Thermodynamics and phase diagram of the Polyakov-Nambu-Jona-Lasinio model
in collaboration with Joerg Aichelin

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Outline

- Motivation: QCD Phase Diagram
- Introduction to Polyakov-Nambu-Jona-Lasinio Model

(P)NJL-like models contain much more than quarks (and static gluons)!

- Collective excitations: Mesons, Diquarks & Baryons
  - Equation of State

- Thermodynamics at $\mu = 0$: mean-field results & meson-like fluctuations
- Thermodynamics at $\mu \neq 0$: mean-field *prelim.* results
- Summary and Outlook
QCD phase diagram: Hadron and quark-gluon plasma phases

Heavy-ion collision to explore them (figure taken from T. Nayak, 2012).
We would like to be able to explore the confined and deconfined phases (and the phase transition itself) from a **single** framework.

QCD phase diagram: Hadron and quark-gluon plasma phases

**Simplest** model to describe the properties of **both** phases (hadronization eventually?)

One good candidate is

*(Polyakov)-Nambu-Jona–Lasinio Model*
NJL Lagrangian

From QCD, the NJL model is inspired on the limit $t \to 0$ of the gluon exchange interaction Lagrangian: color current interaction

$$L_{\text{int}} = -g \left[ \bar{q}_i \gamma^\mu T^a \delta_{ij} q_j \right] \left[ \bar{q}_k \gamma_\mu T^a \delta_{kl} q_l \right]$$

Flavor: $i, j = 1 \ldots N_f = 3$ ; $T^a$: color representations $a = 1 \ldots N_c^2 - 1 = 8$.

NJL for mesons

Exchange Lagrangian (after Fierz transformation): accounts for color singlet vertex

\[ \mathcal{L}_{ex} = G (\bar{q}_i \tau^{a}_{ij} \mathbb{I}_c i \gamma_5 q_j) (\bar{q}_k \tau^{a}_{kl} \mathbb{I}_c i \gamma_5 q_l) ; \quad G = (N_c^2 - 1)/N_c^2 g \]

\( \mathbb{I}_c \): color singlet interaction
\( \tau^a \): flavor generators \( a = 1...8 \) \( (N_f = 3) \).
Exchange Lagrangian (after Fierz transformation): accounts for color singlet vertex

\[ \mathcal{L}_{ex} = G \left( \bar{q}_i \, \tau^a_{ij} \, \mathbb{I}_c \, i \gamma_5 \, q_j \right) \left( \bar{q}_k \, \tau^a_{kl} \, \mathbb{I}_c \, i \gamma_5 \, q_l \right) ; \quad G = \frac{N_c^2 - 1}{N_c^2 g} \]

\( \mathbb{I}_c \): color singlet interaction
\( \tau^a \): flavor generators \( a = 1 \ldots 8 \) (\( N_f = 3 \)).

The effective Lagrangian should share the global symmetries of (massless) QCD:

**Symmetries of massless NJL model**

\[ SU_V(3) \otimes SU_A(3) \otimes U_V(1) \otimes U_A(1) \]

In our scheme, chiral symmetry is explicitly broken to \( SU_V(3) \) by the bare quark masses. We keep an isospin \( SU_V(2) \) symmetry. The \( U_A(1) \) is broken by quantum effects...
$U_A(1)$ breaking

$U_A(1)$ symmetry is broken by the **axial anomaly**. The anomaly is the responsible for the mass difference between the $\eta$ and $\eta'$. (mixing between flavor octet and singlet).

\[
\mathcal{L}_{t \text{ Hooft}} = H \det_{ij} [\bar{q}_i (1 - \gamma^5) q_j] - H \det_{ij} [\bar{q}_i (1 + \gamma^5) q_j]
\]

For $N_f = 3$ it represents a six-quark contact interaction.

$H$ is fixed by the $\eta - \eta'$ mass difference.
Gluon (static) properties are implemented in the model through an effective potential for the Polyakov loop:

\[
\frac{U(T, \Phi, \bar{\Phi})}{T^4} = -\frac{b_2(T)}{2} \Phi \bar{\Phi} - \frac{b_3}{6} (\Phi^3 + \bar{\Phi}^3) + \frac{b_4}{4} (\Phi \bar{\Phi})^3
\]

\[b_2(T) = a_0 + a_1 \frac{T_0}{T} + a_2 \left( \frac{T_0}{T} \right)^2 + a_3 \left( \frac{T_0}{T} \right)^3\]

where \(a_0 = 6.75, a_1 = -1.95, a_2 = 2.625, a_3 = -7.44, b_3 = 0.75, b_4 = 7.5\) from fit to lattice-QCD results of Yang-Mills and \(T_0 = 190\) MeV.

\(\Phi\) is the order parameter of the deconfinement transition:

\[\Phi = \frac{1}{N_c} \text{Tr}_c \langle \mathcal{P} \exp \left( -\int_0^\beta d\tau A_0(\mathbf{x}, \tau) \right) \rangle\]
Gap equation for quark masses

The dressed quark masses are calculated through the **gap equation** in the mean-field approximation.

\[ m_q = m_{q0} - 4G \langle \bar{q}q \rangle + 2H \langle \bar{q}'q' \rangle \langle \bar{q}''q'' \rangle \]

**Quark condensate**

\[ \langle \bar{q}q \rangle = -iN_c \text{Tr} \, S_q \quad , \quad S_q : \text{quark propagator} \]
Quark masses in the isospin limit

Quark masses as a function of the temperature (zero chemical potential)

The limit $m_q \to m_{q0}$ at large $T$ is a signal of the chiral symmetry restoration.
Hadrons can also be generated by the PNJL model!

Key results of

Flavor dependence of baryon melting temperature in effective models of QCD

J.M. Torres-Rincon, J. Aichelin and B. Sintes
The $\bar{q}q$ interaction is used as the kernel for the Bethe-Salpeter equation

$$T(p) = \mathcal{K} + i \int \frac{d^4 k}{(2\pi)^4} \mathcal{K} S \left( k + \frac{p}{2} \right) S \left( k - \frac{p}{2} \right) T(p)$$
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The solution of the $T$–amplitude at finite temperature

$$T(p) = \frac{\mathcal{K}}{1 - \mathcal{K}\Pi(p)} , \quad \Pi(p_0, p) = -T \sum_n \int \frac{d^3 k}{(2\pi)^3} S \left( k + \frac{p}{2} \right) S \left( k - \frac{p}{2} \right)$$

$$\equiv \Pi$$
Mesons appear as bound states/resonances of $\bar{q}q$ scattering, i.e. **poles of the $T$–matrix**.

The position of the pole gives the **mass** and **width** of the meson as a function of the temperature.

$$1 - \mathcal{K} \, \Pi(p_0 = M_{\text{meson}} - i\Gamma_{\text{meson}}/2, \mathbf{p} = 0) = 0$$
Mesons at finite temperature

Pseudoscalar mesons in the PNJL model at $\mu = 0$

Pion mass (blue line) and decay width (grey band)
We can also combine two quarks to form a DIQUARK.

\[
T(p) = \frac{2G_{DIQ}}{1 - 2G_{DIQ}[\Pi(p)]}
\]

\[
\Pi(p) = i \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ \bar{\Omega}(k + \frac{p}{2}) \Omega^T \left( \frac{p}{2} - k \right) \right]
\]

The procedure is analogous to mesons taking care of the different reps, spin structure and charge conjugations.
Modelization of baryons

Two-body equation from (3-body) Fadeev equation (see Reinhardt 1990; Buck et al. 1992 for details)

\[ G(P) = G_0 + G_0 Z G(P) \]

Solution

\[ G(P) = \frac{G_0}{1 - G_0 Z} \]

Baryon mass

\[ 1 - G_0 Z(P_0 = M_{baryon}, P = 0) = 0 \]
Modelization of baryons

Solution and Baryon mass

\[ G(P) = \frac{G_0}{1 - G_0 Z} \rightarrow 1 - G_0 Z(P_0 = M_{\text{baryon}}, P = 0) = 0 \]

Baryon masses in $8_f$ and $10_f$ representations, as a function of temperature.
Thermodynamics and phase diagram of the PNJL model
Equation of state

We want to compute the thermodynamical properties of the PNJL model at finite $T$ and $\mu_q$

Grand-canonical potential of the model: $\Omega(T, \mu_q)$

The computation is usually organized within large-$N_c$ limit considerations and $\Omega$ is expanded in powers of $1/N_c$ (see Klevansky (1992), Hüfner et al. (1993)).

$$\Omega = \sum \ldots$$

- The LO term contains the non-interacting contribution which is $O(N_c)$
- At $O(N_c)$ we also get the Hartree diagram
- Fock diagram is topologically equivalent in the NJL limit but already $O(1)$
At order $\mathcal{O}(N_c)$ the expression for the grand-canonical potential is

$$
\Omega_{PNJL}(\Phi, \bar{\Phi}, m_i, T, \mu_i) = 2G \sum_{i=u,d,s} \langle \bar{\psi}_i \psi_i \rangle^2 - 4H \prod_i \langle \bar{\psi}_i \psi_i \rangle - 2N_c \sum_i \int \frac{d^3 p}{(2\pi)^3} E_i - 2T \sum_i \int \frac{d^3 p}{(2\pi)^3} \left[ \text{tr}_c \log(1 + L e^{-(E_i - \mu_i)/T}) + \text{tr}_c \log(1 + L^\dagger e^{-(E_i + \mu_i)/T}) \right] + U(T, \Phi, \bar{\Phi})
$$

with $L$ the Polyakov loop and $E_i = \sqrt{p^2 + m_i^2}$.

Note that $G \sim \mathcal{O}(1/N_c)$, $H \sim \mathcal{O}(1/N_c^2)$, $\langle \bar{\psi}_i \psi_i \rangle \sim \mathcal{O}(N_c)$

(see details e.g. in Klevansky (1992) for the NJL model or Ratti, Thaler, Weise (2006) for PNJL model)
Equation of state at $\mu_q = 0$

The pressure of the system

\[
P(T) = - [ \Omega(T) - \Omega(0) ]
\]

- We need to subtract the vacuum contribution of $\Omega$.
- We remove the cutoff ($\Lambda \to \infty$) to all UV-convergent integrals (SB limit).
- We use polynomial ansatz for the YM effective potential $U$
  (other choices are possible).
- For $N_f = 3$, only the parameter $T_0$ is readjusted
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Other thermodynamical quantities at $\mu_q = 0$

$$s = \frac{dP}{dT} ; \quad e = sT - P ; \quad C_V = T \frac{ds}{dT} ; \quad \Theta = \epsilon - 3P ; \quad c_s = \frac{s}{C_V}$$

- $s$: entropy density
- $e$: energy density
- $c_s$: speed of sound
- $C_V$: specific heat
- $\Theta$: trace anomaly
Preliminary Results at $\mu_q = 0$

Thermodynamics at $\mathcal{O}(N_c)$ in the mean-field approximation and $\mu_q = 0$.

How it compares with lattice-QCD calculations?
Comparison to Lattice-QCD calculations

PNJL model at MF did a really good job with “OLD” lattice-QCD data ...
Comparison to Lattice-QCD calculations

PNJL model at MF did a really good job with “OLD” lattice-QCD data ...

but it does not describe well the CURRENT lattice-QCD data (S. Borsanyi et al. 2010).
Comparison to Lattice-QCD calculations

Any parametrization underestimates the pressure at low temperature, and push it very quickly to the Stefan-Boltzmann limit.
Any parametrization underestimates the pressure at low temperature, and push it very quickly to the Stefan-Boltzmann limit.
Low-temperature range

- Hadronic states are missing at low temperature
- In fact, they should be dominant in this limit
- The first contributions come from $\pi, K, \eta...$
- ...but PNJL can account for these states!

Mesonic fluctuations on top of the mean-field pressure
This means to include the diagrams at $O(1)$ in the $1/N_c$ expansion:

$\text{Ring summation} + \text{\ldots}$

(see Hüfner, Klevansky, Zhuang (1994,1995), Blaschke et al. (2014)...)

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How can we take into account these states in $\Omega(T)$?

- Mesonic fluctuations on top of the mean-field pressure
- This means to include the diagrams at $O(1)$ in the $1/N_c$ expansion:

  Ring summation

(see H"ufner, Klevansky, Zhuang (1994,1995), Blaschke et al. (2014)...
The $\mathcal{O}(1)$ contribution to the thermodynamic potential:

### Mesonic contribution

\[
\Omega_{\text{fluc}}^{\text{meson}} = \sum_M \frac{N_M}{2} T \int \frac{d^3 p}{(2\pi)^3} \sum_n \log \left( 2G t_M^{-1} \right)
\]

- The summation sum runs over all *Mesons*
- $N_M$ is the degeneracy of each state (e.g. $N_\pi = 3$ for pions)
- $t_M$ is the meson propagator coming from the BS equation

\[
t_M(i\omega_n, \mathbf{p}) = \frac{\mathcal{K}}{1 - \mathcal{K} \Pi_M(i\omega_n, \mathbf{p})}
\]

### Main message

The mesonic contribution to $\Omega$ is directly related to the resummed quark-antiquark *scattering amplitude* in the medium.
Mesonic contribution

The contribution can be rewritten in terms of the phase-shift

\[ e^{2i\delta_M(\omega,p)} = \frac{1 - \mathcal{K}\Pi_M(\omega + i\epsilon, p)}{1 - \mathcal{K}\Pi_M(\omega - i\epsilon, p)} \]

The thermodynamic potential

\[ \Omega_{\text{fluc}, M}^{\text{meson}} = -N_M \int \frac{d^3 p}{(2\pi)^3} \int_{-p^2}^{\infty} ds \frac{1}{2\sqrt{s + p^2}} \left( 1 + \frac{2}{e^{\beta\sqrt{s+p^2}} - 1} \right) \delta_M(s) \]

The analytic structure of the polarization function will determine the contribution to \( \Omega^{\text{meson}} \). In particular a nonzero phase-shift appears at any cut on the real axis (unitarity and Landau cuts).
Results at $\mu_q = 0$

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Thermodynamics and phase diagram of the PNJL model
Results at $\mu_q = 0$
Results at $\mu_q = 0$

![Graph showing results for different PNJL models and lattice QCD](image-url)

- **PNJL M.F.**
- **PNJL M.F.+$\pi$**
- **PNJL M.F.+$\pi$+K**
- **Lattice QCD**

The graph illustrates the pressure $P$ divided by $T^4$ as a function of temperature $T$ (in MeV), comparing different models and lattice QCD data.
Results at $\mu_q = 0$

$\frac{P}{T^4}$ vs $T$ (MeV)

- PNJL M.F.
- PNJL M.F.+π
- PNJL M.F.+π+K
- PNJL M.F.+π+K+K

lattice QCD

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Thermodynamics and phase diagram of the PNJL model
Results at $\mu_q = 0$

Note that more and more states are needed at intermediate temperatures.
Results at $\mu_q = 0$

Note that more and more states are needed at intermediate temperatures (eventually also baryon should be added).
At large-$T$ we have a gas of quasi-massless quarks (chiral limit) and the pressure gets very close to the SB limit.

**Solutions we are considering**

1. Include perturbative-QCD contribution from fermion Fourier modes $k \in (\Lambda, \infty)$ (Turko, Blaschke et al. 2013)

2. Consider backreaction of quarks into the gluonic Yang-Mills potential

$$\frac{U_{YM}}{T^4}(\Phi, \bar{\Phi}, T) \rightarrow \frac{U_{glue}}{T_{YM}^4}(\Phi, \bar{\Phi}, T_{YM}(T))$$

where the relation between glue and YM potentials comes an FRG study (Hass, Stiele et al. 2013)

WORK IN PROGRESS...
Finally, we are working on the thermodynamics at finite chemical potential (very preliminary results!)

**Scenario**

\[ \mu_q = \mu_u = \mu_d; \mu_s = 0 \quad \rightarrow \quad \mu_B = 3\mu_q, \mu_S = 0 \]
Finally, we are working on the thermodynamics at finite chemical potential (very preliminary results!)

Scenario

\[ \mu_q = \mu_u = \mu_d; \mu_s = 0 \rightarrow \mu_B = 3\mu_q, \mu_s = 0 \]

First we need to determine the phase boundary of the model, because for \( \mu_q \neq 0 \) one finds several solutions of the gap equation.

(First-order transition)

(Phase-Transition Temperature)

(be careful: this example is done with a different parametrization)
Critical point coordinates: \((T_c, \mu_{qc}) \simeq (169 \text{ MeV}, 245 \text{ MeV})\)
Very preliminary results at finite $\mu_q$ in the mean-field approximation.
Summary

Conclusions:

- PNJL model provides quark, meson, diquark and baryon masses (and widths) at finite \( T \) and \( \mu \).
- Thermodynamics at \( \mu_q = 0 \) taking into account mesonic fluctuations (beyond mean-field).
- Recent lattice-QCD EoS is reasonably good described by the model.
- Straightforward extension to finite chemical potential: first-order boundary and critical point.

Outlook:

- Improve matching to lattice-QCD at \( \mu_q = 0 \) and predict for \( \mu_q \neq 0 \) (feedback to lattice-QCD).
- \( \mu_q \neq 0 \): Incorporate vectors mesons, diquark condensation, Polyakov eff. potential.
- Generalized susceptibilities (fluctuations of conserved quantities).
- Signatures of critical behavior around \( (T_c, \mu_{qc}) \).
- Collision and hadronization cross sections (transport coefficients at finite \( T \) and \( \mu \)).
- ...
- Full implementation in a heavy-ion collision evolution routine.
THANKS FOR YOUR ATTENTION!
We use imaginary time formalism \((0 \leq it \leq \beta = 1/T)\) with the prescription

\[
k_0 \rightarrow i\omega_n, \quad \int \frac{d^4k}{(2\pi)^4} \rightarrow iT \sum_{n \in \mathbb{Z}} \int \frac{d^3k}{(2\pi)^3}
\]

with \(i\omega_n = i\pi T(2n + 1)\) the fermionic Matsubara frequencies.
We use imaginary time formalism \((0 \leq i\tau \leq \beta = 1/T)\) with the prescription

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\]

with \(i\omega_n = i\pi T(2n + 1)\) the fermionic Matsubara frequencies.

The quark chemical potential can be introduced by the Lagrangian

\[
\mathcal{L}_\mu = \sum_{ij} \bar{q}_i \mu_{ij} \gamma_0 q_j
\]

where

\[
\mu_{ij} = \text{diag} \left( \mu_u, \mu_d, \mu_s \right)
\]
\[ \Pi_{12}^P(p_0, p) = -\frac{N_c}{8\pi^2} \left\{ A(m_1) + A(m_2) + \left[ (m_1 - m_2)^2 - p_0^2 + p^2 \right] B_0(p, m_1, m_2, p_0) \right\} \]

with

\[ A(m_1) = 16\pi^2 T \sum_n \int \frac{d^3 k}{(2\pi)^3} \frac{1}{(i\omega_n)^2 - k^2 - m_1^2} \]

and

\[ B_0(p, m_1, m_2, i\nu_m \to p_0 + i\epsilon) = 16\pi^2 T \sum_n \int \frac{d^3 k}{(2\pi)^3} \frac{1}{(i\omega_n)^2 - k^2 - m_1^2} \]

\[ \times \frac{1}{(i\omega_n - i\nu_m)^2 - (p - k)^2 - m_2^2} \]
Combine $G$ and $K$ at mean-field level into an effective coupling, e.g. for pion:

\[
G_{\text{eff}} = G + \frac{1}{2} H \langle \bar{s} s \rangle
\]

The complete Feynmann rule for the 4-point function

\[
\mathcal{K} = \Omega \ 2G_{\text{eff}} \ \bar{\Omega}
\]

where $\Omega$ encodes color, flavor and spin factors

\[
\Omega = \mathbb{I}_c \otimes \tau^a \otimes \{1, i\gamma_5, \gamma_\mu, \gamma_5\gamma_\mu\}
\]
We take the inverse of the T-matrix

\[ T^{-1}(p^2) = \frac{1 - \mathcal{K}\Pi(p^2)}{\mathcal{K}} \]

and make a Taylor expansion around the pole mass of the meson/diquark \( p^2 = m^2 \)

\[ T^{-1}(p^2) = \left. \frac{\partial T^{-1}(p^2)}{\partial p^2} \right|_{p^2=m^2} (p^2 - m^2) + \cdots \]

\[ T^{-1}(p^2) = -\left. \frac{\partial \Pi(p^2)}{\partial p^2} \right|_{p^2=m^2} (p^2 - m^2) + \cdots \]

Finally, let us define an effective coupling constant

\[ g_{\text{eff}}^2 \equiv \frac{1}{\left. \frac{\partial \Pi(p^2)}{\partial p^2} \right|_{p^2=m^2}} \]

(1)

to finally obtain the meson propagator from the T–matrix

\[ t(p^2) \sim \frac{-g_{\text{eff}}^2}{p^2 - m^2} \]
Restoration of chiral symmetry at large temperatures

Graph showing the restoration of chiral symmetry as a function of temperature (T) and mass. The graph includes lines for different particles: $a_1$, $\rho$, $f_0=\sigma$, and $\pi$.
We can also combine two quarks to form a DIQUARK.

However, it cannot be color singlet $I_C$

$\rightarrow$ they are not interesting as observable states

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We focus on **SCALAR** and **AXIAL** diquarks to build baryons.
(the other diquarks have masses much higher and not suitable to form baryons)
We repeat the calculation with diquarks $\langle q^T \Omega q \rangle$.

There are many types of diquarks according to $\Omega = \Omega_c \otimes \Omega_f \otimes \Omega_{Dirac}$:

$$\Omega_c \in 3 \otimes 3 = \bar{3} \oplus 6$$

$$\Omega_f \in 3 \otimes 3 = \bar{3} \oplus 6$$

$$\Omega_{Dirac} \in 1/2 \otimes 1/2 = 0 \oplus 1$$

With the constraint of Pauli principle: total antisymmetry

$$\Omega^T = -\Omega$$
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Result for [Scalar] and (Axial) diquarks ($q$ denotes either $u$ or $d$ quark)

They will be used to construct the members of the baryon octet and decuplet, respectively.

$$3 \otimes (\bar{3} \oplus 6) = 1 \oplus 8 \oplus 8 \oplus 10$$
Pseudoscalar and vector diquarks have too large masses and they present decay widths even at $T = 0$.

Therefore, we are not assume these states to take part of baryons.
The formal equation for the baryon wavefunction $\mathcal{X}$ (flavor octet) reads (Reinhardt 1990; Buck et al. 1992)

\[
\mathcal{X}_j^\bar{j}(P, q) - \int \frac{d^4 k}{(2\pi)^4} Z_{jk}^\bar{k}(q, k) \ G_{0, k\bar{k}}(P, q) \ \mathcal{X}_k^\bar{k}(P, k) \big|_{P^2 = M_{Baryon}^2} = 0
\]

Quark index $j$, diquark index $\bar{j}$. The two-body quark-diquark propagator reads

\[
G_{0, k\bar{k}}(P, q) = S_k(P/2 + q) \ i t_k(P/2 - q)
\]

And the interaction subdiagram with two diquark-quark-quark vertices

\[
Z_{jk}^\bar{k}(q, k) = \Omega_{jl}^\bar{k} \ S_l(-q - k) \ \Omega_{lk}^\bar{j}
\]
It is convenient to use normalized projectors on to the physical baryons.

\[(\mathcal{P}_{ij}^{A})^\dagger \mathcal{P}_{ij}^{A'} = \delta^{*,AA'}\]

In the octet representation of \(SU(3)\) we can construct them from the Gell-Mann matrices

\[
\begin{align*}
\mathcal{P}_{ij}^{p} &= \frac{1}{2} \left( \lambda^4 - i\lambda^5 \right)_{ij} ; & \mathcal{P}_{ij}^{n} &= \frac{1}{2} \left( \lambda^6 - i\lambda^7 \right)_{ij} \\
\mathcal{P}_{ij}^{\Lambda} &= \mathcal{P}_{ij}^{8} = \sqrt{\frac{1}{2}} \lambda^8_{ij} ; & \mathcal{P}_{ij}^{\Sigma^{0}} &= \mathcal{P}_{ij}^{3} = \sqrt{\frac{1}{2}} \lambda^3_{ij} \\
\mathcal{P}_{ij}^{\Sigma^{\pm}} &= \frac{1}{2} \left( \lambda^1 \mp i\lambda^2 \right)_{ij} ; & \mathcal{P}_{ij}^{\Xi^{0}} &= \frac{1}{2} \left( \lambda^6 + i\lambda^7 \right)_{ij} \\
\mathcal{P}_{ij}^{\Xi^{-}} &= \frac{1}{2} \left( \lambda^4 + i\lambda^5 \right)_{ij} ; & \mathcal{P}_{ij}^{0} &= \sqrt{\frac{1}{3}} \mathbb{I}_{ij}
\end{align*}
\]

Where the singlet flavor mixes with the \(\Lambda - \Sigma^{0}\), producing a coupled channel problem (note that the \(\Lambda, \Sigma\) and singlet contains the same quark content). Because of this fact, the projectors are not totally orthogonal (*).

Similar for the decuplet representation of \(SU(3)\) to obtain the \(\Delta, \Sigma^{*}, \Xi^{*}, \Omega\).