Entropy-limited hydrodynamics: a novel approach to relativistic hydrodynamics

Neutron star mergers: From gravitational waves to nucleosynthesis

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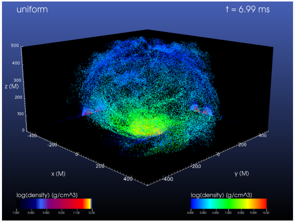
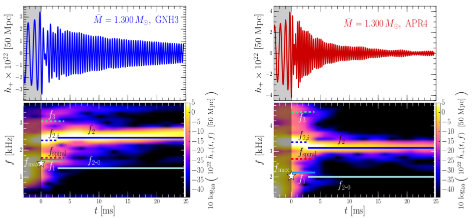
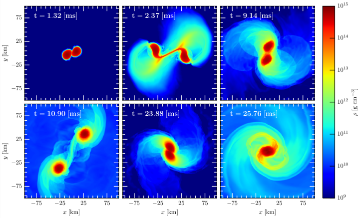
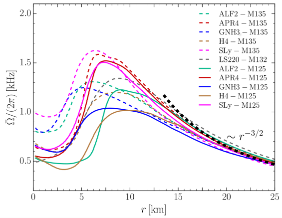
Numerical simulations









Powerful tools... but they don't come for free.

Numerical methods

Besides physical research, there is research on better numerical methods.
An ideal numerical scheme should be:

- Accurate
- Fast
- Parallelizable
- Scalable

... and hopefully easy to implement.

Euler equations

The relativistic Euler equations:

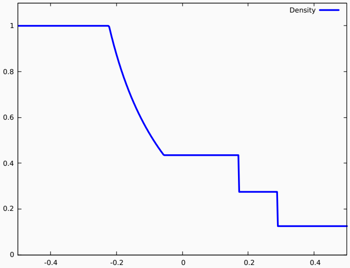
$$\nabla_\mu(\rho u^\mu) = 0$$
$$u^\mu \nabla_\mu u_\nu + \frac{1}{\rho h} P^\mu{}_\nu \nabla_\mu p = 0$$
$$u^\mu \nabla_\mu e + \rho h \nabla_\mu u^\mu = 0\,,$$

where $h = (e + p)/\rho$ and $P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$, closed by an EoS $p = p(\rho, e)$.

Cast them into a flux-conservative formulation (the "Valencia formulation"):

$$\partial_t \boldsymbol{U} + \partial_i \boldsymbol{F}^i = \boldsymbol{S}\,.$$

Several issues arise when solving these equations, stemming from their non-linearity, most importantly the generation of shocks.



High-resolution shock-capturing techniques

The standard way: use of HRSC techniques to preserve stability without sacrificing accuracy. HRSC methods feature:

- second order of accuracy (or higher)
- sharp resolution of discontinuities
- no oscillations

e.g. PPM, ENO, WENO, MP5...

However successful, these methods can potentially suffer from a few shortcomings...

ELH: entropy limited hydrodynamics

Flux-limiter approach

$$f_{i+1/2} = \theta f^{HO}_{i+1/2} + (1-\theta) f^{LF}_{i+1/2}$$

$f^{HO}_{i+1/2}$ is a high order, but unfiltered approximation of the flux

$f^{LF}_{i+1/2}$ is the Lax-Friedrichs flux, which is only first order but stable

We want:

$$\theta \simeq \begin{cases} 1 & \text{when the flow is smooth} \\ 0 & \text{in troubled regions} \end{cases}$$

With $\nu \in [0,1]$ a troubled cell indicator, we define therefore:

$$\theta = \min[\tilde{\theta}, 1-\nu]$$

where $\tilde{\theta}$ guarantees the positivity of the density.

Entropy viscosity

Consider the physical specific entropy s, which for the ideal-gas EoS $p = (\Gamma - 1)\rho\epsilon$ equals $s = \log\left(\frac{\epsilon}{\rho^{\Gamma-1}}\right)$.

The second principle of thermodynamics can be written:

$$R = \nabla_\mu(s\rho u^\mu) \geq 0$$

Therefore one expects the entropy production rate (or entropy residual) R to be a Dirac delta centered at the location of shocks, $R = \delta(\mathbf{x} - \mathbf{x}^s)$.

This suggests to define the viscosity as proportional to the entropy residual:

Entropy viscosity

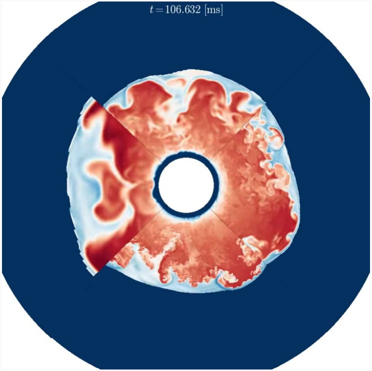
$$\nu = \min[c_e \Delta |R|, c_{max}]$$

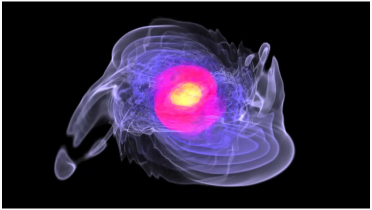
Note the presence of two arbitrary constants, c_e and c_{max}.

Implementation: the `WhiskyTHC` code

We implemented the ELH scheme in the `WhiskyTHC` code (Radice et al.), which features:

- finite differences flux reconstruction...
- applied to components or characteristics variables...
- with upwinding;
- a positivity preserving limiter.





Implementation: entropy viscosity

In the 3+1 decomposition of GR, one can write:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$\Downarrow$$

$$R = \frac{\rho W}{\alpha}\left[\partial_t s + (\alpha v^i - \beta^i)\partial_i s\right]$$

Computed by 2nd order one-sided finite-differences on the past timelevels:

$$\partial_t s = \frac{1}{2\Delta t}(3s^n - 4s^{n-1} + s^{n-2})$$

Computed by nth+1 order central finite-differences, *e.g.* :

$$\partial_x s = \frac{1}{12}s_{i-2} - \frac{2}{3}s_{i-1} + \frac{2}{3}s_{i+1} - \frac{1}{12}s_{i+2}$$

Tests: smooth wave

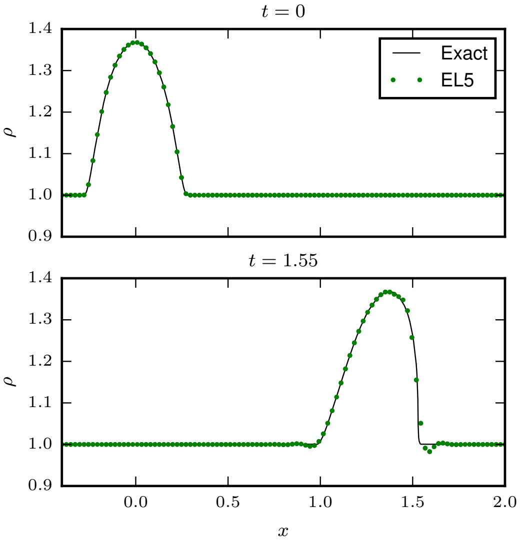


Figure 1: Density profiles at initial time and $t = 1.55$

A non-linear, smooth hydrodynamical wave, propagating to the right and tilting in the direction of its motion, until a caustic is produced.

Tests: smooth wave

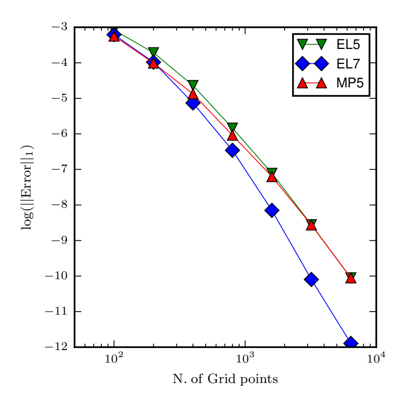


Figure 2: L_1-norm of the error at time $t = 0.8$

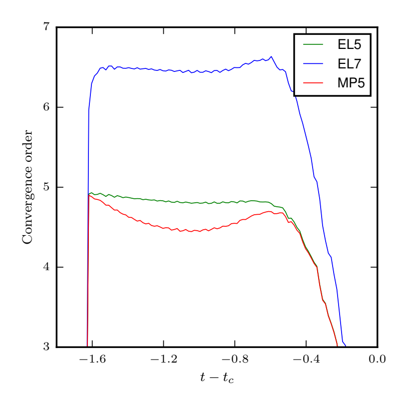


Figure 3: Convergence order as function of time to caustic

Tests: shock tube

Sod's relativistic shock tube test, with initial data:

$$(\rho_l, v_l, p_l) = (1, 0, 1), \quad (\rho_r, v_r, p_r) = (0.125, 0, 0.1).$$

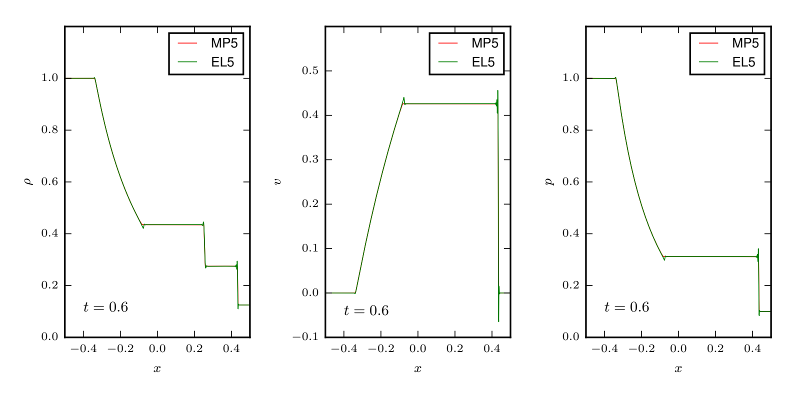


Figure 4: Density, velocity and pressure profiles at time $t = 0.8$

Tests: Cowling TOV

TOV star in the Cowling approximation (*i.e.* the spacetime is fixed, only the matter is evolved).

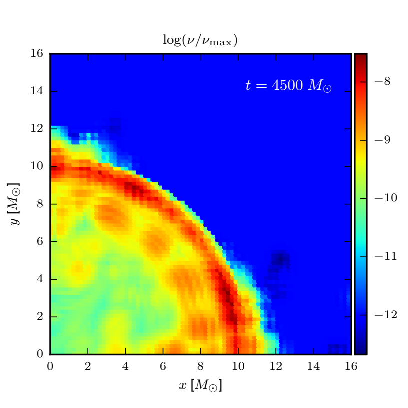


Figure 5: Viscosity distribution on *xy* plane at $t = 4500\ M_\odot$

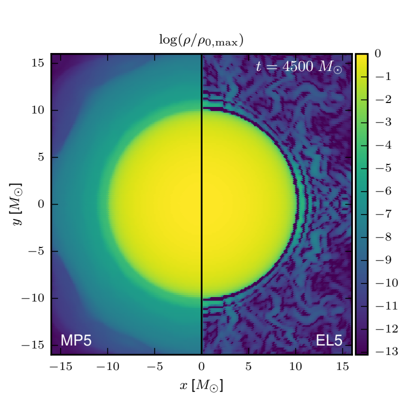


Figure 6: Density distribution on *xy* plane at $t = 4500\ M_\odot$

Tests: Cowling TOV

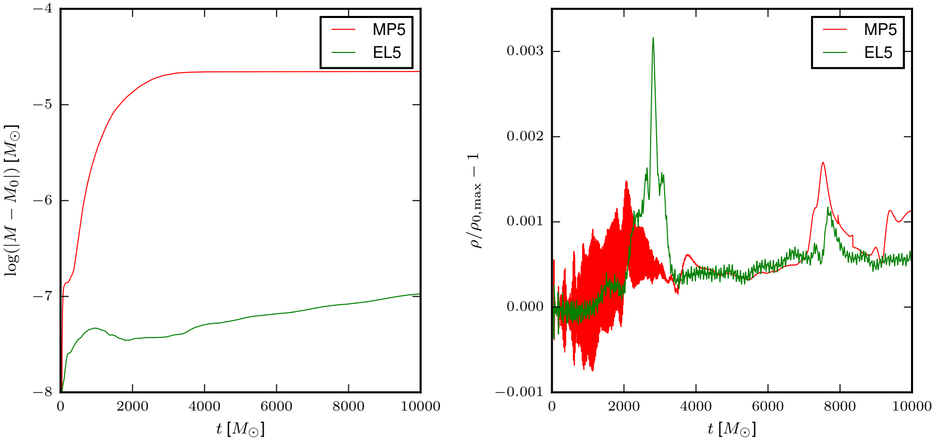


Figure 7: Baryonic mass conservation violation and evolution of the central density as a function of time

Tests: dynamical TOV

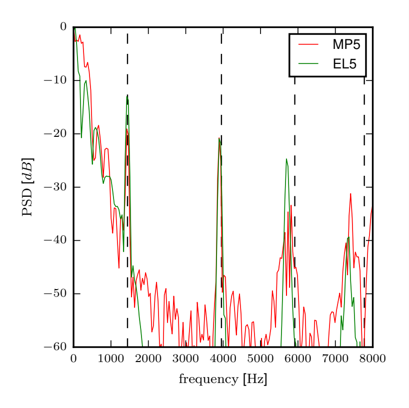


Figure 8: Continuous lines: Power spectral density of the central density evolution. Dashed lines: physical oscillation eigenfrequencies

Tests: migration test

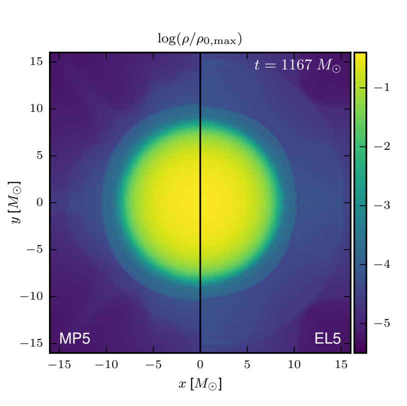


Figure 9: Density distribution on xy plane at $t = 1167\ M_\odot$ (*i.e.* during the 7th contraction cycle)

Tests: migration test

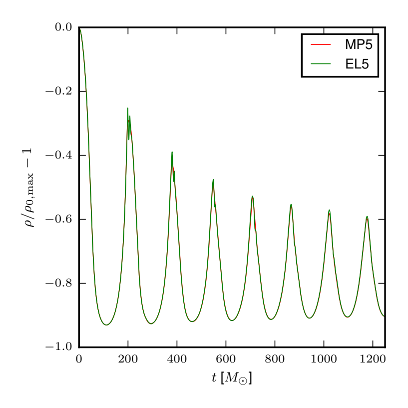


Figure 10: Central density evolution

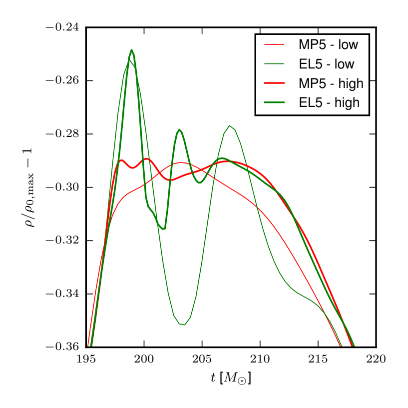


Figure 11: Central density evolution, zoom on the first maximum. Thick lines: low resolution; Thin lines: high resolution

Tests: rotating star

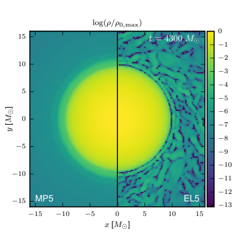


Figure 12: Density distribution on xy plane at $t = 4300\ M_\odot$

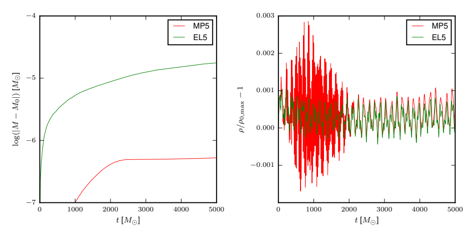


Figure 13: Baryonic mass conservation violation and evolution of the central density as a function of time

Tests: collapse to a black hole

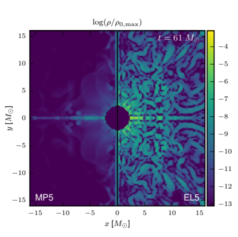


Figure 14: Density distribution on xy plane at $t = 61\ M_\odot$ (*i.e.* just after collapse)

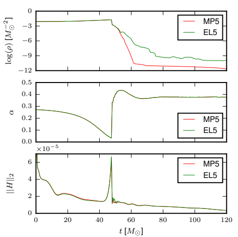


Figure 15: From the top: central density, minimum lapse and Hamiltonian constraint as function of time

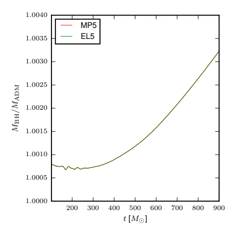


Figure 16: Black hole mass evolution

Conclusions

The entropy limited hydrodynamics scheme is an interesting, robust alternative to the common HRSC schemes. It addresses the issues of

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- Speed
- Ease of implementation and extendability

with no tuning of the free parameters.

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Future prospects

- application to binary neutron stars with nuclear equation of state
- exploit the simplicity of the scheme to efficiently run on MICs/GPUs
- coupling to discontinuous Galerkin scheme
- use of truly multidimensional methods

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Thank you!