Analytical solutions and attractors of higher-order viscous hydrodynamics for Bjorken flow

Amaresh Jaiswal

NISER Bhubaneswar, India

From QCD matter to hadrons
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Collaborators: Chandrodoy Chattopadhyay, Sunil Jaiswal, Subrata Pal
(To appear on arXiv soon)

Relativistic ideal fluids

- The energy-momentum tensor of an ideal fluid can be written in terms of the available tensor degrees of freedom:

$$T^{\mu\nu}_{(0)} = c_1 u^\mu u^\nu + c_2 g^{\mu\nu}$$

- In local rest frame, i.e., $u^\mu = (1,\, 0,\, 0,\, 0)$,

$$T^{\mu\nu}_{(0)} = \mathrm{diag}(\epsilon,\, P,\, P,\, P) \Rightarrow c_1 = \epsilon + P,\, c_2 = -P.$$

- Energy-momentum tensor for the ideal fluid, $T^{\mu\nu}_{(0)}$ is

$$T^{\mu\nu}_{(0)} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu}\ ; \quad \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

- $\Delta^{\mu\nu} u_\mu = \Delta^{\mu\nu} u_\nu = 0$ and $\Delta^{\mu\nu}\Delta^{\alpha}_{\nu} = \Delta^{\mu\alpha}$, hence serves as a projection operator on the space orthogonal to the fluid velocity u^μ.
- Similarly, $N^{\mu}_{(0)} = n u^\mu$.
- Fluids are in general dissipative; dissipation needs to be included.

Ideal and dissipative hydrodynamics

- Dissipation can be included in the energy momentum tensor and conserved current as

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} - \Pi\Delta^{\mu\nu} + \pi^{\mu\nu} \;; \quad N^{\mu} = N^{\mu}_{(0)} + n^{\mu}$$

Ideal	Dissipative
$T^{\mu\nu} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu}$	$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$
$N^\mu = nu^\mu$	$N^\mu = nu^\mu + n^\mu$
Unknowns: $\underbrace{\epsilon,\; P,\; n,\; u^\mu}_{1+1+\;1+\;3} = 6$	$\underbrace{\epsilon,\; P,\; n,\; u^\mu,\; \Pi,\; \pi^{\mu\nu},\; n^\mu}_{1+1+\;1\;+\;3\;+\;1\;+\;5\;+\;3} = 15$
Equations: $\underbrace{\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu N^\mu = 0, \quad EOS}_{4\quad+\quad1\quad+\quad1} \quad = 6$	
Closed set of equations	9 more equations required

- Landau frame chosen: $T^{\mu\nu}u_\nu = \epsilon u^\mu$.

Dissipative equations [L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 1987]

- Second law in covariant form: $\partial_\mu S^\mu \geq 0$, where

$$S^\mu = s\, u^\mu \quad ; \quad s = \frac{\epsilon + P - \mu n}{T}.$$

- Demanding second-law from this entropy current,

$$\Pi = -\zeta\theta, \quad n^\alpha = \lambda T \nabla^\alpha(\mu/T), \quad \pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle},$$

  where,

$$\theta \equiv \partial_\mu u^\mu, \quad \nabla^\alpha \equiv \Delta^{\alpha\beta}\partial_\beta, \quad \nabla^{\langle\mu}u^{\nu\rangle} \equiv (\nabla^\mu u^\nu + \nabla^\nu u^\mu)/2 - \Delta^{\mu\nu}\theta/3.$$

- The transport coefficients $\eta, \zeta, \lambda \geq 0$.
- In the non-relativistic limit, above equations reduces to the Navier-Stokes equations.
- Beautiful and simple but flawed! Exhibits acausal behavior.

Maxwell-Cattaneo law

- One possible way out is the “Maxwell-Cattaneo” law,

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle}.$$

- A relaxation-type equation with relaxation time τ_π: restores causality.
- Brings rich structure to the evolution.
- Consider Bjorken flow with $\pi \equiv -\tau^2\pi^{\eta\eta}$, $\bar{\pi} \equiv \pi/(\epsilon + P)$. Energy conservation and shear evolution:

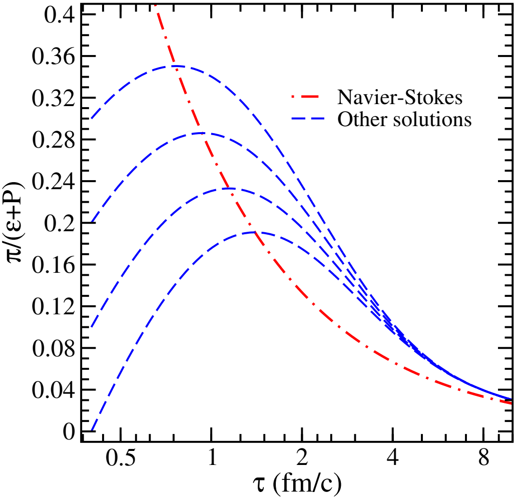
$$\frac{1}{\epsilon\tau^{4/3}}\frac{d(\epsilon\tau^{4/3})}{d\tau} = \frac{4}{3}\frac{\bar{\pi}}{\tau}, \qquad \frac{d\bar{\pi}}{d\tau} + \frac{\bar{\pi}}{\tau_\pi} = \frac{4}{15\tau}.$$

- Can be solved analytically for constant τ_π to give

$$\bar{\pi} = e^{-\tau/\tau_\pi}\int \frac{4e^{\tau/\tau_\pi}}{15\tau}d\tau + \alpha\, e^{-\tau/\tau_\pi},$$

  where α is constant of integration.
- Existence of attractor behaviour: $\dfrac{d\bar{\pi}}{d\alpha} = e^{-\tau/\tau_\pi}$.

Attractor behaviour for Maxwell-Cattaneo equation



Proper time evolution of $\bar{\pi}$ for Maxwell-Cattaneo equation.

Attractors in hydrodynamics and microscopic theories

- Attractors and its implications were first explored in [Heller and Spalinski, Phys. Rev. Lett. 115 (7), 072501 (2015); 1503.07514].
- Attractors can be found in a variety of settings including AdS/CFT simulations of non-equilibrium dynamics, simple kinetic models, and QCD- based kinetic approaches [Romatschke Phys. Rev. Lett. 120 (2018) 012301; Spalinski, Phys. Lett. B776 (2018) 468; Denicol and Noronha, 1711.01657; Strickland, JHEP2018, 128; 1809.01200].
- More about attractors and its implications on applicability of hydrodynmaics: see Mike's talk today.

Second-order hydrodynamics from kinetic theory

- Variants of Maxwell-Cattaneo equation can be derived from kinetic theory for a system close to equilibrium, $f = f_0 + \delta f$.

$$T^{\mu\nu}(x) = \int dp\ p^\mu p^\nu\, f(x,p), \qquad \pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int dp\, p^\alpha p^\beta\, \delta f.$$

- Boltzmann equation in the relxn. time approx. is solved iteratively:

$$p^\mu \partial_\mu f = -\frac{u\cdot p}{\tau_R}(f - f_0) \;\Rightarrow\; f = f_0 - (\tau_R/u\cdot p)\, p^\mu \partial_\mu f$$

- Expand f about its equilibrium value: $f = f_0 + \delta f^{(1)} + \delta f^{(2)} + \cdots$,

$$\delta f^{(1)} = -\frac{\tau_R}{u\cdot p}\, p^\mu \partial_\mu f_0\ , \quad \delta f^{(2)} = \frac{\tau_R}{u\cdot p} p^\mu p^\nu \partial_\mu\left(\frac{\tau_R}{u\cdot p}\partial_\nu f_0\right) .$$

- Substituting $\delta f = \delta f^{(1)} + \delta f^{(2)}$ [**AJ**, PRC 87, 051901(R) (2013)],

$$\dot\pi^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi\sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma}, \qquad \beta_\pi = \frac{4P}{5}.$$

[G. S. Denicol, T. Koide and D. H. Rischke, PRL 105, 162501 (2010)]

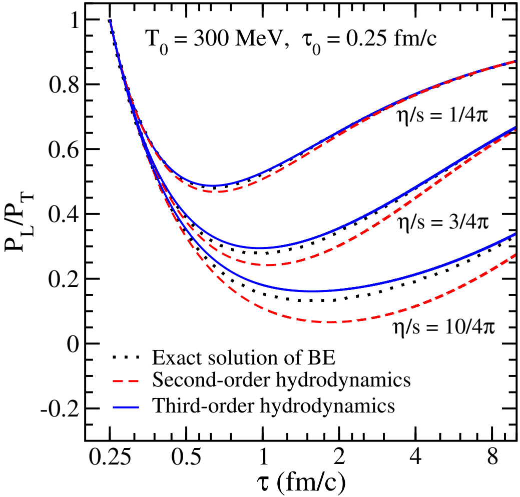
Higher-order hydrodynamics

- Third-order equation for shear stress tensor [**AJ**, PRC 88, 021903(R) (2013)]:

$$
\begin{aligned}
\dot{\pi}^{\langle\mu\nu\rangle} = & -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \frac{4}{3}\pi^{\mu\nu}\theta - \frac{10}{63}\pi^{\mu\nu}\theta^2 \\
& + \tau_\pi\Big[\frac{50}{7}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\sigma_{\rho\gamma} - \frac{76}{245}\pi^{\mu\nu}\sigma^{\rho\gamma}\sigma_{\rho\gamma} - \frac{44}{49}\pi^{\rho\langle\mu}\sigma^{\nu\rangle\gamma}\sigma_{\rho\gamma} \\
& - \frac{2}{7}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\omega_{\rho\gamma} - \frac{2}{7}\omega^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma} + \frac{26}{21}\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma}\theta - \frac{2}{3}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma}\theta\Big] \\
& - \frac{24}{35}\nabla^{\langle\mu}\left(\pi^{\nu\rangle\gamma}\dot{u}_\gamma\tau_\pi\right) + \frac{6}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^\gamma\pi^{\langle\mu\nu\rangle}\right) + \frac{4}{35}\nabla^{\langle\mu}\left(\tau_\pi\nabla_\gamma\pi^{\nu\rangle\gamma}\right) \\
& - \frac{2}{7}\nabla_\gamma\left(\tau_\pi\nabla^{\langle\mu}\pi^{\nu\rangle\gamma}\right) - \frac{1}{7}\nabla_\gamma\left(\tau_\pi\nabla^\gamma\pi^{\langle\mu\nu\rangle}\right) + \frac{12}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^{\langle\mu}\pi^{\nu\rangle\gamma}\right).
\end{aligned}
$$

- Improved accuracy compared to second-order equations.
- Neccessary for incorporation of colored noise in fluctuating hydro evolution [J. Kapusta and C. Young, Phys. Rev. C 90, 044902 (2014)].

Third-order theory: A better description of microscopics



Proper time evolution of pressure anisotropy: $P_L/P_T = (P - \pi)/(P + \pi/2)$.

Bjorken flow

- For boost-invariant longitudinal expansion, $v^z = \frac{z}{t}$, $v^x = v^y = 0$.
- Milne coordinate system: proper time $\tau = \sqrt{t^2 - z^2}$ and space-time rapidity $\eta_s = \tanh^{-1}(z/t)$.

$$\frac{d\epsilon}{d\tau} = -\frac{1}{\tau}\left(\frac{4}{3}\epsilon - \pi\right), \qquad \frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} + \frac{1}{\tau}\left[\frac{4}{3}\beta_\pi - \left(\lambda + \frac{4}{3}\right)\pi - \chi\frac{\pi^2}{\beta_\pi}\right],$$

- The coefficients are: $\beta_\pi = \dfrac{4P}{5}, \quad \lambda = \dfrac{10}{21}, \quad \chi = \dfrac{72}{245}$.
- In terms of normalized shear stress $\bar{\pi} \equiv \pi/(\epsilon + P)$,

$$\frac{1}{\epsilon\tau^{4/3}}\frac{d(\epsilon\tau^{4/3})}{d\tau} = \frac{4}{3}\frac{\bar{\pi}}{\tau}, \qquad \frac{d\bar{\pi}}{d\tau} = -\frac{\bar{\pi}}{\tau_\pi} + \frac{1}{\tau}\left(a - \lambda\bar{\pi} - \gamma\bar{\pi}^2\right)$$

where $a = 4/15$, $\gamma = 5\chi + (4/3) = 412/147$ and $\tau_\pi \propto 1/T$.

Case 1: Analytical solutions for constant relaxation time

- The equation to be solved: $\dfrac{d\bar{\pi}}{d\tau} = -\dfrac{\bar{\pi}}{\tau_\pi} + \dfrac{1}{\tau}\left(a - \lambda\bar{\pi} - \gamma\bar{\pi}^2\right).$
- Assume a constant relaxation time [Denicol and Noronha arXiv:1711.01657].
- Make variable transformation: $\dfrac{1}{y}\dfrac{dy}{d\tau} = \gamma\dfrac{\bar{\pi}}{\tau} \quad \Rightarrow \quad \bar{\pi} = \dfrac{\tau}{\gamma y}\dfrac{dy}{d\tau}.$
- To obtain a linear ODE: $\dfrac{d^2y}{d\tau^2} + \left(\dfrac{1+\lambda}{\tau} + \dfrac{1}{\tau_\pi}\right)\dfrac{dy}{d\tau} - \dfrac{a\gamma}{\tau^2}y = 0.$
- Solution in terms of Whittaker functions $M_{k,m}(\tau)$ and $W_{k,m}(\tau)$:

$$\bar{\pi}(\tau) = \frac{(2k+2m+1)M_{k+1,m}(\tau/\tau_\pi) - 2\alpha W_{k+1,m}(\tau/\tau_\pi)}{2\gamma\ \left[M_{k,m}(\tau/\tau_\pi) + \alpha W_{k,m}(\tau/\tau_\pi)\right]},$$

where $k = -\frac{\lambda+1}{2}$, $m = \frac{1}{2}\sqrt{4a\gamma + \lambda^2}$ and α is constant of integration.

Emergent attractor behavior

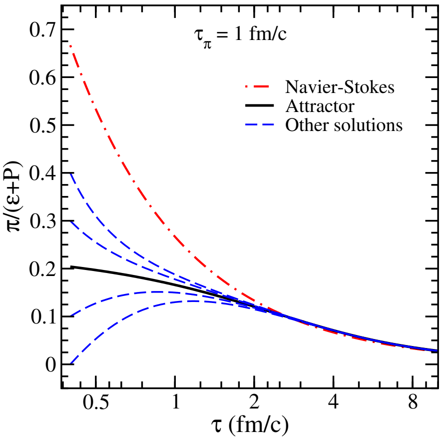
- To prove the existence of attractor, we look at late time behavior of the analytical solution.
- We find that for large τ

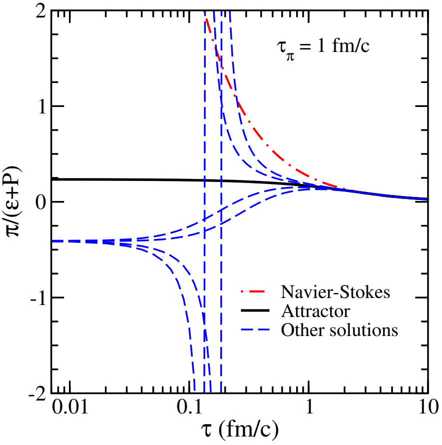
$$\frac{\partial \bar{\pi}}{\partial \alpha} \propto \frac{e^{-\tau}}{\hat{\tau}}.$$

- The information of initial state is damped exponentially: suggestive of attractor behaviour.
- Next we find the attractor solution.
- We propose that the attractor solution corresponds to the value of α for which

$$\lim_{\tau \to 0} \frac{\partial \bar{\pi}}{\partial \alpha} = \infty.$$

Results for constant relaxation time





Attractor behaviour and the attractor solution.

Case 2: Temperature from ideal hydrodynamic evolution

- The equation to be solved: $\dfrac{d\bar{\pi}}{d\tau} = -\dfrac{\bar{\pi}}{\tau_\pi} + \dfrac{1}{\tau}\left(a - \lambda\bar{\pi} - \gamma\bar{\pi}^2\right)$.
- From kinetic theory, $\tau_\pi = 5\left(\dfrac{\eta}{s}\right)\dfrac{1}{T}$.
- To be absolutely consistent, one should consider the temperature evolution from: $\dfrac{1}{\epsilon\tau^{4/3}}\dfrac{d(\epsilon\tau^{4/3})}{d\tau} = \dfrac{4}{3}\dfrac{\bar{\pi}}{\tau}$
- We approximate the temperature evolution from ideal hydro evolution.

$$\tau T^3 = \text{const.} \;\Rightarrow\; T = T_0\left(\frac{\tau_0}{\tau}\right)^{1/3} \Rightarrow\; \tau_\pi = \frac{\tau^{1/3}}{c},\; \text{where } c = \frac{T_0\tau_0^{1/3}}{5(\eta/s)}.$$

- We make successive change of variables $x^3 = \tau$ and $\dfrac{1}{y}\dfrac{dy}{dx} = 3\gamma\dfrac{\bar{\pi}}{x}$ to obtain the linear ODE

$$\frac{d^2y}{dx^2} + \left(\frac{3\lambda+1}{x} + 3cx\right)\frac{dy}{dx} - \frac{9a\gamma}{x^2}y = 0.$$

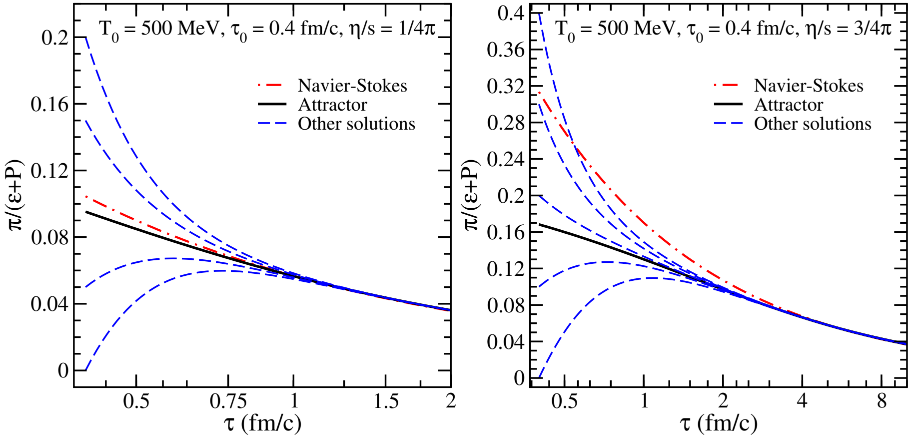
Case 2: Analytical solution

- The solution is obtained in terms of the Kummer (confluent) Hypergeometric functions:

$$
\begin{aligned}
\bar{\pi}(\tau) = & -\frac{2m+\lambda}{2\gamma} + \Bigg[8m\tau^{2m}\,{}_1F_1\left(\frac{3}{4}(2m-\lambda); 1+3m; -\frac{3c}{2}\tau^{2/3}\right) \\
& - \frac{3c(2m-\lambda)}{1+3m}\tau^{2/3+2m}\,{}_1F_1\left(1+\frac{3}{4}(2m-\lambda); 2+3m; -\frac{3c}{2}\tau^{2/3}\right) \\
& + \alpha 3c(2m+\lambda)\tau^{2/3}\,{}_1F_1\left(1-\frac{3}{4}(2m+\lambda); 2-3m; -\frac{3c}{2}\tau^{2/3}\right)\Bigg] \Bigg/ \\
& \Bigg[4\gamma\tau^{2m}\,{}_1F_1\left(\frac{3}{4}(2m-\lambda); 1+3m; -\frac{3c}{2}\tau^{2/3}\right) \\
& + 4\gamma\alpha\;{}_1F_1\left(-\frac{3}{4}(2m+\lambda); 1-3m; -\frac{3c}{2}\tau^{2/3}\right)\Bigg].
\end{aligned}
$$

- Here α is the constant of integration.

Results for Case 2



Attractor behaviour for different η/s.

Case 3: Temperature from vsicous hydro evolution

- The equation to be solved: $\frac{d\bar{\pi}}{d\tau} = -\frac{\bar{\pi}}{\tau_\pi} + \frac{1}{\tau}\left(a - \lambda\bar{\pi} - \gamma\bar{\pi}^2\right).$
- From kinetic theory, $\tau_\pi = 5\left(\frac{\eta}{s}\right)\frac{1}{T}$.
- We approximate the temperature evolution from Navier-Stokes viscous hydro evolution.

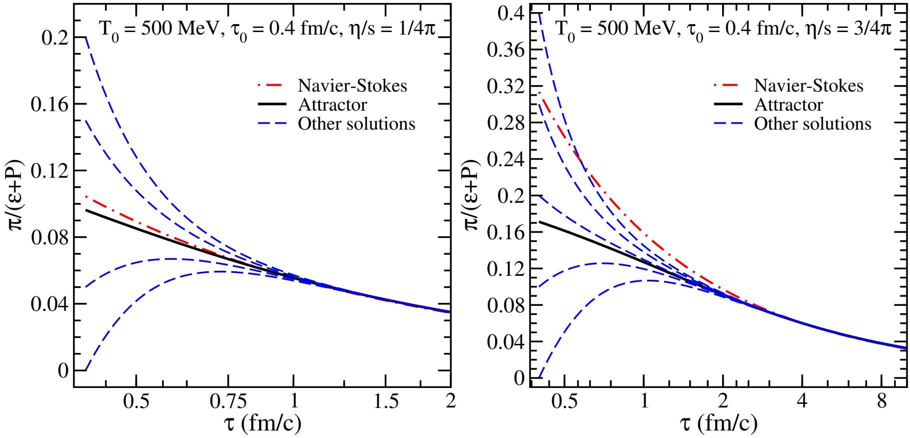
$$T = T_0\left(\frac{\tau_0}{\tau}\right)^{1/3}\left[\left\{1 + \frac{2(\eta/s)}{3\tau_0 T_0}\right\} - \frac{2(\eta/s)}{3\tau_0 T_0}\left(\frac{\tau_0}{\tau}\right)^{2/3}\right].$$

- The relaxation time can then be obtained as

$$\tau_\pi = \frac{\tau}{c_1\tau^{2/3} - c_2} \quad \text{where} \quad c_1 = \frac{T_0\tau_0^{1/3}}{5(\eta/s)} + \frac{2}{15\tau_0^{2/3}}, \quad c_2 = \frac{2}{15}$$

- We again make successive change of variables $x^3 = \tau$ and $\frac{1}{y}\frac{dy}{dx} = 3\gamma\frac{\bar{\pi}}{x}$ to obtain the linear ODE which is formally similar to that in the previous case and therefore analytically solvable.

Results for Case 3



Attractor behaviour for different η/s.

Case 4: Constant Knudsen number

- The equation to be solved: $\frac{d\bar{\pi}}{d\tau} = -\frac{\bar{\pi}}{\tau_\pi} + \frac{1}{\tau}\left(a - \lambda\bar{\pi} - \gamma\bar{\pi}^2\right).$
- We consider constant Knudsen number, $\frac{\tau_\pi}{\tau} = f$.
- The equations reduces to

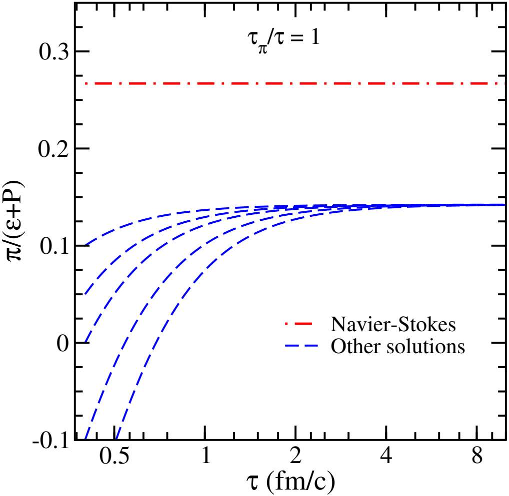
$$\frac{d\bar{\pi}}{d\tau} = \frac{1}{\tau}\left[a - \left(\lambda + \frac{1}{f}\right)\bar{\pi} - \gamma\bar{\pi}^2\right]$$

- The above equation is variable separable for which the solution is

$$\bar{\pi}(\tau) = \frac{-1 - f\lambda + z\tanh\left(\frac{z(\alpha f + \log(\tau))}{2f}\right)}{2\gamma f}$$

where $z = \sqrt{4a\gamma f^2 + (1 + f\lambda)^2}$ and α is the constant of integral.

Solutions for constant Knudsen number case



The solutions do not converge to Navier-Stokes solution.

Summary and outlook

- Analytical solutions of third-order ‘hydrodynamics’ for Bjorken expansion for several cases.
- Criteria for existence of attractor behaviour.
- Citeria for identifying the attractor solution.
- One can further look for convergence/divergence of gradient expansion and slow roll approximation in these cases.