1 Remarks on potential scattering

What does one learn when one studies the scattering of hadrons, and fits the scattering data in some model? Since one can adjust the parameters of a potential model one may think that one learns what the potential is like. This is true in an ideal case, where one knows the scattering phase shift from threshold to infinite energy, as well as the asymptotic coefficients of the wave function of bound states, if any. Furthermore, one must assume that the interaction is a local potential. However, in reality the energy range over which one has data is always limited, and consequently the potential is is not unique. This is because the energies that can be achieved in experiments is limited, and because a potential picture breaks down at high energies, where inelastic channels become important and/or the relevant degrees of freedom change.

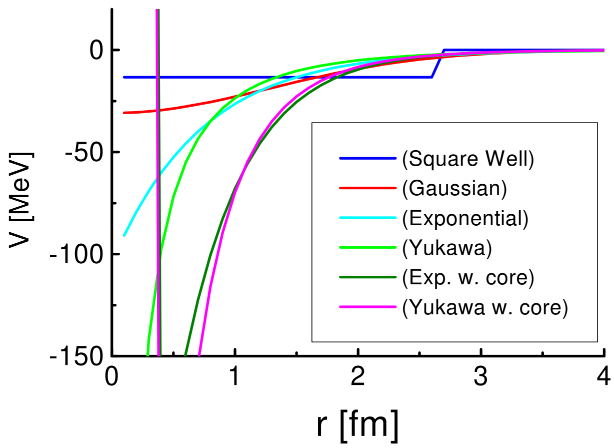

Figure 1: NN-potentials that reproduce the low-energy scattering data in the ${}^1S_0$ channel.

As an illustration of this, we show in fig. **??** six NN potentials that all reproduce the low energy nucleon-nucleon scattering data in the ${}^1S_0$ channel. What is meant by this is that they all yield the same values for the scattering length a and the effective range parameter r_0, which is defined by the effective-range expansion

$$k \cot \delta = -\frac{1}{a} + \frac{1}{2}k^2 r_0 + \cdots \tag{1}$$

Thus, in this simple example we see that the ambiguity in the potential is enormous, if one fits only the low-energy data. The ambiguity is reduced as the energy range is increased, but a certain amount of ambiguity always remains. There is an infinite amount of so called phase-equivalent potentials! Hence the goal of this type of work is not to determine the potential but rather to determine the scattering amplitude. As we will see later, the physics that one wants to address, like e.g. the properties of a certain hadron in nuclear matter, can be formulated completely in terms of scattering amplitudes.

Now, one may ask, why not use the scattering data directly without the intermediate step of fitting a model to the data? A model is useful, because if it is a good model it summarizes a large amount of scattering data in a few parameters. Furthermore, such a model yields the scattering amplitude at subthreshold energies, which which cannot be obtained directly from the data. This is of importance for the discussion of hadron properties in matter.

2 Breit-Wigner description of resonances

In this section we describe the Breit-Wigner resonance formula, and modifications thereof. Recall the form of the scattering amplitude for a given partial wave (24)

$$f_\ell = \frac{1}{2ik}\left(\eta_\ell e^{2i\delta_\ell} - 1\right). \tag{2}$$

We assume for simplicity that we have a pure resonance located at the energy $E = E_R$, without any back-ground phase. (In this section E refers to the energy in the cm frame.) Thus, the resonance corresponds to a phase shift of $\delta_\ell(E_R) = \pi/2$. Furthermore, we assume that only one channel is open, so that only elastic scattering is possible. Thus, $\eta_\ell = 1$ and

$$f_\ell = \frac{1}{k} e^{i\delta_\ell} \sin\delta_\ell. \tag{3}$$

Using

$$e^{-i\delta_\ell} = \cos\delta_\ell - i\sin\delta_\ell \tag{4}$$

and simple manipulations one can rewrite the scattering amplitude in the form

$$f_\ell = \frac{1}{k}\frac{1}{\cot\delta_\ell - i}. \tag{5}$$

Now, for energies close to to the resonance, the phase shift is close to $\pi/2$ and $\cot\delta_\ell \simeq 0$. We expand the denominator of f_ℓ in a Taylor series. Using

$$\begin{aligned}\cot\delta(E) &= \cot\delta(E_R) + (E - E_R)\left[\frac{d}{dE}\cot\delta(E)\right]_{E=E_R} + \cdots \\ &\simeq -(E - E_R)\frac{2}{\Gamma}.\end{aligned} \tag{6}$$

We have defined $2/\Gamma = [d(\cot\delta)/dE]_{E=E_R}$. Γ turns out to be the width of the resonance. Terms of higher order in the expansion can be neglected if $\Gamma << E_R$, i.e., when the width of the resonance is small compared to its energy. How broad resonances can be treated will be discussed below. By using eq. **??**, one obtains the phase shift in the Breit-Wigner approximation

$$\delta_{BW}(E) = \arctan\left[\frac{\Gamma/2}{E_R - E}\right]. \tag{7}$$

Inserting the expansion in the scattering amplitude one finds

$$f_\ell(E) = \frac{1}{k}\frac{\Gamma/2}{E_R - E - i\Gamma/2}, \tag{8}$$

and using

$$\sigma_\ell = 4\pi|f_\ell|^2, \tag{9}$$

where we have defined the contribution of the ℓ'th partial wave to the elastic cross section (see (27)),

$$\sigma_\ell(E) = \frac{4\pi}{k^2}(2\ell+1)\underbrace{\frac{\Gamma^2/4}{(E - E_R)^2 + \Gamma^2/4}}_{\sin^2\delta_\ell}. \tag{10}$$

This is the famous Breit-Wigner formula for a resonant cross section. The width Γ is defined such that the cross section at $|E - E_R| = \pm\Gamma/2$ is half its maximum value. Note that last term, which can be identified with $\sin^2\delta_\ell$, equals unity at the resonance energy $E = E_R$, i.e., where $\delta_\ell = \pi/2$.

The Breit-Wigner formula is very useful for describing the scattering amplitude and cross section in cases where a resonance dominates the cross section. However, often background processes cannot be neglected, so the Breit-Wigner amplitude must be supplemented with a model for these processes.

The Breit-Wigner formula has been generalized in many ways. However, before we discuss this, let us see how well it does in describing a baryonic resonance like the Δ_{33} resonance in πN scattering. In fig. **??** we show a comparison of this simple form, for $E_R = 1232$ MeV and $\Gamma = 120$ MeV, with the empirical phase shift We note that the phase shift is well described only near the resonance

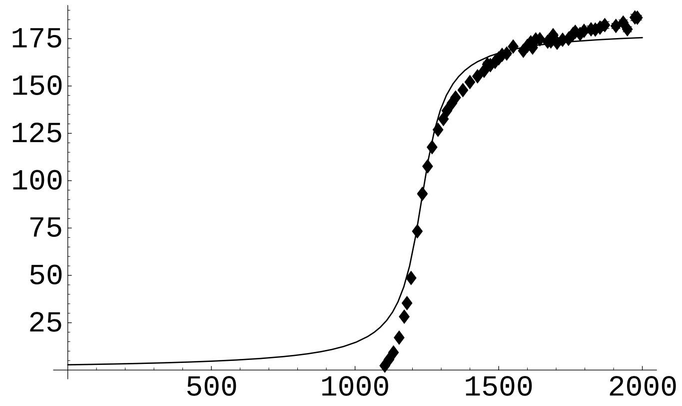

Figure 2: The scattering phase shift in the p_{33} channel as a function of the cm-energy in MeV.

energy. The threshold is not correctly described and neither is the high-energy part. A correct description of the threshold can be achieved by using a phase-space corrected width rather than a constant one. Using the phase-space techniques discussed in the first part of these lectures and the fact that we are considering a p-wave channel one finds

$$\Gamma(E) = \Gamma_0 \left(\frac{E_R}{E}\right)^2 \left(\frac{k(E)}{k(E_R)}\right)^3 . \tag{11}$$

The correcponding phase shift is shown in fig. **??**. Now the threshold is correctly described, but on the high-energy side the description is worse! The problem is that we have assumed that the form of the matrix element is const. $\times\, k$, which means that the width keeps growing with energy. This corresponds to the bare interaction of pointlike objects in field theory. Since the nucleon and the pion are neither bare nor pointlike, the matrix element decreases at large momenta (small distances). This is often taken into account phenomenologically by introducing formfactors. In fig. **??** the phase shift obtained by multiplying the width by a factor $(\Lambda^2 + k(E_R)^2)/(\Lambda^2 + k(E)^2)$ is

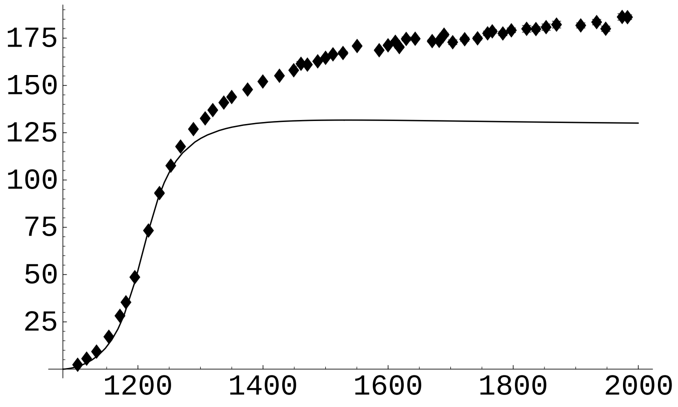

Figure 3: Same as the previous figure, but with a phase-space corrected width.

shown. A considerable improvement is obtained already with this trivial form factor. Better fits exist in the literature.

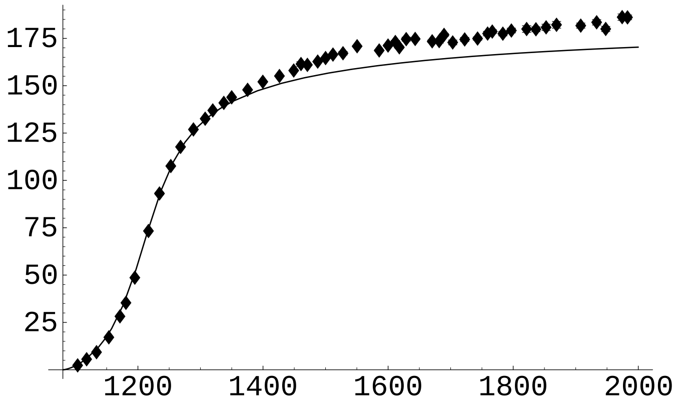

Figure 4: Same as the previous figure, but with a form factor.

The Breit-Wigner form for the cross section has been generalized to describe relativistic particles. One then usually writes the amplitude in the form

$$f_\ell(\sqrt{s}) = \frac{1}{k} \frac{\sqrt{s}\Gamma}{m_R^2 - s - i\sqrt{s}\Gamma}. \tag{12}$$

In some papers one finds a factor m_R in front of Γ rather than $\sqrt{s}$. This corresponds to a trivial redefinition of the width, which plays no role in microscopic calculations or when the width is phase-space corrected, but may well play a role when the width is approximated by a constant.

For particles with spin s_1 and s_2, the spin-averaged Breit-Wigner cross section is then of the form

$$\sigma_\ell(s) = \frac{4\pi}{k^2}\frac{(2J+1)}{(2s_1+1)(2s_2+1)}\frac{s\Gamma^2}{(s-m_R^2)^2+s\Gamma^2}, \tag{13}$$

where J is the total spin of the resonance. When one considers particular spin states in the initial and final state, i.e. an experiment with a polarized beam and target, the corresponding Clebsch-Gordan coefficients must be included. Similarly, particular isospins in the initial and final states can be accounted for. Thus e.g. the spin-averaged Breit-Wigner cross section for the reaction $\pi^- p \to \pi^- p$ scattering through the $I = 3/2, J = 3/2$ Δ resonance is

$$\sigma_\ell(s) = \left(\frac{1}{\sqrt{3}}\right)^2 \frac{4\pi}{k^2}\frac{4}{2}\frac{s\Gamma_\Delta^2}{(s-m_\Delta^2)^2+s\Gamma_\Delta^2}. \tag{14}$$

For the reaction $\pi^+ p \to \pi^+ p$ the Clebsch-Gordan coefficient, which in the above example is $(1/\sqrt{3})^2$, equals unity.

Finally, we generalize the Breit-Wigner formula to the case when several channels are open, i.e., when inelastic reactions are possible. Denote the partial width of the resonance R to decay into channel a by Γ_a and the total width by Γ_R. Then the cross section for scattering from channel a to channel b through the resonance R is given by

$$\sigma_{a\to b}(s) = \frac{4\pi}{k^2}\frac{(2J+1)}{(2s_1+1)(2s_2+1)}\frac{s\Gamma_a\Gamma_b}{(s-m_R)^2+s\Gamma_R^2}, \tag{15}$$

while the total cross section is given by

$$\sigma_a(s) = \frac{4\pi}{k^2}\frac{(2J+1)}{(2s_1+1)(2s_2+1)}\frac{s\Gamma_a\Gamma_R}{(s-m_R)^2+s\Gamma_R^2}. \tag{16}$$

Thus, the peak value of the total cross section is proportional to the branching ratio for the resonance to decay into the channel a.

3 Inelastic collisions

In this section we discuss inelastic collisions to specific channels. For elastic scattering we found

$$f_\ell = \frac{1}{2ik}\left(\underbrace{\eta_\ell e^{2i\delta_\ell}}_{S_\ell} - 1\right), \tag{17}$$

where S_ℓ is the S-matrix for the elastic channel. If only elastic scattering is possible, $\eta_\ell = 1$, while if inelastic channels are open, η_ℓ is less than unity. Consider now a reaction $a \to b$, where the states a and b are both assumed to contain two particles, but the particle species may be changed in the reaction. The asymptotic form of the wave function of the initial state is just as in the discussion of elastic scattering

$$\psi_a \overset{r\to\infty}{\longrightarrow} e^{ik_a z} + \frac{1}{r} f_{aa}(\theta) e^{ik_a r}. \tag{18}$$

The cross section for elastic scattering is then as before

$$\frac{d\sigma_{aa}}{d\Omega_i} = |f_{aa}(\theta)|^2 \tag{19}$$

On the other hand, for inelastic scattering $a \to b$, where $b \neq a$, the wave function for the final state is somewhat different

$$\psi_b \overset{r\to\infty}{\longrightarrow} \frac{1}{r} f_{ba}(\theta) \sqrt{\frac{\mu_b}{\mu_a}} e^{ik_b r}, \tag{20}$$

where μ_a and μ_b are the reduced masses in the initial and final channels respectively. We use the notations $f_{ba} = f_{a\to b}$ interchangeably. The square root factor in (**??**)is a convention, which leads to a convenient normalization of the scattering amplitude. (Note that this form applies to a non-relativistic system. In a relativistic treatment, $\mu_a = \omega_{1a}\omega_{2a}/(\omega_{1a}+\omega_{2a})$ etc., where ω_{1a} and ω_{2a} are the energies of the two incident particles in the cm frame. The relation between the S-matrix and the scattering matrix, eq. **??** below, remains unchanged.)

The leading contribution to the rate for particles to scatter in the solid angle $d\Omega_b$ is given by

$$\frac{k_b}{\mu_b}|\psi_b|^2 r^2 d\Omega_b \overset{r\to\infty}{\longrightarrow} \frac{k_b}{\mu_a}|f_{ba}(\theta)|^2 d\Omega_b \tag{21}$$

while the current of incident particles is k_a/μ_a (in a relativistic treatment the current is $k_a(\omega_{1a} + \omega_{2a})/\omega_{1a}\omega_{2a}$). Consequently the cross section for the inelastic reaction $a \to b$ is

$$\frac{d\sigma_{ba}}{d\Omega_b} = \frac{k_b}{k_a}|f_{ba}(\theta)|^2. \tag{22}$$

Also the inelastic scattering amplitudes can be expanded in partial waves

$$f_{ba}(\theta) = \sum_\ell (2\ell+1) f_{ba}^{(\ell)} P_\ell(\cos\theta). \tag{23}$$

The corresponding S-matrix is given by

$$S_{ba}^{(\ell)} = \delta_{ba} + 2i\sqrt{k_b k_a} f_{ba}^{(\ell)}. \tag{24}$$

Note that in the elastic channel (**??**) reduces to the relation (**??**).

The total cross section for elastic scattering is given by

$$\sigma_{aa} = \frac{\pi}{k_a^2}\sum_\ell (2\ell+1)|S_{aa}^{(\ell)} - 1|^2, \tag{25}$$

while the cross section for the reaction $a \to b$ ($b \neq a$) is

$$\sigma_{ba} = \frac{\pi}{k_a^2}\sum_\ell (2\ell+1)|S_{ba}^{(\ell)}|^2. \tag{26}$$

The reaction cross section is obtained by summing over all inelastic channels

$$\sigma_r = \sum_{b\neq a}\sigma_{ba} = \frac{\pi}{k_a^2}\sum_\ell (2\ell+1)\sum_{b\neq a}|S_{ba}^{(\ell)}|^2. \tag{27}$$

Unitarity of the S-matrix ($\sum_b |S_{ba}^{(\ell)}|^2 = 1$) implies that

$$\sigma_r = \frac{\pi}{k_a^2}\sum_\ell (2\ell+1)(1 - |S_{aa}^{(\ell)}|^2). \tag{28}$$

Finally, the total cross section is the sum of the elastic and reaction cross sections

$$\sigma_{tot} = \sigma_{aa} + \sigma_r = \frac{2\pi}{k_a^2}\sum_\ell (2\ell+1)(1 - \mathrm{Re}(S_{aa})). \tag{29}$$

Using (**??**) and (**??**) as well as $P_\ell(1) = 1$ we recover the optical theorem

$$\sigma_{tot} = \frac{4\pi}{k_a}\sum_\ell (2\ell+1)\mathrm{Im} f_{aa}^{(\ell)} = \frac{4\pi}{k_a}\mathrm{Im} f_{aa}(\theta = 0). \tag{30}$$

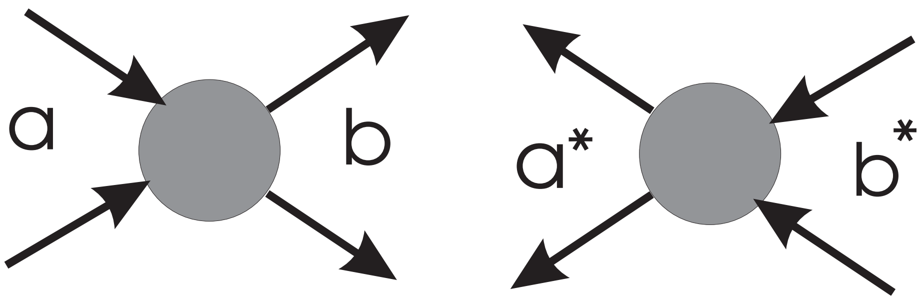

Figure 5: The reaction $a \to b$ and its timereversed counterpart.

4 Detailed balance

Consider an inelastic reaction $a \to b$, illustrated in fig. ??. Timereversal invariance implies that the S-matrix for this reaction equals that of the time reversed one $b^\star \to a^\star$, i.e.,

$$S_{ba} = S_{a^\star b^\star}. \tag{31}$$

Here $a^\star$ and $b^\star$ are the time reversed states to a and b, respectively. This means that the momenta and the spins are reversed. Using eq. ?? one finds that also the corresponding scattering amplitudes are equal

$$f_{ba} = f_{a^\star b^\star}. \tag{32}$$

The cross sections are

$$d\sigma_{ba} = |f_{ba}|^2 \frac{k_b}{k_a} d\Omega_b \tag{33}$$

and

$$d\sigma_{a^\star b^\star} = |f_{a^\star b^\star}|^2 \frac{k_a^\star}{k_b^\star} d\Omega_a^\star \tag{34}$$

Now, using (??) one finds

$$\frac{d\sigma_{ba}}{d\Omega_b} \frac{k_a}{k_b} = \frac{d\sigma_{a^\star b^\star}}{d\Omega_{a^\star}} \frac{k_{b^\star}}{k_{a^\star}} \tag{35}$$

This is the detailed balance relation. More explicitly, this give a relation between the cross section for the inelastic scattering of two particles with relative momentum $\vec{k}_a$ and spins $\vec{s}_{1a}$ and $\vec{s}_{2a}$ into a state with different particles species of relative momentum $\vec{k}_b$ and spins $\vec{s}_{1b}$ and $\vec{s}_{2b}$ and the cross section for scattering of the latter type with momentum $-\vec{k}_b$ and spins $-\vec{s}_{1b}$ and $-\vec{s}_{2b}$ into particles of the former type with relative momentum $-\vec{k}_a$ and spins $-\vec{s}_{1a}$ and $-\vec{s}_{2a}$. If one averages over the spins in the initial states and sums over those in the final states, one obtains the spin-averaged cross sections. If we also integrate over the final direction and average over the initial one, the difference between the transition $a \to b$ and $a^\star \to b^\star$ no longer exist. We define the cross section

$$\overline{\sigma_{ba}} = \frac{1}{4\pi(2s_{1a}+1)(2s_{2a}+1)} \sum_{(spins)} \int\int \frac{d\sigma_{ba}}{d\Omega_b} d\Omega_b d\Omega_a. \tag{36}$$

Now, if space is homogeneous (no magnetic fields etc.)

$$\int \frac{d\sigma_{ba}}{d\Omega_b} d\Omega_b \tag{37}$$

is independent of the angles Ω_a and the integral over $d\Omega_a$ yields just an overall factor 4π. Thus, $\overline{\sigma_{ba}}$ is just the spin-averaged total cross section

$$\overline{\sigma_{ba}} = \frac{1}{(2s_{1a}+1)(2s_{2a}+1)} \sum_{(spins)} \int \frac{d\sigma_{ba}}{d\Omega_b} d\Omega_b. \tag{38}$$

One then finds the common form of the detailed-balance relation

$$g_a\, k_a^2\, \overline{\sigma_{ba}} = g_b\, k_b^2\, \overline{\sigma_{ab}}, \tag{39}$$

where $g_a = (2s_{1a}+1)(2s_{2a}+1)$ and $g_b = (2s_{1b}+1)(2s_{2b}+1)$ are the spin-degeneracy factors for the initial and final states. A comparison with (**??**), shows that the Breit-Wigner cross section for inelastic scattering satisfies the detailed balance relation.

Now, let us consider as an example the case where below some energy E_0 only elastic scattering in channel a is possible, while above the threshold energy E_0 another channel b with heavier particles opens up. What can one say about the different cross sections for energies close to the threshold energy? Close to threshold s-wave scattering dominates, so in the following $\ell = 0$ is assumed and all higher partial waves are neglected. Below threshold $S_{bb} = 1$ since elastic scattering in the b channel is impossible. Slightly above threshold $S_{bb} = e^{2i\delta_b}$, where the phase shift δ_b is complex, since inelastic scattering $b \to a$ is possible. The parameter η in eq. (**??**) is related to the imaginary part of δ; $\eta_b = e^{-2\mathrm{Im}\delta_b}$. Near threshold, the phase shift varies linearly with the momentum $\delta_b = -k_b a$. Here a is the scattering length, which in this case is complex $a = \alpha + i\beta$. For $k_b|a| << 1$

$$S_{bb} \simeq 1 - 2ik_b\alpha + 2k_b\beta. \tag{40}$$

Since the modulus of $S_{bb} < 1$ the imaginary par $\beta < 0$. Substituting (**??**) into (**??**) and (**??**) one finds

$$\begin{aligned} \sigma_{bb} &= 4\pi|a|^2, \\ \sigma_{ab} &= \frac{4\pi}{k_b}|\beta| - 4\pi|a|^2. \end{aligned} \tag{41}$$

Hence, the inelastic cross section for the heavier system scattering into the lighter one, i.e., $b \to a$ diverges at threshold $\sim k_b^{-1}$, while the elastic cross section for $b \to b$ assumes a value which is independent of the momentum.

Now, we can obtain the cross section for the reverse reaction $a \to b$ by using the detailed balance relation

$$\sigma_{ba} \sim \frac{k_b^2}{k_a^2}\sigma_{ab} \sim \frac{k_b}{k_a^2}|\beta|. \tag{42}$$

Since k_a is not small at the threshold, we can take it to be a constant. Thus, the cross section for the production of the heavier particles is proportional to the imaginary part of the $b \to b$ scattering length and vanishes linearly with the momentum k_b.

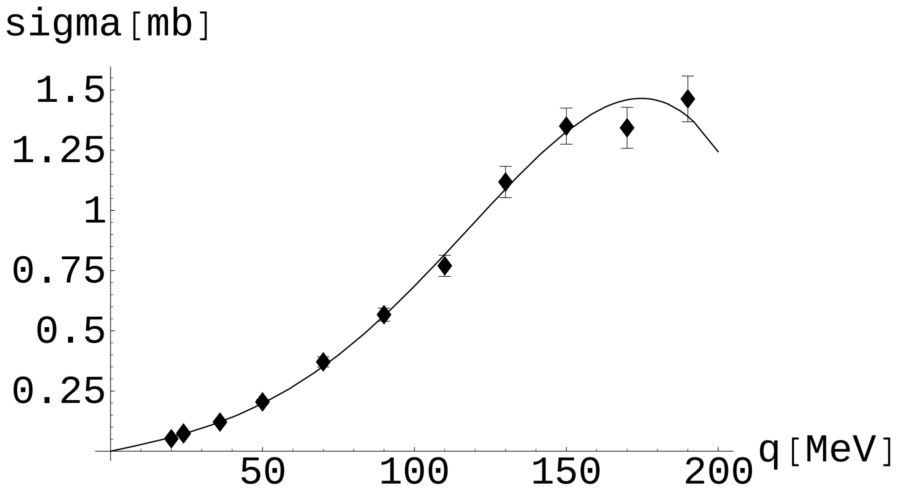

Figure 6: The cross section for $\pi^- p \to \omega n$. The line shows a fit to the data.

As a more concrete example we consider the reaction $\pi^- p \to \omega n$. The empirical cross section for this reaction is shown in fig. **??**. The detailed balance relation tells us that the cross section for the inverse reaction $\omega n \to \pi^- p$ is

$$\sigma_{\omega n \to \pi^- p} = \frac{2}{6}\frac{k_\pi^2}{k_\omega^2}\sigma_{\pi^- p \to \omega n}, \tag{43}$$

where the factor 2/6 is the ratio of the spin degeneracies. In fig. **??** we show the resulting cross section. The singularity at threshold shows that there is an s-wave component present in the cross section.

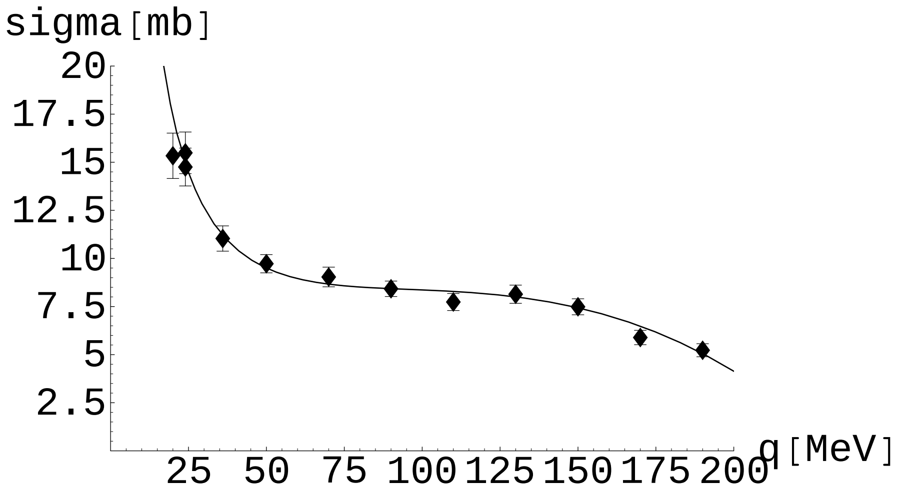

Figure 7: The cross section for $\omega n \to \pi^- p$ obtained using detailed balance. The line shows the inverse cross section corresponding to the fit of the previous figure.

As indicated above, one can also extract information on the imaginary part of the elastic scattering amplitude for the heavy system, in this case for $\omega n \to \omega n$. Using detailed balance and unitarity one finds

$$\sigma_{\pi^- p \to \omega n} = 12\pi \frac{k_\omega}{k_\pi^2}\mathrm{Im} f^{(\pi^- p)}_{\omega n \to \omega n} \tag{44}$$

where $\mathrm{Im} f^{(\pi^- p)}_{\omega n \to \omega n}$ denotes the imaginary part of the $\omega n \to \omega n$ scattering amplitude that is due to the $\pi^- p$ channel. In fig. **??** we show the scattering amplitude obtained from the data of fig. **??** together with the corresponding fit.

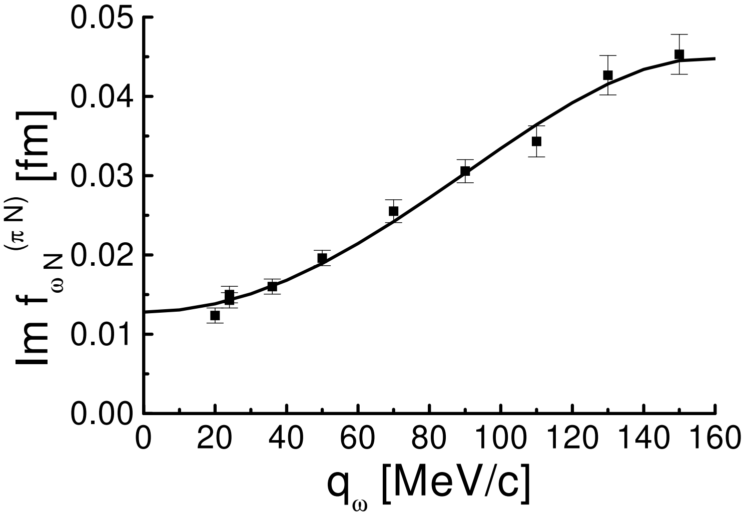

Figure 8: The imaginary part of the $\omega n \to \omega n$ scattering amplitude due to the $\pi^- p$ channel. The line shows the fit to the data, and corresponds to the lines in the two previous figures.

As will be discussed later, knowledge of the hadron-nucleon scattering amplitude is important for obtaining the properties of the hadron in nuclear matter. The imaginary part of the scattering amplitude gives rise to an increased width of the hadron in the nuclear medium.

We note that $\mathrm{Im} f^{(\pi^- p)}_{\omega n \to \omega n}$ is unexpectedly small. The corresponding scattering amplitude for the η meson is more than an order of magnitude larger, $\mathrm{Im} f^{(\pi^- p)}_{\eta n \to \eta n} \simeq 0.25$ fm. It is presently not understood why $\mathrm{Im} f^{(\pi^- p)}_{\omega n \to \omega n}$ is so small.