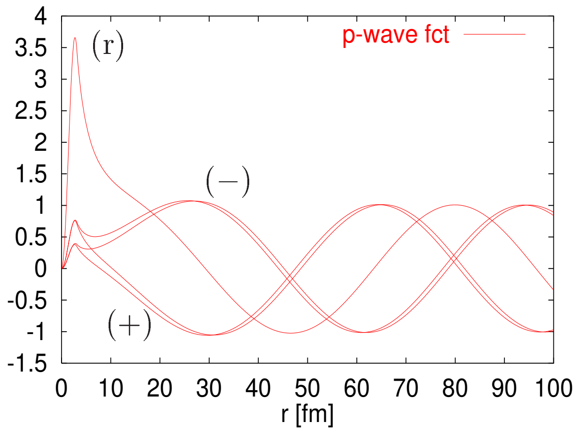
Cross sections for the two cases to the left (red) compared to the maximum s-wave cross section (blue).

In neither of the situations one obtaines a typical resonance behavior. However the phase shifts have a strong k-dependence close to $k = 0$, which leads to very large cross sections, since the scattering length $a = \mathrm{d}\delta(k)/\mathrm{d}k|_{k=0}$ become very large. The two examples (a) and (b) are typical for the ${}^1s_0$ and ${}^3s_1$ phase-shifts in nucleon-nucleon scattering, with an almost bound state in the first case and the deuteron bound state in case (b).

Typical resonance structures appear for p and higher partial waves.



p-wave phase-shift $\delta_1(k)$ (red) and amplitude A (green) for two cases, (a) a shallow potential (b) a potential with a resonance.



p-wave radial wave function $\chi_1(r)$ for 5 cases: (r) at resonance and (-) and (+) for $\pm 1.5\%$ and $\pm 2.5\%$ below and above resonance.

The parameters of the potential used in the figures are $R = 3.1$ fm and $\kappa_0 = 1 fm^{-1}$. One sees that only very close to the resonance the wave function is significatly enhanced in the interior of the potential well. Below and above resonance the phase shifts are close to zero and π respectively.

4.6 Potential with barrier

In many particle physics text books one finds the statement that a signature for a resonance is that the phase shift goes through the values of $(n+\frac{1}{2})\pi$ with positive slope. This statement is not true in this generality. In order to clarify this point we discuss the case of a potential well of radius R and depth V_0 with a thin barrier at R, for simplicity approximated by a δ-function at R, i.e.

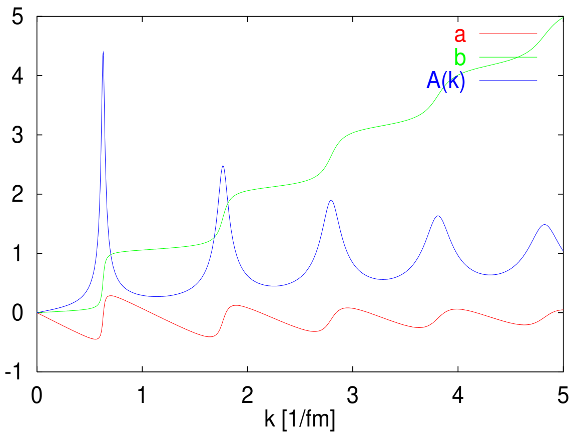
$$V(r) = -V_0\Theta(R-r) + V_B\delta(r-R). \tag{57}$$

For this case the s-wave radial equation can still be solved in closed form in terms of sin and cosin functions. Thus

$$\tan\delta_0(k) = \frac{V_B\sin\kappa R + \kappa\cos\kappa R}{k\sin\kappa R} \tag{58}$$

$$A(k) = \frac{k}{\sqrt{(\kappa\cos\kappa R + V_B\sin\kappa R)^2 + (k\sin\kappa R)^2}} \tag{59}$$

$$\kappa = \sqrt{k^2+\kappa_0^2}, \qquad \kappa_0^2 = 2\mu V_0$$



s-wave phase-shift for the potential with barrier. a) (green) total phase shift $\delta_0(k)/\pi$, b) (red) resonance part of phase shift $\delta_{\rm res}(k)/\pi$, and and amplitude A (blue) for two cases, (a) a shallow potential (b) a potential with a resonance.

Depending on the strength of the barrier, this leads to a couple of sharp resonance structures, as can be seen from the behavior of the interior amplitude A as a function of wave number k. The resulting phase shifts, however, has a peculiar behavior, it sharply increases close to the resonance and then falls smoothly until it sharply raises again at the next resonance. From the amplitude $A(k)$ one reads off that the wave function in the interior of the potential well is mostly small off the resonances. That is in these off-resonance cases the system behaves like a repulsive sphere. Since the respulsive sphere leads to a phase shift $\delta_{\rm rep.sphere} = -kR$, we subtract this part from $\delta_0(k)$ in order to isolate the resonance part of the phase shift $\delta_{\rm res} = \delta_0(k) - \delta_{\rm rep.sphere}$. The resulting curve (b) shows the typical phase shift behavior for resonances namely that it increases by pi at each resonance. Each of these resonance "steps" can be parametrized in terms of a Breit-Wigner form.