# Kinetic theory of massive spin-1/2 particles from the Wigner-function formalism

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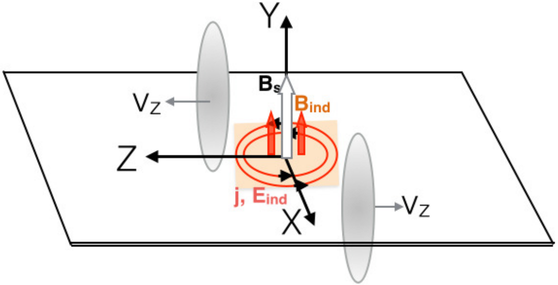
# Introduction I

- Early stage of non-central heavy-ion collisions: large orbital angular momenta and strong electromagnetic fields.
- Electromagnetic field strengths decrease fast.




(Fig. from V. Roy, S. Pu, L. Rezzolla, and D.H. Rischke, PRC 96 (2017) 054909)

# Introduction II

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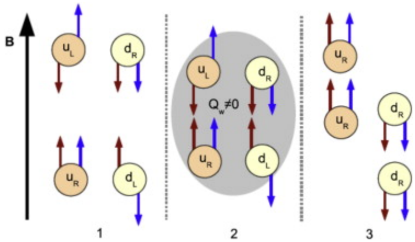
- Chiral vortical effect (CVE): charge currents induced by vorticity.
- Chiral magnetic effect (CME): charge currents induced by magnetic fields.




Dmitri E. Kharzeev, Larry D. McLerran, and Harmen J. Warringa, NPA 803 (2008) 227-253

- Has been studied in massless case.
  J-Y. Chen, D. T. Son, and M. Stephanov, PRL 115 (2015) 021601;
  Y. Hidaka, S. Pu, D-L. Yang, PRD 95 (2017) 091901;
  A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, arXiv:1801.03640 [hep-th]

# Motivation

- What we want: kinetic theory and fluid dynamics for massive spin-1/2 particles in inhomogeneous electromagnetic fields.
- For massive spin-0 particles, second-order dissipative magnetohydrodynamics has already been studied.
  G. S. Denicol, X-G Huang, E. Molnar, G. M. Monteiro, H. Niemi, J. Noronha, D. H. Rischke, and Q. Wang, PRD 98 (2018) 076009
- Starting point: quantum field theory, Dirac equation.
- Strategy: use Wigner functions to derive kinetic theory.
  $\rightarrow$ Semi-classical expansion.
  $\rightarrow$ Comparison to massless case.
- Goal: determine fluid-dynamical equations of motion from resulting Boltzmann equation.

## Conventions and Definitions

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- Natural units, $c = k_B = 1$, but keep $\hbar$ explicitly.
- To simplify notation: only write positive-energy parts of solutions.
- Polarization direction $n^\mu$: space-like unit vector parallel to axial-vector current.
- Spin quantization direction: unit vector, purely spatial in particle rest frame.
  "spin up", $s = +$: projection of spin onto quantization direction positive.
  "spin down", $s = -$: projection of spin onto quantization direction negative.
  Here: chosen to be identical to polarization direction.

$$\bar{u}_s \gamma^\mu \gamma^5 u_s = 2ms\, n^\mu$$

## Spin tensor vs. dipole-moment tensor

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- Dipole-moment tensor:

$$s\Sigma^{\mu\nu} = \frac{1}{2m}\bar{u}_s \frac{i}{2}[\gamma^\mu, \gamma^\nu] u_s = -s\frac{1}{m}\epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta$$

  called "spin tensor" in
  U. Heinz, PLB 144 (1984) 228,
  J-Y. Chen, D. T. Son, and M. Stephanov, PRL 115 (2015), 021601
  Y. Hidaka, S. Pu, D-L. Yang, PRD 95 (2017) 091901
  S.R. De Groot, Relativistic Kinetic Theory. Principles and Applications (1980)

- Spin tensor:
  rank-3 tensor $S^{\lambda,\mu\nu}$ such that total angular momentum

$$J^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\mu T^{\lambda\mu} + \hbar S^{\lambda,\mu\nu}$$

## Wigner functions

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- Quantum analogue of classical distribution function.
- Contains information about quantum state of system.
- Off-equilibrium: two-point function depends not only on relative coordinate $y$, but also on central coordinate $x$.
- Wigner transformation of two-point function:
  H.-Th. Elze, M. Gyulassy, and D. Vasak, Ann. Phys. 173 (1987) 462

$$W(x,p) = \int \frac{d^4\,y}{(2\pi)^4} e^{-\frac{i}{\hbar}p\cdot y} \langle : \bar{\Psi}(x+\frac{y}{2})\Psi(x-\frac{y}{2}) : \rangle \,,$$

- Integral over $y$: uncertainty principle.

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$$W(x,p) = \int \frac{d^4 y}{(2\pi)^4} e^{-\frac{i}{\hbar} p\cdot y} \langle : \bar{\Psi}(x+\frac{y}{2}) U(x+\frac{y}{2}, x) U(x, x-\frac{y}{2}) \Psi(x-\frac{y}{2}) : \rangle ,$$

with gauge link

$$U(b,a) \equiv P \exp\left( -\frac{i}{\hbar} \int_a^b dz^\mu \, A_\mu(z) \right)$$

to ensure gauge invariance.
- Integral over $y$: uncertainty principle.

# Transport equation

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- From Dirac equation: transport equation for Wigner function:
  H.-Th. Elze, M. Gyulassy, and D. Vasak, Ann. Phys. 173 (1987) 462

$$(\gamma_\mu K^\mu - m) W(X,p) = 0,$$

with

$$\begin{aligned} K^\mu &\equiv \Pi^\mu + \frac{1}{2} i\hbar \nabla^\mu, \\ \nabla^\mu &\equiv \partial_x^\mu - j_0(\Delta) F^{\mu\nu} \partial_{p\nu}, \\ \Pi^\mu &\equiv p^\mu - \hbar \frac{1}{2} j_1(\Delta) F^{\mu\nu} \partial_{p\nu}, \end{aligned}$$

$\Delta = \frac{1}{2}\hbar \partial_p \cdot \partial_x$ with $\partial_x$ only acting on $F^{\mu\nu}$ and $j_0(r) = \sin(r)/r$, $j_1(r) = [\sin(r) - r\cos(r)]/r^2$ spherical Bessel functions.

- Exact quantum kinetic equation for Wigner function for massive spin 1/2-particles and inhomogeneous fields!
- Only assumption: vanishing collision kernel, external classical gauge fields.

## Strategy

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- Decompose $W$ into generators of Clifford algebra.

$$W = \frac{1}{4}\left(\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right).$$

- Insert into transport equation.
- Get system of 32 coupled (differential) equations.
- Equations for $\mathcal{F}$ (scalar, "distribution function") and $\mathcal{S}_{\mu\nu}$ (tensor, "dipole moment") decouple from rest.
- Solve by expanding in powers of $\hbar$, assuming that Wigner function gradients, em field strengths and em field gradients are sufficiently small.
- Determine $\mathcal{V}_\mu$ ("vector current"), $\mathcal{A}_\mu$ ("polarization"), $\mathcal{P}$ from $\mathcal{S}_{\mu\nu}, \mathcal{F}$.

# Zeroth-order Wigner function

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- To zeroth order:

$$(p_\mu \gamma^\mu - m) W(x,p) = 0.$$

Wigner function is on-shell!

- Momentum variable $p$ is physical momentum of particle.
- Direct calculation yields

$$\begin{aligned}
\mathcal{F}^{(0)}(x,p) &= m\,\delta(p^2 - m^2) V(x,p), \\
\mathcal{A}^{(0)}_\mu(x,p) &= m\, n_\mu\, \delta(p^2 - m^2) A(x,p), \\
\mathcal{P}^{(0)}(x,p) &= 0, \\
\mathcal{V}^{(0)}_\mu(x,p) &= p_\mu\, \delta(p^2 - m^2) V(x,p), \\
\mathcal{S}^{(0)}_{\mu\nu}(x,p) &= m\, \Sigma_{\mu\nu}\, \delta(p^2 - m^2) A(x,p),
\end{aligned}$$

with

$$\begin{aligned}
V(x,p) &\equiv \frac{2}{(2\pi\hbar)^3} \sum_s f_s(x,p), \\
A(x,p) &\equiv \frac{2}{(2\pi\hbar)^3} \sum_s s f_s(x,p).
\end{aligned}$$

- Solution fulfills zeroth-order transport equation for Wigner function.

# Next-to-leading order

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- To first order, Wigner function is no longer on-shell!
- Momentum variable of directly calculated Wigner function is not equal to physical momentum of particle → useless!
- Use transport equation for Wigner function to determine first-order solution!
- Generalized on-shell conditions:

$$
\begin{aligned}
(p^2 - m^2)\mathcal{F} &= \frac{1}{2}\hbar F^{\mu\nu}\mathcal{S}_{\mu\nu}, \\
(p^2 - m^2)\mathcal{S}_{\mu\nu} &= \hbar F_{\mu\nu}\mathcal{F}.
\end{aligned}
$$

- with constraint:

$$
p_\mu \mathcal{S}^{\mu\nu} = -\frac{\hbar}{2}\nabla^\nu \mathcal{F}.
$$

# $\mathcal{F}$ and $\mathcal{S}^{\mu\nu}$ up to order $\hbar$

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- General solution:

$$\mathcal{F} = m\left[V\,\delta(p^2-m^2) - \hbar\frac{1}{2}F^{\mu\nu}\Sigma_{\mu\nu}A\,\delta'(p^2-m^2)\right] + \mathcal{O}(\hbar^2)\,,$$

$$\mathcal{S}_{\mu\nu} = m\left[\tilde{\Sigma}_{\mu\nu}\delta(p^2-m^2) - \hbar F_{\mu\nu}V\delta'(p^2-m^2)\right] + \mathcal{O}(\hbar^2)\,,$$

with

$$\tilde{\Sigma}_{\mu\nu} \equiv \Sigma_{\mu\nu}A + \frac{\hbar}{2}\chi_{\mu\nu}$$

- $p_\nu\Sigma^{\mu\nu} = 0 \quad \rightarrow$ dipole-moment tensor, (remember: $\Sigma^{\mu\nu} = -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\alpha n_\beta$,)
- $p_\nu\chi^{\mu\nu} = \nabla^\mu V \quad \rightarrow$ induced by vorticity.

## $\mathcal{V}^\mu$, $\mathcal{P}$, and $\mathcal{A}^\mu$ up to order $\hbar$

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- From $\mathcal{F}$ and $\mathcal{S}_{\mu\nu}$ determine

$$\begin{aligned}
\mathcal{P} &= \hbar \frac{1}{4m} \epsilon^{\mu\nu\alpha\beta} \nabla_\mu \left[ p_\nu \Sigma_{\alpha\beta} A \, \delta(p^2 - m^2) \right] + \mathcal{O}(\hbar^2) \,, \\
\mathcal{V}_\mu &= \delta(p^2 - m^2) \left[ p_\mu V + \hbar \frac{1}{2} \nabla^\nu \Sigma_{\mu\nu} A \right] \\
&\quad - \hbar \left[ \frac{1}{2} p_\mu F^{\alpha\beta} \Sigma_{\alpha\beta} + \Sigma_{\mu\nu} F^{\nu\alpha} p_\alpha \right] A \, \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2) \,, \\
\mathcal{A}_\mu &= -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} p^\nu (\Sigma^{\alpha\beta} A + \frac{\hbar}{2} \chi^{\alpha\beta}) \, \delta(p^2 - m^2) \\
&\quad + \hbar \tilde{F}_{\mu\nu} p^\nu V \, \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2) \,.
\end{aligned}$$

- Polarization currents, induced by vorticity $\rightarrow$ CVE!
- Off-shell currents, induced by em fields $\rightarrow$ CME!

## Boltzmann equation for massive spin-1/2 particles

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- Generalized Boltzmann equation:

$$p\cdot\nabla\mathcal{F}=\hbar\frac{1}{2}\partial_x^\lambda F^{\nu\rho}(\partial_{p\lambda}\mathcal{S}_{\nu\rho}+\partial_{p\rho}\mathcal{S}_{\nu\lambda}).$$

- Taylor expansion:

$$\delta\left(p^2-m^2-\hbar\frac{s}{2}F^{\mu\nu}\Sigma_{\mu\nu}\right)=\delta(p^2-m^2)-\hbar\frac{s}{2}F^{\mu\nu}\Sigma_{\mu\nu}\delta'(p^2-m^2)+O(\hbar^2),$$

- After some calculation:

$$\sum_s\delta\left(p^2-m^2-\frac{s}{2}\hbar F^{\alpha\beta}\Sigma_{\alpha\beta}\right)\left\{p^\mu\partial_{x\mu}f_s+\partial_{p\mu}\left[F^{\mu\nu}p_\nu+\hbar\frac{1}{4}s\Sigma^{\nu\rho}(\partial^\mu F_{\nu\rho})\right]f_s\right\}=0.$$

- Modified on-shell condition!
- Recover first Mathisson-Papapetrou-Dixon equation!
  W. Israel, General Relativity and Gravitation 9 (1978) 451

## Time evolution of spin in classical limit

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- From kinetic equation for dipole moment to zeroth order:

$$m\frac{d}{d\tau}\Sigma^{\mu\nu} = \Sigma^{\lambda\nu}F^{\mu}_{\ \lambda} - \Sigma^{\lambda\mu}F^{\nu}_{\ \lambda},$$

  where $\tau$ is worldline parameter with $\frac{d}{d\tau} = \dot{x}^{\mu}\frac{\partial}{\partial x^{\mu}} + \dot{p}^{\mu}\frac{\partial}{\partial p^{\mu}}$, where $\dot{x} \equiv \frac{\partial x}{\partial \tau}$.
- Recover second Mathisson-Papapetrou-Dixon equation!
  W. Israel, General Relativity and Gravitation 9 (1978) 451
- After some calculation:

$$m\frac{d}{d\tau}n^{\mu} = F^{\mu\nu}n_{\nu}.$$

- Recover BMT equation!
  V. Bargmann, L. Michel, and V. L. Telegdi, PRL 2 (1959) 435-436

# Excursion: Massive vs. massless dipole-moment tensor

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- Non-relativistic dipole-moment tensor:

$$\Sigma^{ij} = \epsilon^{ijk} n^k ,$$

where $n^k$ is spin three-vector.

- For massive particles: define the spin in rest frame.
Spin vector $n^\mu$ is additional degree of freedom.
U. Heinz, PLB 144 (1984) 228

$$\Sigma^{\mu\nu} = -\frac{1}{m} \epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta .$$

- Massless limit cannot by obtained by simply taking $m \to 0$.
- For massless particles, there is no rest frame.
Define spin in arbitrary frame with four-velocity $u^\mu$.
Spin vector is always parallel to momentum.
J-Y. Chen, D. T. Son, and M. Stephanov, PRL 115 (2015) 021601

$$\Sigma_u^{\mu\nu} = -\frac{1}{p \cdot u} \epsilon^{\mu\nu\alpha\beta} u_\alpha p_\beta .$$

## Currents for $m = 0$

- Replace massive by massless dipole-moment tensor $\Sigma^{\mu\nu} \to \Sigma_u^{\mu\nu}$.
- Attention: $\delta(p^2 - m^2)/m \to \delta(p^2)/(p \cdot u)$.
- Constraint for $\chi_{\mu\nu}$ solved for

$$\chi_{\mu\nu} = \frac{1}{p \cdot u} \left( u_\nu \nabla_\mu - u_\mu \nabla_\nu \right) V \, .$$

with $u_\beta$ four-velocity of arbitrary frame.

- Define right- and left-handed currents $J_\mu^\pm \equiv \frac{1}{2}(\mathcal{V}_\mu^{m=0} \pm \mathcal{A}_\mu^{m=0})$
- Find:

$$J_\mu^\pm = \left[ p_\mu \delta(p^2) \pm \frac{1}{2} \hbar \epsilon_{\mu\nu\alpha\beta} p^\nu F^{\alpha\beta} \delta'(p^2) \pm \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha u^\beta}{p \cdot u} \delta(p^2) \nabla^\nu \right] f_\pm .$$

Agrees with previously known massless solution!
Y. Hidaka, S. Pu, D-L. Yang, PRD 95 (2017) 091901;
A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, arXiv:1801.03640 [hep-th]

## Global equilibrium I

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- Up to now: distribution function not determined.
  Most simple case: global equilibrium.
- Equilibrium distribution function:

$$f_s^{eq} = \left(e^{g_s} + 1\right)^{-1},$$

with $g$ linear combination of conserved quantities charge, momentum, and angular momentum:

$$g_s = \beta\pi \cdot U - \beta\mu_s + \frac{\hbar}{4} s\Sigma^{\mu\nu}\partial_\mu(\beta U_\nu).$$

Here, $\pi_\mu \equiv p_\mu + A_\mu$ is canonical momentum, $U$ is fluid velocity, $\beta \equiv \frac{1}{T}$ is inverse temperature, and $\mu_s$ is chemical potential.

- To zeroth order

$$f_s^{(0)} = \left(e^{g_{s0}} + 1\right)^{-1},$$

with

$$g_{s0} = \beta(\pi \cdot U - \mu_s).$$

# Global equilibrium II

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- By Taylor expansion of distribution function:

$$V^{(1)\mu} = \frac{2}{(2\pi\hbar)^3}\sum_s \Big[\delta(p^2-m^2)\left(p^\mu - m\hbar\frac{s}{2}\tilde{\omega}^{\mu\nu}n_\nu\partial_{\beta\pi\cdot U}\right) + \hbar s\tilde{F}^{\mu\nu}n_\nu\delta'(p^2-m^2) + \hbar\frac{s}{2m}\delta(p^2-m^2)\epsilon^{\nu\mu\alpha\beta}p_\alpha\nabla_\nu n_\beta\Big] f_s^{(0)}.$$

- Thermal vorticity tensor: $\omega_{\mu\nu} \equiv \frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu)$.
- Dual thermal vorticity tensor: $\tilde{\omega}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\omega^{\alpha\beta}$.

# Conclusions

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- Found transport equation for distribution function and polarization for massive spin-1/2 particles in inhomogeneous electromagnetic fields.
- Recovered classical equations of motion.
- Found a way of obtaining massless limit.
- Showed agreement of our solution to previously known massless solution in this limit.
- Gave explicit expressions for current in global equilibrium.

# Outlook

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- Generalized Boltzmann equation still has to be solved.
- Collisions have to be included.
  $\rightarrow$ Boltzmann equation without assumption of local equilibrium.
- Derive equations of motion for dissipative quantities.
  $\rightarrow$ Method of moments.

Back-up

# Diagonal spin basis

- Distribution function $f_{rs}$ is Hermitian matrix in spin space.
- Can be diagonalized by Unitary transformation:

$$f_{rs} = D_{rr'} \tilde{f}_{s'} \delta_{r's'} D^{\dagger}_{s's} .$$

- Redefine spinors

$$\tilde{u}_s \equiv \sum_{s'} u_{s'} D_{s's} .$$

- Define

$$sn^{\mu} \equiv \bar{\tilde{u}}_s \gamma^{\mu} \gamma^5 u_s .$$

- Only diagonal part contributes!

# Equilibrium conditions

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- "Homogeneous" part of the Boltzmann equation fulfilled if:

$$\begin{aligned} \mu_s &= \text{const}, \\ \partial_\nu \beta_\mu + \partial_\mu \beta_\nu &= 0, \\ \mathcal{L}_\beta F_{\mu\nu} &= 0. \end{aligned}$$

- "Inhomogeneous" part of Boltzmann equation: additional conditions to make global equilibrium possible, e.g. $\mu_{s=+} - \mu_{s=-} = 0$.
- In general: Axial-vector current not conserved for massive particles $\rightarrow$ no associated charge $\mu_5 \equiv \mu_{s=+} - \mu_{s=-}$ can be added in equilibrium distribution.